Padmakar-Ivan index of some types of perfect graphs

Manju Sankaramalil Chithrabhanu1,*, Kanagasabapathi Somasundaram2

1Department of Mathematics, Amrita School of Arts and Science, Kochi, Amrita Vishwa Vidyapeetham, India
2Department of Mathematics, Amrita School of Engineering, Coimbatore, Amrita Vishwa Vidyapeetham, India

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Abstract

The Padmakar-Ivan (PI) index of a graph $G$ is defined as $PI(G) = \sum_{e \in E(G)} (|V(G)| - N_G(e))$, where $N_G(e)$ is the number of equidistant vertices for the edge $e$. A graph is perfect if for every induced subgraph $H$, the equation $\chi(H) = \omega(H)$ holds, where $\chi(H)$ is the chromatic number and $\omega(H)$ is the size of a maximum clique of $H$. In this paper, the PI index of some types of perfect graphs is obtained. These types include co-bipartite graphs, line graphs, and prismatic graphs.

Keywords: PI index; co-bipartite graphs; line graphs; prismatic graphs.

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1. Introduction

All graphs considered in this paper are finite, simple and connected. For a graph $G$, the distance between two vertices $x, y$ is denoted by $d(x, y)$. A vertex $w$ is equidistant for an edge $e = xy$ if $d(x, w) = d(y, w)$. For an edge $e \in E(G)$, denote by $D_G(e)$ the set of all equidistant vertices in $G$. In particular, $D_i(e)$ denotes the set of vertices at distance $i$ for $e$. Also, we denote $|D_G(e)| = N_G(e)$.

The vertex Padmakar-Ivan (PI) index of a graph $G$ is a topological index, defined as

$$PI(G) = \sum_{e = uv \in E(G)} (n_u(e) + n_v(e)),$$

where $n_u(e)$ denotes the number of those vertices of $G$ whose distance from the vertex $u$ is smaller than the distance from the vertex $v$ and $n_v(e)$ denotes the number of those vertices of $G$ whose distance from $v$ is smaller than the distance from $u$. Since $n_u(e) + n_v(e) = |V(G)| - N_G(e)$, the PI index can be rewritten as

$$PI(G) = \sum_{e \in E(G)} (|V(G)| - N_G(e)).$$

The PI index was proposed by Khadikar [10] in 2000. Khadikar and his coauthors investigated the chemical and biological applications of this index in [11]. Khalifeh [12] introduced a vertex version of the PI index and using this notion, they computed exact expression for the PI index of Cartesian product of graphs. John and Khadikar established a method for calculating the PI index of benzenoid hydrocarbons using orthogonal cuts in [9]. Gutman and Ashrafi [6] obtained the PI index of phenylenes and their hexagonal squeezes. The PI index of bridge graphs and chain graphs was studied in [13]. Das and Gutman [3] obtained a lower bound on the PI index of a connected graph in terms of the number of vertices, edges, pendent vertices, and the clique number, and also they characterized the extremal graphs. There are different types of topological indices; for example distance-based topological indices, degree-based topological indices, etc. Topological indices has many applications in the field of mathematical chemistry. Trinajstić and Zhou introduced the sum-connectivity index and found several basic properties in [16]. Many topological indices and their applications are thoroughly explored in [15]. Ilić and Milosavljević introduced the weighted vertex PI index and established some of its basic properties in [7]. The weighted PI index of a graph $G$ is given as

$$PI_w(G) = \sum_{e = uv \in E(G)} (d_G(u) + d_G(v)) (|V(G)| - N_G(e)).$$

Gopika et al. [5] obtained the weighted PI index of the direct and strong product for certain types of graphs. Indulal et al. [8] studied the graphs satisfying the equation $PI(G) = PI(G - e)$.
A graph is perfect if for every induced subgraph \( H \), the equation \( \chi(H) = \omega(H) \) holds, where \( \chi(H) \) is the chromatic number and \( \omega(H) \) is the size of a maximum clique of \( H \). A claw-free graph is a graph in which no vertex has three pairwise nonadjacent neighbours. Every claw-free graph is a perfect graph. A survey on claw-free graphs is given in [4]. Chudnovsky and Seymour studied the structure of claw-free graphs thoroughly in a series of seven papers from 2007 to 2012. For example, in the first paper [1] of this series, they studied the orientable prismatic graphs and in the second paper [2], they studied non-orientable prismatic graphs. In this paper, we obtain the PI index of some classes of perfect graphs, including co-bipartite graphs, line graphs, and prismatic graphs.

2. Co-bipartite graphs

An edge \( e = xy \) of a graph \( G \) is said to be an equidistant edge for a vertex \( a \in V(G) \) if \( d(a, x) = d(a, y) \). The edge \( e \) is at distance \( r \) for a vertex \( a \) if \( d(a, x) = d(a, y) = r \). The set of all equidistant edges of \( a \) is \( D_G(a) = \{e = xy \in E(G) : d(a, x) = d(a, y)\} \) and we take \( N_G(a) = |D_G(a)| \). It is easy to see that \( \sum_{e \in E(G)} N_G(e) = \sum_{a \in V(G)} N_G(a) \).

**Lemma 2.1.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then, \( PI(G) = mn - \sum_{a \in G} N_G(a) \).

**Proof:**

\[
PI(G) = \sum_{e \in E(G)} (|V(G)| - N_G(e)) = \sum_{e \in E(G)} |V(G)| - \sum_{e \in E(G)} N_G(e) = mn - \sum_{a \in G} N_G(a).
\]

Let \( G(U, V) \) be a bipartite graph with partite sets \( U \) and \( V \). A co-bipartite graph is the complement of a bipartite graph \( G(U, V) \) and it is denoted as \( \overline{G} \). In \( \overline{G} \), the vertices in \( U \) and the vertices in \( V \) forms two disjoint cliques. Every co-bipartite graph is a perfect graph. The diameter of a connected co-bipartite graph is either 2 or 3.

Consider a bipartite graph \( G(U, V) \) with \( |U| = n \) and \( |V| = m \). Let \( \Delta_1 \) and \( \Delta_2 \) be the maximum degree in \( U \) and \( V \) respectively, where \( \Delta_1 \leq m \) and \( \Delta_2 \leq n \). Let \( U_1 = \{u \in U : d(u) < m\} \) and \( U_2 = \{u \in U : d(u) = m\} \). It is noted that \( U = U_1 \cup U_2 \). Similarly, \( V = V_1 \cup V_2 \), provided that the degree of every vertex in \( V_1 \) is less than \( n \) and the degree of every vertex in \( V_2 \) is \( n \). Let \( U_1 = \{u_1, u_2, ..., u_p\}, U_2 = \{u_{p+1}, u_{p+2}, ..., u_n\}, V_1 = \{v_1, v_2, ..., v_q\} \) and \( V_2 = \{v_{q+1}, v_{q+2}, ..., v_m\} \). Let \( d(u_i) = f_i \) for \( i = 1, 2, ..., p \) and \( d(v_j) = g_j \) for \( i = 1, 2, ..., q \). We denote \( \sum_{i=1}^p f_i \) by \( f^* \), \( \sum_{j=1}^q g_j \) by \( g^* \) and \( g^* \).

**Theorem 2.1.** Let \( G(U, V) \) be a bipartite graph. Then \( PI(\overline{G}) = n(n-1) + m(m-1) + mn(2p+q) - pq(m+n-p-q) - mp(p-1) - nq(q+1) + f(2m-n-1) - 2(f^* + g^*) + g(3n-1) \).

**Proof:** Let \( U = U_1 \cup U_2 \) and \( V = V_1 \cup V_2 \), where \( U_1 = \{u_1, u_2, ..., u_p\}, U_2 = \{u_{p+1}, u_{p+2}, ..., u_n\}, V_1 = \{v_1, v_2, ..., v_q\}, V_2 = \{v_{q+1}, v_{q+2}, ..., v_m\} \), \( d(u_i) = f_i \) if \( i \leq p \), \( d(u_i) = m \) if \( i > p \), \( d(v_j) = g_j \) if \( j \leq q \), and \( d(v_j) = n \) if \( j > q \). The degrees in \( \overline{G} \) (see Figure 1) are given as

\[
d(u_i) = \begin{cases} (m-f_i) + (n-1) & \text{if } i = 1, 2, ..., p \\ (n-1) & \text{if } i > p \end{cases}
\]

and

\[
d(v_j) = \begin{cases} (n-g_j) + (m-1) & \text{if } j = 1, 2, ..., q \\ (m-1) & \text{if } j > q. \end{cases}
\]

We partition \( E(\overline{G}) \) with \( E_1, E_2 \) and \( E_3 \), where \( E_1 \) is the set of edges in the clique with vertices in \( U \), \( E_2 \) is the set of edges in the clique with vertices in \( V \) and \( E_3 = \{(u, v) : u \in U, v \in V\} \).

![Figure 1: The graph \( \overline{G} \) used in the proof of Theorem 2.1.](image)

For a vertex \( u \in U \), it is easy to see that

\[
N_{E_1}(u) = \frac{(n-1)(n-2)}{2}.
\]
A vertex \( v_i \in V_1 \) has \((n - g_i)\) neighbours in \( U \) and the remaining \( g_i \) vertices are at distance 2, which means that
\[
N_{E_1}(v_i) = \frac{(n - g_i)(n - g_i - 1)}{2} + \frac{g_i(g_i - 1)}{2}.
\]
Similarly, a vertex \( v \in V_2 \) has no neighbours in \( U \) and
\[
d(u_i, v) = \begin{cases} 2 & \text{if } u_i \in U_1, \\ 3 & \text{if } u_i \in U_2. \end{cases}
\]
Also,
\[
N_{E_1}(v) = \frac{p(p - 1)}{2} + \frac{(n - p)(n - p - 1)}{2}
\]
and
\[
\sum_{e \in E_1} N_G(e) = \frac{n(n - 1)(n - 2)}{2} + \sum_{i=1}^{q} \left(\frac{(n - g_i)(n - g_i - 1)}{2} + \frac{g_i(g_i - 1)}{2}\right) + (m - q) \left(\frac{p(p - 1)}{2} + \frac{(n - p)(n - p - 1)}{2}\right).
\]
Similarly, for edges in \( E_2 \), one has
\[
\sum_{e \in E_2} N_G(e) = \frac{m(m - 1)(m - 2)}{2} + \sum_{j=1}^{p} \left(\frac{(m - f_j)(m - f_j - 1)}{2} + \frac{f_j(f_j - 1)}{2}\right)
\]
\[
+ (n - p) \left(\frac{q(q - 1)}{2} + \frac{(m - q)(m - q - 1)}{2}\right) + (n + m)(m - f_1) + (m - f_2) + ... + (m - f_p)
\]
\[
- \left(\sum_{j=1}^{p}(m - f_j)(m - f_j - 1) + \sum_{i=1}^{q}(n - g_i)(n - g_i - 1)\right)
\]
\[
= \frac{n(n - 1)(m + n)}{2} - \frac{n(n - 1)(n - 2)}{2} - \sum_{i=1}^{q} \left(\frac{(n - g_i)(n - g_i - 1)}{2} + \frac{g_i(g_i - 1)}{2}\right)
\]
\[
- (m - q) \left(\frac{p(p - 1)}{2} + \frac{(n - p)(n - p - 1)}{2}\right)
\]
\[
+ \frac{m(m - 1)(m + n)}{2} - \frac{m(m - 1)(m - 2)}{2} - \sum_{j=1}^{p} \left(\frac{(m - f_j)(m - f_j - 1)}{2} + \frac{f_j(f_j - 1)}{2}\right)
\]
\[
- (n - p) \left(\frac{q(q - 1)}{2} + \frac{(m - q)(m - q - 1)}{2}\right) + (n + m)(m - f_1) + (m - f_2) + ... + (m - f_p)
\]
\[
- \left(\sum_{j=1}^{p}(m - f_j)(m - f_j - 1) + \sum_{i=1}^{q}(n - g_i)(n - g_i - 1)\right)
\]
\[
= n(n - 1)(m + 2) - \sum_{i=1}^{q}(n - g_i)(n - g_i - 1) - \sum_{i=1}^{q} g_i(g_i - 1) - (m - q)(p(p - 1) + (n - p)(n - p - 1))
\]
\[
+ (m(m - 1)(n + 2) - \sum_{j=1}^{p}(m - f_j)(m - f_j - 1) - \sum_{j=1}^{p} f_j(f_j - 1) - (n - p)(q(q - 1) + (m - q)(m - q - 1))
\]
\[
+ 2(m + n)\sum_{j=1}^{p}(m - f_j) - 2\sum_{j=1}^{p}(m - f_j)(m - f_j - 1) - 2\sum_{i=1}^{q}(n - g_i)(n - g_i - 1)
\]
\[
= 2n^2 - 2n + 4mpq + 2p^2n - 2nmp + 2p^2q + 2m^2 - 2m + 2mmpq - 2mpq + 2pq^2
\]
\[
- 2n^2q + 2nq + 2pm - 2n\sum_{j=1}^{p} f_j + 4m\sum_{j=1}^{p} f_j - 4\sum_{j=1}^{p} f_j^2 - 2\sum_{j=1}^{p} f_j + 6n\sum_{i=1}^{q} g_i - 4\sum_{i=1}^{q} g_i^2 - 2 \sum_{i=1}^{q} g_i
\]
\[
= n(n - 1) + m(m - 1) + mn(2p + q) - pq(m + n - q - p) - mp(p - 1)
\]
\[
- nq(q + n - 1) + f(2m - n - 1) - 2(f^* + g^*) + g(3n - 1).
\]
A bipartite graph $G(U, V)$ is $(x, y)$-biregular if each vertex in $U$ has degree $x$ and each vertex in $V$ has degree $y$.

**Corollary 2.1.** If $G(U, V)$ is a $(x, y)$-biregular graph then $PI(G) = (n + m)(n + m - 1) + 2my(n + m - (x + y + 1))$.

**Proof.** From Theorem 2.1, we have

$$PI(G) = n(n - 1) + m(m - 1) + mn(2p + q) - pq(m + n - p - q) - mp(p - 1) - nq(q + n - 1) + f(2m - n - 1) - 2(f + g) + g(3n - 1).$$

Here $p = n, q = m, f = nx, g = my, f^* = nx^2$, and $g^* = my^2$. Thus,

$$PI(G) = n^2 - n + m^2 - m + 2mn + 2m^2x - n^2x - nx - 2nx^2 - 2my^2 + 3my - my$$

$$= (n + m)^2 - (n + m) + 2nym + 2m^2y - 2mxy - 2my^2 - 2my$$

$$= (n + m)(n + m - 1) + 2my(n + m - (x + y + 1)).$$

\[\square\]

**Corollary 2.2.** If $G$ is a $k$-regular bipartite graph with $2n$ vertices then $PI(G) = 2n \left[ 2n(k + 1) - (2k^2 + k + 1) \right]$.

**Proof.** In Theorem 2.1, by taking $n = m$ and $x = y = k$, one gets

$$PI(G) = 2n(2n - 1) + 2nk(2n - (2k + 1))$$

$$= 2n(2n(k + 1) - (2k^2 + k + 1)).$$

\[\square\]

Two particular examples of Corollary 2.2 are $PI(C_{2n}) = 2n(6n - 11)$ and $PI(K_{n,n}) = 2n(n - 1)$.

**Corollary 2.3.** If $G$ is a $k$-regular bipartite graph with $2n$ vertices then

$$PI_w(G) = 4n(2n - k - 1) \left( 2n(k + 1) - (2k^2 + k + 1) \right).$$

**Proof.** We know that the weighted PI index of a regular graph is a multiple of its PI index. Therefore,

$$PI_w(G) = 2(2n - k - 1) PI(G) = 4n(2n - k - 1) \left( 2n(k + 1) - (2k^2 + k + 1) \right).$$

\[\square\]

3. **Line graphs of some classes of graphs**

Let $G$ be a graph with $n$ vertices and $m$ edges. Its line graph denoted by $L(G)$, is a simple graph whose vertices are the edges of $G$ and two vertices are adjacent in $L(G)$ if the corresponding edges are adjacent in $G$. Let $T$ be a tree with $n$ vertices. Every vertex $v$ in $T$ with degree $i, i > 2$, forms a star $K_{1,i}$, in $T$, we denote it by $S_i$. Let $S$ be the collection of all stars in $T$. If we delete edges of all stars in $T$, the remaining edges of $T$ are parts of paths. Some paths have both of its end vertices common with the stars; we call them as central paths and the remaining have one end vertex shared with stars (paths) and the other end vertex is a pendant vertex; we call them leaf paths. We denote the central path with the length $l$ by $P_l$ and pendant path with the length $l$ by $P_l^\ast$. As we know that line graphs of stars are complete graphs and line graphs of paths are paths. Each star $S_i$ in $T$ is transformed to a clique with $K_i$ in $L(T)$. The central path $P_l$ has $l$ edges, so it is transformed to the path with $l$ vertices having length $l - 1$ and each of its end vertices is connected with a vertex of a clique in $L(T)$, so it has $l - 1 + 2 = l + 1$ edges. Each leaf path $P_l^\ast$ is transformed to a path with $l$ vertices and $l - 1$ edges, and it is connected with a vertex of $L(T)$, so it has $l$ edges.

**Theorem 3.1.** Let $T$ be a tree with $n$ vertices then

$$PI(L(T)) = (n - 1)(n - 2).$$

**Proof.** Let $T$ be a tree with $n$ vertices. Assume that the edge set $E(T)$ is the union of $m$ stars $S_{k_i}, r$ central paths $P_{f_i}$, and $s$ pendant paths $P_{g_i}^\ast$. Let us assume that

$$S = \cup_{i=1}^m S_{k_i} \quad \text{and} \quad P' = \cup_{i=1}^r P_{f_i} \cup \cup_{i=1}^s P_{g_i}^\ast.$$
Therefore, we claim that
\[ \sum_{A \text{ vertex in common}} (|V(T)| - N_T(e)) = (k_i - 1) (n - 1). \]

Let \( e \) be an edge of \( K_i \) in \( L(T) \) and let \( v \in V(K_i) \) be equidistant to \( e \). If we delete all the edges of \( K_i \), then \( L(T) \) has more than one component. All vertices in the component \( W \) containing \( v \) are also equidistant to \( e \). If we consider all the edges and vertices of \( K_i \), then
\[
\sum_{e \in E(K_i)} (|V(T)| - N_T(e)) = (k_i - 1) (n - 1).
\]

Also, since each edge of a path is a cut edge, there is no equidistant vertex corresponding to those edges. Each \( P_{j+1} \) contributes \((f_i + 1)(n - 1)\) and each \( P_{g_i}^* \) contributes \( g_i(n - 1) \) to the PI index of \( L(T) \). Thus,
\[
PI(L(T)) = \sum_{e \in E(L(T))} (|V(L(T))| - N_{L(T)}(e))
\]
\[
= \sum_{e \in E(\cup S_k)} (|V(L(T))| - N_{L(T)}(e)) + \sum_{e \in E(\cup P_{j+1})} (|V(L(T))| - N_{L(T)}(e)) + \sum_{e \in E(\cup P_{g_i}^* \cup P_{j+1})} (|V(L(T))| - N_{L(T)}(e))
\]
\[
= \sum_{i=1}^{m} (k_i - 1)(n - 1) + \sum_{i=1}^{r} (f_i + 1)(n - 1) + \sum_{i=1}^{s} g_i(n - 1)
\]
\[
= (n - 1) \left( \sum_{i=1}^{m} k_i - m + \sum_{i=1}^{r} f_i + r + \sum_{i=1}^{s} g_i \right)
\]
\[
= (n - 1) ((n - 1) - m + r).
\]

Since each \( P_j \) lies between two \( S_k \), it holds that \( r = m - 1 \). Therefore,
\[
PI(L(T)) = (n - 1) (n - 1 - 1) = (n - 1)(n - 2) = PI(T) - 2(n - 1).
\]

Let \( K_n \) be the complete graph with \( n \) vertices. The graph \( L(K_n) \) is the edge disjoint union of \( n \) cliques \( A_1, A_2, A_3, ..., A_n \), each of which has order \( n - 1 \). Also, each vertex of \( L(K_n) \) is a part of exactly two cliques and any two cliques in \( L(K_n) \) have exactly one vertex in common.

**Theorem 3.2.** \( PI(L(K_n)) = n(n - 1)(n - 2) \).

**Proof.** The edge set of \( L(K_n) \) can be partitioned as
\[
E(L(K_n)) = \bigcup_{i=1}^{n} E(A_i),
\]
where \( A_i \)s are cliques of order \( n - 1 \). Let \( e = uv \) be an arbitrary edge in \( L(K_n) \), then \( e \in A_i \) for some \( i \). All the remaining vertices in \( A_i \) are at distance one, so \( V(A_i) \setminus \{u, v\} \subseteq D_1(e) \). Since each vertex belongs to exactly two cliques, \( u \in A_j \) and \( v \in A_h \), for some \( i \notin \{j, h\} \). Also, two cliques have exactly one vertex in common, say \( w \), which is different from \( u \) and \( v \). So, \( d(u, w) = d(v, w) = 1 \) implies that \( w \in D_1(e) \). Moreover, the number of vertices at distance 2 is
\[
\frac{n(n - 1)}{2} - (d(u) + d(v) - D_1(e)) = \frac{n(n - 1)}{2} - (4(n - 2) - (n - 2)).
\]

Therefore,
\[
N_L(K_n)(e) = (n - 2) + \frac{n(n - 1)}{2} - 3(n - 2) = \frac{n(n - 1)}{2} - 2(n - 2).
\]
and hence
\[
PI(L(K_n)) = \sum_{e \in E(L(K_n))} (|V(L(K_n))| - N_{L(K_n)}(e))
\]
\[
= \sum_{e \in E(L(K_n))} \left( \frac{n(n-1)}{2} - \left( \frac{n(n-1)}{2} - 2(n-2) \right) \right)
\]
\[
= 2 \left( \sum_{v \in V(K_n)} \frac{d^2(v)}{2} - m \right) (n-2)
\]
\[
= \left( \sum_{v \in V(K_n)} d^2(v) - 2m \right) (n-2)
\]
\[
= (n(n-1)^2 - n(n-1)) (n-2) = n(n-1)(n-2)^2 = PI(K_n)(n-2)^2.
\]

Next, we consider the complete bipartite graph \( K_{n,m} = G(U,V) \) with \(|U| = n \) and \(|V| = m \). Its line graph \( L(G) \) is the edge disjoint union of \( m+n \) cliques, where \( m \) cliques have order \( n \) and \( n \) cliques have order \( m \). Each vertex in \( L(G) \) belongs to exactly two cliques, one of which has order \( n \) and the other is of order \( m \). Two cliques of the same order have no vertex in common.

**Theorem 3.3.** \( PI(L(K_{n,m})) = mn(2mn - (m+n)). \)

**Proof.** Take \( G = K_{n,m}. \) Its edge set can be partitioned as \( E(G) = E(\cup K_n) \cup E(\cup K_m) \). Take an arbitrary edge \( e \in E(L(G)) \). Then there are two possibilities.

**Case 1.** \( e = xy \) is an edge of a clique \( K_n \) of order \( n \).

All the vertices of \( K_n \) other than the end vertices of \( e \) are at distance 1. There is no other vertex at distance 1. (If there exists a vertex \( z \) at distance 1, then the edge \( xz \) belongs to a clique of order \( m \) and \( zy \) belongs to another clique of the same order. So, the vertex \( w \) belongs to exactly two cliques of order \( m \), it is not possible). Thus,

\[
mm - (d(x) + d(y) - (n-2)) = mn - (2(n+m-2) - (n-2))
\]

are the number of vertices at distance 2. So,

\[
N_{L(G)}(e) = mn - (2(n+m-2) - (n-2)) + (n-2) = mn - 2(n+m-2) + 2(n-2).
\]

**Case 2.** \( e \) is an edge of a clique \( K_m \) of order \( m \).

In the same way as in Case 1, one gets

\[
N_{L(G)}(e) = mn - 2(n+m-2) + 2(m-2).
\]

Therefore,

\[
PI(L(G)) = \sum_{e \in E(L(G))} (|V(L(G))| - N_{L(G)}(e))
\]
\[
= mn \left( \frac{mn(m+n-2)}{2} \right) - \left( \sum_{e \in E(\cup K_n)} N_{L(G)}(e) + \sum_{e \in E(\cup K_m)} N_{L(G)}(e) \right)
\]
\[
= mn \left( \frac{mn(m+n-2)}{2} \right) - (mn - 2(n+m-2)) \left( \frac{mn(m+n-2)}{2} \right) - mn[(n-1)(n-2) + (m-1)(m-2)]
\]
\[
= mn((m+n-2)^2 - (n-1)(n-2) - (m-1)(m-2))
\]
\[
= mn(2mn - (m+n)).
\]
4. Prismatic graphs

Chudnovsky and Seymour studied different structural properties of claw-free graphs in a series of seven papers. In their first paper [1] of this series, they studied the orientable prismatic graphs and in the second paper [2] they studied non-orientable prismatic graphs. A graph \( G \) is prismatic if for every triangle \( T \) in \( G \), every vertex not in \( T \) has exactly one neighbour in \( T \). Core of a prismatic graph is the union of all triangles in \( G \). Total coloring of prismatic graphs are discussed in [14]. Here, we consider a particular class of prismatic graphs, namely rigid prismatic graphs. A prismatic graph \( G \) with core \( W \) is rigid if

- there does not exist two distinct vertices \( u \) and \( v \), not in the core, with the same neighbouring set in \( W \),
- every two non-adjacent vertices have a common neighbour in the core.

**Theorem 4.1.** If \( G \) is a rigid prismatic graph with \( p \) triangles and \( n \) vertices, then its PI index is

\[
PI(G) = M_1(G) + 2np - \sum_{(u,v)\in E(W)} (d(u) + d(v)).
\]

**Proof.** Let \( G \) be a rigid prismatic graph with \( p \) triangles, \( n \) vertices, and \( m \) edges. Since every two non-adjacent vertices of \( G \) have a common neighbour in the core, its diameter is 2. The edge set of \( G \) can be partitioned as \( E(G) = E(W) \cup E_1 \cup E_2 \cup E_3 \), where \( E_1 = \{(u,v) \notin E(W) \mid u,v \in W\} \), \( E_2 = \{(u,v) \mid \text{either } u \in W \text{ or } v \in W\} \), and \( E_3 = \{(u,v) \mid u,v \notin W\} \).

\[
PI(G) = \sum_{e \in E(W)} (|V(G)| - N_G(e)) + \sum_{e \in E_1} (|V(G)| - N_G(e)) + \sum_{e \in E_2} (|V(G)| - N_G(e)) + \sum_{e \in E_3} (|V(G)| - N_G(e)).
\]

Since each triangle contributes \( 2n \) to \( PI(G) \), one has

\[
\sum_{e \in E(W)} (|V(G)| - N_G(e)) = 2np.
\]

Since each edge in \( E_i \), \( i = 1, 2, 3 \), is not a part of a triangle, it holds that

\[
\sum_{e \in E_i} (|V(G)| - N_G(e)) = \sum_{(u,v) \in E_i} (n - (n - (d(u) + d(v)))) = \sum_{(u,v) \in E_i} (d(u) + d(v))
\]

and thus,

\[
PI(G) = 2np + \sum_{(u,v) \in E_1 \cup E_2 \cup E_3} (d(u) + d(v)) = 2np + \sum_{(u,v) \in E(G) \setminus E(W)} (d(u) + d(v))
\]

\[
= 2np + \sum_{(u,v) \in E(G)} (d(u) + d(v)) - \sum_{(u,v) \in E(W)} (d(u) + d(v))
\]

\[
= M_1(G) + 2np - \sum_{(u,v) \in E(W)} (d(u) + d(v)).
\]

\[\square\]

Figure 2: The rotator.

For illustration of Theorem 4.1, we consider two non-orientable prismatic graphs: rotator and twister. The rotator and twister are shown in 2 and 3, and their PI indices are 120 and 154, respectively.
Figure 3: The twister.

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References