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Research Article

Padmakar-Ivan index of some types of perfect graphs

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Abstract

The Padmakar-Ivan (PI) index of a graph G is defined as $PI(G) = \sum_{e\in E(G)} (|V(G)| - N_G(e))$, where $N_G(e)$ is the number of equidistant vertices for the edge e. A graph is perfect if for every induced subgraph H, the equation $\chi(H) = \omega(H)$ holds, where $\chi(H)$ is the chromatic number and $\omega(H)$ is the size of a maximum clique of H. In this paper, the PI index of some types of perfect graphs is obtained. These types include co-bipartite graphs, line graphs, and prismatic graphs.

Keywords: PI index; co-bipartite graphs; line graphs; prismatic graphs.

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1. Introduction

All graphs considered in this paper are finite, simple and connected. For a graph G, the distance between two vertices x, y is denoted by $d(x, y)$. A vertex w is equidistant for an edge $e = xy$ if $d(x, w) = d(y, w)$. For an edge $e \in E(G)$, denote by $D_G(e)$ the set of all equidistant vertices in G. In particular, $D_i(e)$ denotes the set of vertices at distance i for e. Also, we denote $|D_G(e)| = N_G(e)$.

The vertex Padmakar-Ivan (PI) index of a graph G is a topological index, defined as

$$PI(G) = \sum_{e=uv\in E(G)} (n_u(e) + n_v(e)),$$

where $n_u(e)$ denotes the number of those vertices of G whose distance from the vertex u is smaller than the distance from the vertex v and $n_v(e)$ denotes the number of those vertices of G whose distance from v is smaller than the distance from u. Since $n_u(e) + n_v(e) = |V(G)| - N_G(e)$, the PI index can be rewritten as

$$PI(G) = \sum_{e\in E(G)} (|V(G)| - N_G(e)).$$

The PI index was proposed by Khadikar [10] in 2000. Khadikar and his coauthors investigated the chemical and biological applications of this index in [11]. Khalifeh [12] introduced a vertex version of the PI index and using this notion, they computed exact expression for the PI index of Cartesian product of graphs. John and Khadikar established a method for calculating the PI index of benzenoid hydrocarbons using orthogonal cuts in [9]. Gutman and Ashrafi [6] obtained the PI index of phenylenes and their hexagonal squeezes. The PI index of bridge graphs and chain graphs was studied in [13]. Das and Gutman [3] obtained a lower bound on the PI index of a connected graph in terms of the number of vertices, edges, pendent vertices, and the clique number, and also they characterized the extremal graphs. There are different types of topological indices; for example distance-based topological indices, degree-based topological indices, etc. Topological indices has many applications in the field of mathematical chemistry. Trinajstić and Zhou introduced the sum-connectivity index and found several basic properties in [16]. Many topological indices and their applications are thoroughly explored in [15]. Ilić and Milosavljević introduced the weighted vertex PI index and established some of its basic properties in [7]. The weighted PI index of a graph G is given as

$$PI_w(G) = \sum_{e=uv\in E(G)} (d_G(u) + d_G(v)) (|V(G)| - N_G(e)).$$

Gopika et al. [5] obtained the weighted PI index of the direct and strong product for certain types of graphs. Indulal et al. [8] studied the graphs satisfying the equation $PI(G) = PI(G - e)$.

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A graph is perfect if for every induced subgraph H, the equation $\chi(H) = \omega(H)$ holds, where $\chi(H)$ is the chromatic number and $\omega(H)$ is the size of a maximum clique of H. A claw-free graph is a graph in which no vertex has three pairwise nonadjacent neighbours. Every claw-free graph is a perfect graph. A survey on claw-free graphs is given in [4]. Chudnovsky and Seymour studied the structure of claw-free graphs thoroughly in a series of seven papers from 2007 to 2012. For example, in the first paper [1] of this series, they studied the orientable prismatic graphs and in the second paper [2], they studied non-orientable prismatic graphs. In this paper, we obtain the PI index of some classes of perfect graphs, including co-bipartite graphs, line graphs, and prismatic graphs.

2. Co-bipartite graphs

An edge $e = xy$ of a graph G is said to be an equidistant edge for a vertex $a \in V(G)$ if $d(a,x) = d(a,y)$. The edge e is at distance r for a vertex a if $d(a,x) = d(a,y) = r$. The set of all equidistant edges of a is $D_G(a) = \{e = xy \in E(G) : d(a,x) = d(a,y)\}$ and we take $N_G(a) = |D_G(a)|$. It is easy to see that $\sum_{e\in E(G)} N_G(e) = \sum_{a\in V(G)} N_G(a)$.

Lemma 2.1. *Let G be a graph with n vertices and m edges. Then, $PI(G) = mn - \sum_{a\in V(G)} N_G(a)$.*

Proof.

$$PI(G) = \sum_{e\in E(G)} (|V(G)| - N_G(e)) = \sum_{e\in E(G)} |V(G)| - \sum_{e\in E(G)} N_G(e) = mn - \sum_{a\in G} N_G(a).$$

□

Let $G(U,V)$ be a bipartite graph with partite sets U and V. A co-bipartite graph is the complement of a bipartite graph $G(U,V)$ and it is denoted as $\overline{G}$. In $\overline{G}$, the vertices in U and the vertices in V forms two disjoint cliques. Every co-bipartite graph is a perfect graph. The diameter of a connected co-bipartite graph is either 2 or 3.

Consider a bipartite graph $G(U,V)$ with $|U| = n$ and $|V| = m$. Let Δ_1 and Δ_2 be the maximum degree in U and V respectively, where $\Delta_1 \le m$ and $\Delta_2 \le n$. Let $U_1 = \{u \in U : d(u) < m\}$ and $U_2 = \{u \in U : d(u) = m\}$. It is noted that $U = U_1 \cup U_2$. Similarly, $V = V_1 \cup V_2$, provided that the degree of every vertex in V_1 is less than n and the degree of every vertex in V_2 is n. Let $U_1 = \{u_1, u_2, ..., u_p\}$, $U_2 = \{u_{p+1}, u_{p+2}, ..., u_n\}$, $V_1 = \{v_1, v_2, ..., v_q\}$ and $V_2 = \{v_{q+1}, v_{q+2}, ..., v_m\}$. Let $d(u_i) = f_i$ for $i = 1, 2, ..., p$ and $d(v_i) = g_i$ for $i = 1, 2, ..., q$. We denote $\sum_{j=1}^{p} f_j$ by f, $\sum_{j=1}^{p} f_j^2$ by f^*, $\sum_{i=1}^{q} g_i$ by g and $\sum_{i=1}^{p} g_i^2$ by g^*.

Theorem 2.1. *Let $G(U,V)$ be a bipartite graph. Then $PI(\overline{G}) = n(n-1) + m(m-1) + mn(2p+q) - pq(m+n-p-q) - mp(p-1) - nq(n+q-1) + f(2m-n-1) - 2(f^* + g^*) + g(3n-1)$.*

Proof. Let $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$, where $U_1 = \{u_1, u_2, ..., u_p\}$, $U_2 = \{u_{p+1}, u_{p+2}, ..., u_n\}$, $V_1 = \{v_1, v_2, ..., v_q\}$, $V_2 = \{v_{q+1}, v_{q+2}, ..., v_m\}$, $d(u_i) = f_i$ if $i \le p$, $d(u_i) = m$ if $i > p$, $d(v_j) = g_j$ if $j \le q$, and $d(v_j) = n$ if $j > q$. The degrees in $\overline{G}$ (see Figure 1) are given as

$$d(u_i) = \begin{cases} (m - f_i) + (n-1) & \text{if } i = 1, 2, ..., p \\ (n-1) & \text{if } i > p \end{cases}$$

and

$$d(v_j) = \begin{cases} (n - g_j) + (m-1) & \text{if } j = 1, 2, ..., q \\ (m-1) & \text{if } j > q. \end{cases}$$

We partition $E(\overline{G})$ with E_1, E_2 and E_3, where E_1 is the set of edges in the clique with vertices in U, E_2 is the set of edges in the clique with vertices in V and $E_3 = \{(u,v) : u \in U, v \in V\}$.

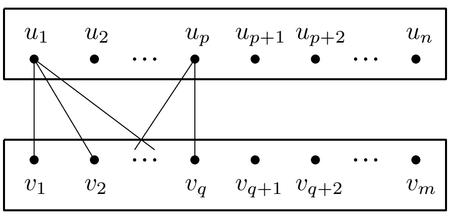

Figure 1: The graph $\overline{G}$ used in the proof of Theorem 2.1.

For a vertex $u \in U$, it is easy to see that

$$N_{E_1}(u) = \frac{(n-1)(n-2)}{2}.$$

A vertex $v_i \in V_1$ has $(n-g_i)$ neighbours in U and the remaining g_i vertices are at distance 2, which means that

$$N_{E_1}(v_i) = \frac{(n-g_i)(n-g_i-1)}{2} + \frac{g_i(g_i-1)}{2}.$$

Similarly, a vertex $v \in V_2$ has no neighbours in U and

$$d(u_i, v) = \begin{cases} 2 & \text{if } u_i \in U_1 \\ 3 & \text{if } u_i \in U_2 \end{cases}.$$

Also,

$$N_{E_1}(v) = \frac{p(p-1)}{2} + \frac{(n-p)(n-p-1)}{2}$$

and

$$\sum_{e\in E_1} N_{\overline{G}}(e) = \frac{n(n-1)(n-2)}{2} + \sum_{i=1}^{q}\left(\frac{(n-g_i)(n-g_i-1)}{2} + \frac{g_i(g_i-1)}{2}\right) + (m-q)\left(\frac{p(p-1)}{2} + \frac{(n-p)(n-p-1)}{2}\right). \quad (1)$$

Similarly, for edges in E_2, one has

$$\begin{aligned}\sum_{e\in E_2} N_{\overline{G}}(e) = &\frac{m(m-1)(m-2)}{2} + \sum_{j=1}^{p}\left(\frac{(m-f_j)(m-f_j-1)}{2} + \frac{f_j(f_j-1)}{2}\right)\\ &+ (n-p)\left(\frac{q(q-1)}{2} + \frac{(m-q)(m-q-1)}{2}\right).\end{aligned} \quad (2)$$

For edges in E_3, a vertex $u \in U_1$ has $(m-f_j)$ neighbours in V and the remaining f_j vertices are at distance 2. Similarly, $v \in V_1$ has $(n-g_i)$ neighbours in U and remaining g_i vertices are at distance 2. That is, $N_{E_3}(N_V(u)) = (m-f_j)(m-f_j-1)$ and $N_{E_3}(N_U(v)) = (n-g_i)(n-g_i-1)$. Therefore,

$$\sum_{e\in E_3} N_{\overline{G}}(e) = \sum_{j=1}^{p}(m-f_j)(m-f_j-1) + \sum_{i=1}^{q}(n-g_i)(n-g_i-1). \quad (3)$$

Now, we combine the three equations (1), (2), and (3) to calculate $PI(\overline{G})$.

$$\begin{aligned}
PI(\overline{G}) &= \sum_{e\in E_1}\left(|V(\overline{G})| - N_{\overline{G}}(e)\right) + \sum_{e\in E_2}\left(|V(\overline{G})| - N_{\overline{G}}(e)\right) + \sum_{e\in E_3}\left(|V(\overline{G})| - N_{\overline{G}}(e)\right)\\
&= \frac{n(n-1)(m+n)}{2} - \frac{n(n-1)(n-2)}{2} - \sum_{i=1}^{q}\left(\frac{(n-g_i)(n-g_i-1)}{2} + \frac{g_i(g_i-1)}{2}\right)\\
&\quad - (m-q)\left(\frac{p(p-1)}{2} + \frac{(n-p)(n-p-1)}{2}\right)\\
&\quad + \frac{m(m-1)(m+n)}{2} - \frac{m(m-1)(m-2)}{2} - \sum_{j=1}^{p}\left(\frac{(m-f_j)(m-f_j-1)}{2} + \frac{f_j(f_j-1)}{2}\right)\\
&\quad - (n-p)\left(\frac{q(q-1)}{2} + \frac{(m-q)(m-q-1)}{2}\right) + (m+n)\left((m-f_1)+(m-f_2)+\ldots+(m-f_p)\right)\\
&\quad - \left(\sum_{j=1}^{p}(m-f_j)(m-f_j-1) + \sum_{i=1}^{q}(n-g_i)(n-g_i-1)\right)\\
&= n(n-1)(m+2) - \sum_{i=1}^{q}(n-g_i)(n-g_i-1) - \sum_{i=1}^{q} g_i(g_i-1) - (m-q)(p(p-1)+(n-p)(n-p-1))\\
&\quad + m(m-1)(n+2) - \sum_{j=1}^{p}(m-f_j)(m-f_j-1) - \sum_{j=1}^{p} f_j(f_j-1) - (n-p)(q(q-1)+(m-q)(m-q-1))\\
&\quad + 2(m+n)\sum_{j=1}^{p}(m-f_j) - 2\sum_{j=1}^{p}(m-f_j)(m-f_j-1) - 2\sum_{i=1}^{q}(n-g_i)(n-g_i-1)\\
&= 2n^2 - 2n + 4mnp - 2p^2m - 2npq + 2p^2q + 2m^2 - 2m + 2mnq - 2nq^2 - 2mpq + 2pq^2\\
&\quad - 2n^2q + 2nq + 2pm - 2n\sum_{j=1}^{p} f_j + 4m\sum_{j=1}^{p} f_j - 4\sum_{j=1}^{p} f_j^2 - 2\sum_{j=1}^{p} f_j + 6n\sum_{i=1}^{q} g_i - 4\sum_{i=1}^{q} g_i^2 - 2\sum_{i=1}^{q} g_i\\
&= n(n-1) + m(m-1) + mn(2p+q) - pq(m+n-q-p) - mp(p-1)\\
&\quad - nq(q+n-1) + f(2m-n-1) - 2(f^* + g^*) + g(3n-1).
\end{aligned}$$

□

A bipartite graph $G(U,V)$ is (x,y)-biregular if each vertex in U has degree x and each vertex in V has degree y.

Corollary 2.1. *If $G(U,V)$ is a (x,y)-biregular graph then* $PI\left(\overline{G}\right) = (n+m)(n+m-1) + 2my\,(n+m-(x+y+1))\,.$

Proof. From Theorem 2.1, we have

$$PI\left(\overline{G}\right) = n(n-1) + m(m-1) + mn(2p+q) - pq(m+n-p-q) - mp(p-1) \\ - nq(q+n-1) + f(2m-n-1) - 2(f*+g*) + g(3n-1)\,.$$

Here $p=n$, $q=m$, $f=nx$, $g=my$, $f^*=nx^2$, and $g^*=my^2$.Thus,

$$\begin{aligned} PI(\overline{G}) &= n^2 - n + m^2 - m + 2mn + 2mnx - n^2x - nx - 2nx^2 - 2my^2 + 3nmy - my \\ &= (n+m)^2 - (n+m) + 2nmy + 2m^2y - 2mxy - 2my^2 - 2my \\ &= (n+m)(n+m-1) + 2my(n+m-(x+y+1)). \end{aligned}$$

□

Corollary 2.2. *If G is a k-regular bipartite graph with $2n$ vertices then* $PI\left(\overline{G}\right) = 2n\left[2n(k+1) - \left(2k^2+k+1\right)\right].$

Proof. In Theorem 2.1, by taking $n=m$ and $x=y=k$, one gets

$$\begin{aligned} PI\left(\overline{G}\right) &= 2n(2n-1) + 2nk(2n-(2k+1)) \\ &= 2n\left(2n(k+1) - \left(2k^2+k+1\right)\right). \end{aligned}$$

□

Two particular examples of Corollary 2.2 are $PI\left(\overline{C_{2n}}\right) = 2n(6n-11)$ and $PI\left(\overline{K_{n,n}}\right) = 2n(n-1)\,.$

Corollary 2.3. *If G is a k-regular bipartite graph with $2n$ vertices then*

$$PI_w\left(\overline{G}\right) = 4n(2n-k-1)\left(2n(k+1) - \left(2k^2+k+1\right)\right).$$

Proof. We know that the weighted PI index of a regular graph is a multiple of its PI index. Therefore,

$$PI_w\left(\overline{G}\right) = 2(2n-k-1)\,PI\left(\overline{G}\right) = 4n(2n-k-1)\left(2n(k+1) - \left(2k^2+k+1\right)\right).$$

□

3. Line graphs of some classes of graphs

Let G be a graph with n vertices and m edges. Its line graph denoted by $L(G)$, is a simple graph whose vertices are the edges of G and two vertices are adjacent in $L(G)$ if the corresponding edges are adjacent in G. Let T be a tree with n vertices. Every vertex v in T with degree $i, i>2$, forms a star $K_{1,i}$ in T, we denote it by S_i. Let S be the collection of all stars in T. If we delete edges of all stars in T, the remaining edges of T are parts of paths. Some paths have both of its end vertices common with the stars; we call them as central paths and the remaining have one end vertex shared with stars (paths) and the other end vertex is a pendent vertex; we call them leaf paths. We denote the central path with the length l by P_l and pendent path with the length l by P_l^*. As we know that line graphs of stars are complete graphs and line graphs of paths are paths. Each star S_i in T is transformed to a clique with K_i in $L(T)$. The central path P_l has l edges, so it is transformed to the path with l vertices having length $l-1$ and each of its end vertices is connected with a vertex of a clique in $L(T)$, so it has $l-1+2=l+1$ edges. Each leaf path P_l^* is transformed to a path with l vertices and $l-1$ edges, and it is connected with a vertex of $L(T)$, so it has l edges.

Theorem 3.1. *Let T be a tree with n vertices then*

$$PI(L(T)) = (n-1)(n-2)\,.$$

Proof. Let T be a tree with n vertices. Assume that the edge set $E(T)$ is the union of m stars S_{k_i}, r central paths P_{f_i}, and s pendant paths $P^*_{g_i}$. Let us assume that

$$S = \cup_{i=1}^{m} S_{k_i} \quad \text{and} \quad P' = \left(\cup_{i=1}^{r} P_{f_i}\right) \cup \left(\cup_{i=1}^{s} P^*_{g_i}\right).$$

Then,

$$|E(T)| = n-1 = \sum_{i=1}^{m} k_i + \sum_{i=1}^{r} f_i + \sum_{i=1}^{s} g_i.$$

We claim that

$$\sum_{e\in E\left(K_{k_i}\right)} \left(|V(T)| - N_T(e)\right) = \left(k_i - 1\right)\left(n-1\right).$$

Let e be an edge of K_{k_i} in $L(T)$ and let $v \in V(K_{k_i})$ be equidistant to e. If we delete all the edges of K_{k_i}, then $L(T)$ has more than one component. All vertices in the component W containing v are also equidistant to e. If we consider all the edges and vertices of K_{k_i}, then

$$\sum_{e\in E\left(K_{k_i}\right)} \left(|V(T)| - N_T\left(e\right)\right) = \left(k_i - 1\right)\left(n-1\right).$$

Also, since each edge of a path is a cut edge, there is no equidistant vertex corresponding to those edges. Each P_{f_i+1} contributes $(f_i+1)(n-1)$ and each $P^*_{g_i}$ contributes $g_i(n-1)$ to the PI index of $L(T)$. Thus,

$$\begin{aligned}
PI\left(L(T)\right) &= \sum_{e\in E(L(T))} \left(|V\left(L(T)\right)| - N_{L(T)}(e)\right) \\
&= \sum_{e\in E\left(\cup S_{k_i}\right)} \left(|V\left(L(T)\right)| - N_{L(T)}(e)\right) + \sum_{e\in E\left(\cup P_{f_i+1}\right)} \left(|V(L(T))| - N_{L(T)}(e)\right) + \sum_{e\in E\left(\cup P^*_{g_{i+1}}\right)} \left(|V(L(T))| - N_{L(T)}(e)\right) \\
&= \sum_{i=1}^{m} \left(k_i-1\right)\left(n-1\right) + \sum_{i=1}^{r} \left(f_i+1\right)\left(n-1\right) + \sum_{i=1}^{s} g_i\left(n-1\right) \\
&= (n-1)\left(\sum_{i=1}^{m}(k_i-1) + \sum_{i=1}^{r}(f_i+1) + \sum_{i=1}^{s} g_i\right) \\
&= (n-1)\left(\sum_{i=1}^{m} k_i - m + \sum_{i=1}^{r} f_i + r + \sum_{i=1}^{s} g_i\right) \\
&= (n-1)\left((n-1) - m + r\right).
\end{aligned}$$

Since each P_{f_i} lies between two S_{k_i}, it holds that $r = m-1$. Therefore,

$$PI\left(L(T)\right) = (n-1)\left(n-1-1\right) = (n-1)\left(n-2\right) = PI\left(T\right) - 2\left(n-1\right).$$

□

Let K_n be the complete graph with n vertices. The graph $L\left(K_n\right)$ is the edge disjoint union of n cliques $A_1, A_2, A_3, ..., A_n$, each of which has order $n-1$. Also, each vertex of $L\left(K_n\right)$ is a part of exactly two cliques and any two cliques in $L\left(K_n\right)$ have exactly one vertex in common.

Theorem 3.2. $PI\left(L(K_n)\right) = n\left(n-1\right)\left(n-2\right)^2$.

Proof. The edge set of $L\left(K_n\right)$ can be partitioned as

$$E\left(L(K_n)\right) = \bigcup_{i=1}^{n} E\left(A_i\right),$$

where $A_i's$ are cliques of order $n-1$. Let $e = uv$ be an arbitrary edge in $L\left(K_n\right)$, then $e \in A_i$ for some i. All the remaining vertices in A_i are at distance one, so $V(A_i) \setminus \{u, v\} \subseteq D_1(e)$. Since each vertex belongs to exactly two cliques, $u \in A_j$ and $v \in A_h$, for some $i \notin \{j, h\}$. Also, two cliques have exactly one vertex in common, say w, which is different from u and v. So, $d(u,w) = d(v,w) = 1$ implies that $w \in D_1(e)$. Moreover, the number of vertices at distance 2 is

$$\frac{n(n-1)}{2} - \left(d(u) + d(v) - D_1(e)\right) = \frac{n(n-1)}{2} - \left(4(n-2) - (n-2)\right).$$

Therefore,

$$N_{L(K_n)}(e) = (n-2) + \frac{n\left(n-1\right)}{2} - 3\left(n-2\right) = \frac{n\left(n-1\right)}{2} - 2\left(n-2\right)$$

and hence

$$\begin{aligned}
PI\left(L(K_n)\right) &= \sum_{e\in E(L(K_n))} \left(|V(L(K_n))| - N_{L(K_n)}(e)\right)\\
&= \sum_{e\in E(L(K_n))} \frac{n\left(n-1\right)}{2} - \left(\frac{n(n-1)}{2} - 2(n-2)\right)\\
&= 2\left(\frac{\sum_{v\in V(K_n)} d^2\left(v\right)}{2} - m\right)(n-2)\\
&= \left(\sum_{v\in V(K_n)} d^2(v) - 2m\right)(n-2)\\
&= \left(n(n-1)^2 - n(n-1)\right)(n-2) = n\left(n-1\right)\left(n-2\right)^2 = PI\left(K_n\right)\left(n-2\right)^2.
\end{aligned}$$

□

Next, we consider the complete bipartite graph $K_{n,m} = G(U,V)$ with $|U| = n$ and $|V| = m$. Its line graph $L(G)$ is the edge disjoint union of $m+n$ cliques, where m cliques have order n and n cliques have order m. Each vertex in $L(G)$ belongs to exactly two cliques, one of which has order n and the other is of order m. Two cliques of the same order have no vertex in common.

Theorem 3.3. $PI(L(K_{n,m})) = mn\left(2mn - (m+n)\right)$.

Proof. Take $G = K_{n,m}$. Its edge set can be partitioned as $E(G) = E\left(\cup K_n\right) \cup E\left(\cup K_m\right)$. Take an arbitrary edge $e \in E\left(L(G)\right)$. Then there are two possibilities.

Case 1. $e = xy$ is an edge of a clique K_n of order n.
All the vertices of K_n other than the end vertices of e are at distance 1. There is no other vertex at distance 1. (If there exists a vertex z at distance 1, then the edge xz belongs to a clique of order m and zy belongs to another clique of the same order. So, the vertex w belongs to exactly two cliques of order m, it is not possible). Thus,

$$mn - (d(x) + d(y) - (n-2)) = mn - (2(n+m-2) - (n-2))$$

are the number of vertices at distance 2. So,

$$N_{L(G)}(e) = mn - \left(2\left(n+m-2\right) - \left(n-2\right)\right) + \left(n-2\right) = mn - 2\left(n+m-2\right) + 2\left(n-2\right).$$

Case 2. e is an edge of a clique K_m of order m.
In the same way as in Case 1, one gets

$$N_{L(G)}(e) = mn - 2\left(n+m-2\right) + 2\left(m-2\right).$$

Therefore,

$$\begin{aligned}
PI(L(G)) &= \sum_{e\in E(L(G))} (|V(L(G))| - N_{L(G)}(e))\\
&= mn\left(\frac{mn(n+m-2)}{2}\right) - \left(\sum_{e\in E(\cup K_n)} N_{L(G)}(e) + \sum_{e\in E(\cup K_m)} N_{L(G)}(e)\right)\\
&= mn\left(\frac{mn(n+m-2)}{2}\right) - (mn - 2(n+m-2))\left(\frac{mn(m+n-2)}{2}\right) - mn[(n-1)(n-2) + (m-1)(m-2)]\\
&= mn((n+m-2)^2 - (n-1)(n-2) - (m-1)(m-2))\\
&= mn(2mn - (m+n)).
\end{aligned}$$

□

4. Prismatic graphs

Chudnovsky and Seymour studied different structural properties of claw-free graphs in a series of seven papers. In their first paper [1] of this series, they studied the orientable prismatic graphs and in the second paper [2] they studied non-orientable prismatic graphs. A graph G is *prismatic* if for every triangle T in G, every vertex not in T has exactly one neighbour in T. Core of a prismatic graph is the union of all triangles in G. Total coloring of prismatic graphs are discussed in [14]. Here, we consider a particular class of prismatic graphs, namely rigid prismatic graphs. A prismatic graph G with core W is rigid if

- there does not exist two distinct vertices u and v, not in the core, with the same neighbouring set in W,
- every two non-adjacent vertices have a common neighbour in the core.

Theorem 4.1. *If G is a rigid prismatic graph with p triangles and n vertices, then its PI index is*

$$PI\left(G\right) = M_1\left(G\right) + 2np - \sum_{(u,v)\in E(W)} \left(d(u) + d(v)\right).$$

Proof. Let G be a rigid prismatic graph with p triangles, n vertices, and m edges. Since every two non-adjacent vertices of G have a common neighbour in the core, its diameter is 2. The edge set of G can be partitioned as $E(G) = E(W) \cup E_1 \cup E_2 \cup E_3$, where $E_1 = \{(u,v) \notin E(W) \mid u,v \in W\}$, $E_2 = \{(u,v) \mid \text{ either } u \in W \text{ or } v \in W\}$, and $E_3 = \{(u,v) \mid u,v \notin W\}$.

$$PI(G) = \sum_{e\in E(W)} \left(|V(G)| - N_G(e)\right) + \sum_{e\in E_1} \left(|V(G)| - N_G(e)\right) + \sum_{e\in E_2} \left(|V(G)| - N_G(e)\right) + \sum_{e\in E_3} \left(|V(G)| - N_G(e)\right).$$

Since each triangle contributes $2n$ to $PI(G)$, one has

$$\sum_{e\in E(W)} \left(|V(G)| - N_G(e)\right) = 2np.$$

Since each edge in E_i, $i = 1,2,3$, is not a part of a triangle, it holds that

$$\sum_{e\in E_i} \left(|V(G)| - N_G(e)\right) = \sum_{(u,v)\in E_i} \left(n - (n - (d(u) + d(v)))\right) = \sum_{(u,v)\in E_i} \left(d(u) + d(v)\right)$$

and thus,

$$\begin{aligned} PI(G) &= 2np + \sum_{(u,v)\in E_1\cup E_2\cup E_3} \left(d(u) + d(v)\right) = 2np + \sum_{(u,v)\in E(G)\setminus E(W)} \left(d(u) + d(v)\right) \\ &= 2np + \sum_{(u,v)\in E(G)} \left(d(u) + d(v)\right) - \sum_{(u,v)\in E(W)} \left(d(u) + d(v)\right) \\ &= M_1(G) + 2np - \sum_{(u,v)\in E(W)} \left(d(u) + d(v)\right). \end{aligned}$$

□

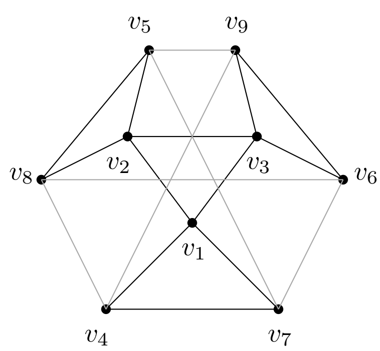

Figure 2: The rotator.

For illustration of Theorem 4.1, we consider two non-orientable prismatic graphs: rotator and twister. The rotator and twister are shown in 2 and 3, and their PI indices are 120 and 154, respectively.

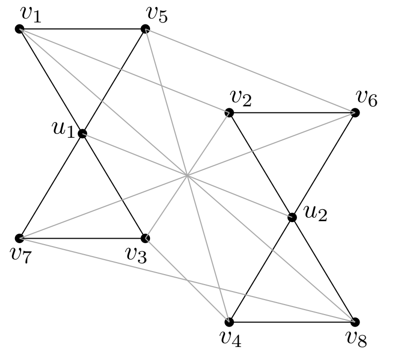

Figure 3: The twister.

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