Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **9** (2022) 80–85
DOI: 10.47443/dml.2021.s213

Research Article

Multiplicative Sombor index of graphs

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(Received: 26 February 2022. Received in revised form: 3 March 2022. Accepted: 19 March 2022. Published online: 28 March 2022.)

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Abstract

The Sombor index of a graph G is defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}$, where $d_G(u)$ denotes the degree of the vertex u of G. Accordingly, the multiplicative Sombor index of G can be defined as $\prod_{SO}(G) = \prod_{uv\in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}$. In this article, some graph transformations which increase or decrease the multiplicative Sombor index are first introduced. Then by using these transformations, extremal values of the multiplicative Sombor index of trees and unicyclic graphs are determined.

Keywords: multiplicative Sombor index; extremal value.

2020 Mathematics Subject Classification: 05C09, 05C92.

1. Introduction

The term "chemical graph theory" was coined by Nenad Trinajstić and it was used as the title of his seminal book [33]. In the new AMS subject classification 2020, the subject number 05C92 in "05C graph theory" is assigned to chemical graph theory. Gutman said that it is a major success of all those colleagues who over several decades worked and/or are working in "chemical topology", a field of research considered by many as worthless.

We only consider simple connected graph G with the vertex set $V(G)$ and edge set $E(G)$. Denote by $N_G(u)$ the set of the vertices that are neighbors of the vertex $u \in V(G)$. Then $|N_G(u)|$ is the degree of the vertex u, denoted by $d_G(u)$ or $d(u)$. We call the vertex u as a pendent vertex if $d(u) = 1$. We call a path $P = u_1u_2\cdots u_k$ in G as a pendent path if $d(u_1) \geq 3$, $d(u_k) = 1$ and $d(u_i) = 2$ for $2 \leq i \leq k-1$. The girth of G is the length of a shortest cycle in G. Denote by P_n and S_n the path and star graphs with n vertices, respectively. All notations and terminology used, but not defined here, can be found in the textbook [3].

The Sombor index [15] was proposed by Gutman, which is defined as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}.$$

Since the publication of [15], the Sombor index has attracted much attention of researchers. For the mathematical properties and chemical applications on the Sombor index or its variants, see [1, 2, 4–14, 16–32, 34, 35] and the references cited therein.

According to the definition of the Sombor index, it is natural to consider the multiplicative version of the Sombor index, defined as

$$\prod_{SO}(G) = \prod_{uv\in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}.$$

The aim of this paper is to begin the research on mathematical properties of the multiplicative Sombor index.

2. Transformations

We first introduce some transformations which will be useful in the proof of main theorems.

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Lemma 2.1. *Let G be a connected graph.*

(1) *if $uv \in E(G)$, then $\prod_{SO}(G) > \prod_{SO}(G-uv)$;*

(2) *if $uv \notin E(G)$, then $\prod_{SO}(G) < \prod_{SO}(G+uv)$.*

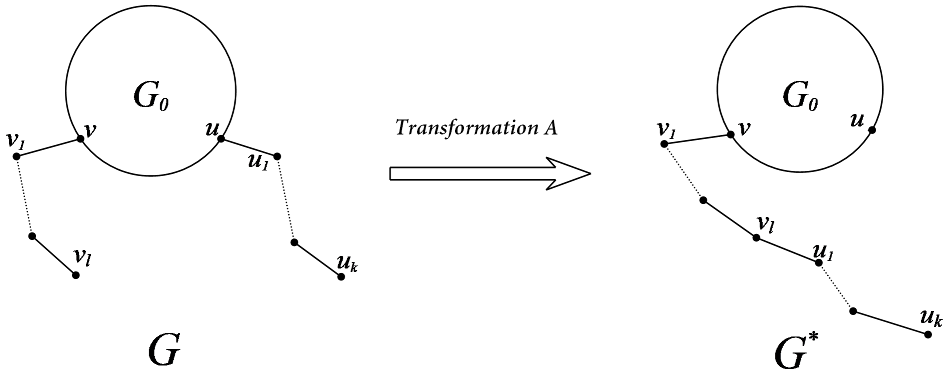

Figure 1: Transformation A.

Lemma 2.2. *Let G and G^* be the graphs shown in Figure 1. We allow $u = v$. If $uu_1u_2\cdots u_k$ and $vv_1v_2\cdots v_l$ are two pendent path in G, and $G^* = G - uu_1 + u_1v_l$, then $\prod_{SO}(G^*) < \prod_{SO}(G)$.*

Proof. We consider two cases.

Case 1. $u = v$.
Let $d_{G_0}(u) = t \geq 1$ and $N_{G_0}(u) = \{x_1, x_2, \cdots, x_t\}$. If $k \geq 2$, $l \geq 2$, then

$$\begin{aligned}\prod\nolimits_{SO}(G) - \prod\nolimits_{SO}(G^*) &= 5\cdot((t+2)^2+4)\cdot 8^{\frac{k+l-4}{2}}\cdot\prod_{i=1}^{t}\sqrt{(d_{G_0}(x_i)+2)^2+t^2}\\ &\quad -\sqrt{5}\cdot\sqrt{(t+1)^2+4}\cdot 8^{\frac{k+l-2}{2}}\cdot\prod_{i=1}^{t}\sqrt{(d_{G_0}(x_i)+1)^2+t^2}\\ &> 8^{\frac{k+l-4}{2}}\cdot\left[5((t+2)^2+4) - 8\sqrt{5}\sqrt{(t+1)^2+4}\right]\\ &> 0.\end{aligned}$$

Similarly, if $k = l = 1$ or $k = 1, l \geq 2$ or $k \geq 2, l = 1$, then we have

$$\prod\nolimits_{SO}(G) - \prod\nolimits_{SO}(G^*) > 0.$$

Case 2. $u \neq v$.

$$\frac{\prod_{SO}(G^*)}{\prod_{SO}(G)} = \frac{\sqrt{d_G^2(u_1)+4}\sqrt{d_G^2(v_{l-1})+4}}{\sqrt{d_G^2(u_1)+d_G^2(u)}\sqrt{d_G^2(v_{l-1})+1}}\prod_{u_i\in N_G(u)\setminus\{u_1\}}\frac{\sqrt{d_G^2(u_i)+(d_G(u)-1)^2}}{\sqrt{d_G^2(u_i)+d_G^2(u)}}.$$

If $l \geq 2$, then $d_G(v_{l-1}) = 2$, $d_G(u_1) = 2$ or 1. Since $d_G(u) \geq 3$, one has

$$\prod_{SO}(G^*) < \prod_{SO}(G).$$

If $l = 1$, then $d_G(v_{l-1}) \geq 3$. Since $d_G(u) \geq 3$, $d_G(u_1) = 2$ or 1.

$$\frac{\prod_{SO}(G^*)}{\prod_{SO}(G)} < \frac{\sqrt{13}}{\sqrt{10}}\frac{\sqrt{d_G^2(u_1)+4}}{\sqrt{d_G^2(u_1)+d_G^2(u)}} < 1.$$

□

Lemma 2.3. *Let G and G^* be the graphs as depicted in Figure 2, and $G^* = G - \{uw_1, uw_2, \cdots, uw_t\} + \{vw_1, vw_2, \cdots, vw_t\}$. Then $\prod_{SO}(G) < \prod_{SO}(G^*)$.*

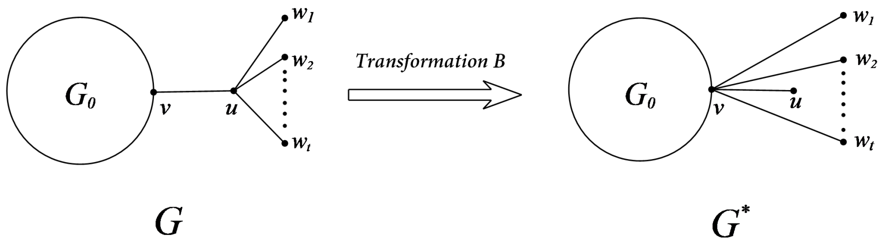

Figure 2: Transformation B.

Proof. Let $d_{G_0}(v) = k \geq 1$ and $N_{G_0}(v) = \{v_1, v_2, \cdots, v_k\}$. Then

$$\begin{aligned}\prod\nolimits_{SO}(G^*) - \prod\nolimits_{SO}(G) &\geq ((k+t+1)^2+1)^{\frac{t+1}{2}} \cdot \prod_{i=1}^{k} \sqrt{d^2_{G_0}(v_i) + (k+t+1)^2} \\ &\quad - \sqrt{(k+1)^2+(t+1)^2} \cdot ((t+1)^2+1)^{\frac{t}{2}} \cdot \prod_{i=1}^{k} \sqrt{d^2_{G_0}(v_i) + (k+1)^2} \\ &> ((k+t+1)^2+1)^{\frac{t+1}{2}} - \sqrt{(k+1)^2+(t+1)^2} \cdot ((t+1)^2+1)^{\frac{t}{2}} \\ &> \sqrt{(k+t+1)^2+1} - \sqrt{(k+1)^2+(t+1)^2} \\ &> 0.\end{aligned}$$

□

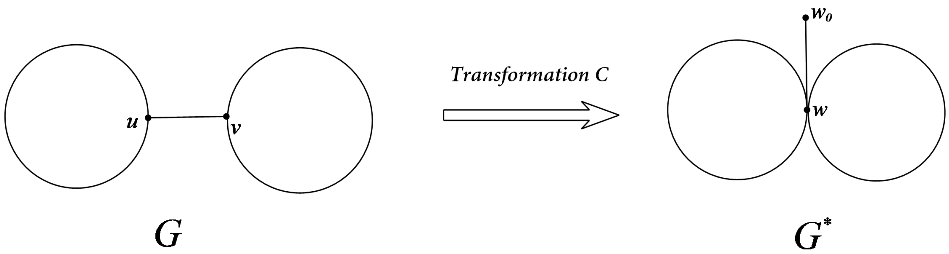

Figure 3: Transformation C.

Lemma 2.4. *Let G and G^* be the graphs as shown in Figure 3. If G^* is the graph obtained from G by identifying the vertices u and v to a new vertex w and adding a pendent vertex w_0 to the vertex w, then*

$$\prod\nolimits_{SO}(G) < \prod\nolimits_{SO}(G^*).$$

Proof. Let $N_G(u) = \{v, u_1, u_2, \cdots, u_k\}$ and $N_G(v) = \{u, v_1, v_2, \cdots, v_l\}$. Then

$$\begin{aligned}\prod\nolimits_{SO}(G^*) - \prod\nolimits_{SO}(G) &\geq \sqrt{(k+t+1)^2+1} \cdot \prod_{i=1}^{k} \sqrt{d^2_G(u_i) + (k+l+1)^2} \cdot \prod_{j=1}^{l} \sqrt{d^2_G(v_j) + (k+l+1)^2} \\ &\quad - \sqrt{(k+1)^2+(l+1)^2} \cdot \prod_{i=1}^{k} \sqrt{d^2_G(u_i) + (k+1)^2} \cdot \prod_{j=1}^{l} \sqrt{d^2_G(v_j) + (l+1)^2} \\ &> \sqrt{(k+t+1)^2+1} - \sqrt{(k+1)^2+(l+1)^2} \\ &> 0.\end{aligned}$$

□

Lemma 2.5. *Let G, G^* be the graphs in Figure 4. Note that $P = v_1 v_2 \cdots v_t v_{t+1}$ is a pendent path, $N_G(v_1) = \{v_2, u, w\}$ and $G^* = G - uv_1 + uv_{t+1}$. Then*

$$\prod\nolimits_{SO}(G) > \prod\nolimits_{SO}(G^*).$$

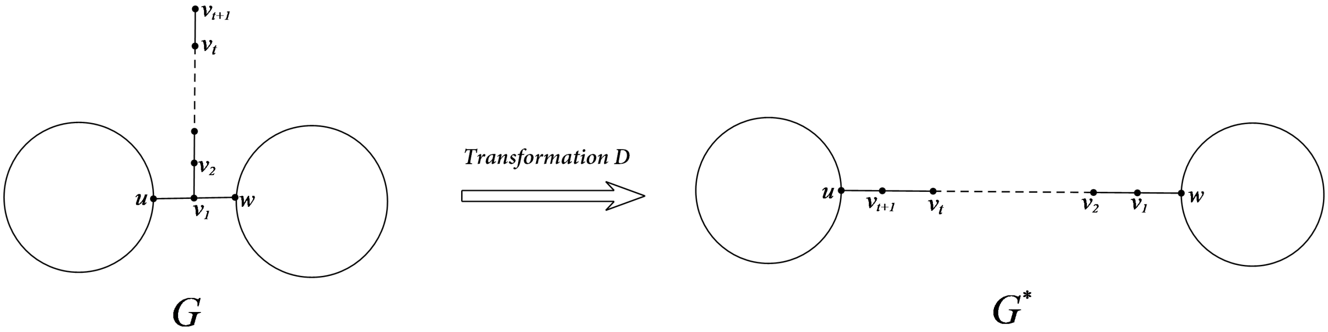

Figure 4: Transformation D.

Proof. If $d_G(u)=1$, then by Lemma 2.2, the conclusion holds. Thus, in the following, we suppose that $d_G(u)\geq 2$. If $t\geq 2$, then

$$\begin{aligned}\prod_{SO}(G)-\prod_{SO}(G^*) &\geq \sqrt{4+9}\cdot\sqrt{1+4}\cdot 8^{\frac{t-2}{2}}\cdot\sqrt{d_G^2(u)+9}\cdot\sqrt{d_G^2(w)+9}-8^{\frac{t}{2}}\cdot\sqrt{d_G^2(u)+4}\cdot\sqrt{d_G^2(w)+4}\\ &= 8^{\frac{t-2}{2}}\left[\sqrt{65}\sqrt{d_G^2(u)+9}\cdot\sqrt{d_G^2(w)+9}-8\sqrt{d_G^2(u)+4}\cdot\sqrt{d_G^2(w)+4}\right]\\ &> 0.\end{aligned}$$

If $t=1$, then

$$\prod_{SO}(G)-\prod_{SO}(G^*) \geq \sqrt{1+9}\sqrt{d_G^2(u)+9}\cdot\sqrt{d_G^2(w)+9}-\sqrt{4+4}\sqrt{d_G^2(u)+4}\cdot\sqrt{d_G^2(w)+4}>0.$$

□

Lemma 2.6. *Let* $f(x)=((x+m)^2+1)^{\frac{x}{2}}\cdot\prod_{i=1}^m\sqrt{(x+m)^2+(d^{(i)})^2}$*, where* $d^{(i)}$ $(i=1,2,\cdots,m)$ *are all nonnegative integers. Then for any positive integers* s*,* t*, the inequality* $f(s+t)f(0)>f(s)f(t)$ *holds.*

Proof. We prove that $\ln f(s+t)+\ln f(0)>\ln f(s)+\ln f(t)$. For this, let $g(x)=\ln f(x)+\ln f(0)-\ln f(x_1)-\ln f(x-x_1)$. Then,

$$\begin{aligned}g'(x) &= \frac{1}{2}\ln((x+m)^2+1)+\frac{x(x+m)}{(x+m)^2+1}+\sum_{i=1}^m\frac{x+m}{(x+m)^2+(d^{(i)})^2}\\ &\quad-\left[\frac{1}{2}\ln((x-x_1+m)^2+1)+\frac{(x-x_1)(x-x_1+m)}{(x-x_1+m)^2+1}+\sum_{i=1}^m\frac{x-x_1+m}{(x-x_1+m)^2+(d^{(i)})^2}\right]\\ &\triangleq h(x)-h(x-x_1),\end{aligned}$$

where

$$h(x)=\frac{1}{2}\ln((x+m)^2+1)+\frac{x(x+m)}{(x+m)^2+1}+\sum_{i=1}^m\frac{x+m}{(x+m)^2+(d^{(i)})^2}.$$

But,

$$\begin{aligned}h'(x) &= \frac{x+m}{(x+m)^2+1}+\frac{m(x+m)^2+2x+m}{((x+m)^2+1)^2}+\sum_{i=1}^m\frac{(d^{(i)})^2-(x+m)^2}{((x+m)^2+1)^2}\\ &= \frac{x+m}{(x+m)^2+1}+\frac{2x+m}{((x+m)^2+1)^2}+\sum_{i=1}^m\frac{(d^{(i)})^2}{((x+m)^2+1)^2}\\ &> 0.\end{aligned}$$

Thus $g'(x)>0$, which implies $\ln f(s+t)+\ln f(0)>\ln f(s)+\ln f(t)$, and consequently, it holds that $f(s+t)f(0)>f(s)f(t)$. □

Lemma 2.7. *Let* G *and* G^* *be the graphs as given in Figure 5. Take* $G^*=G-\{vv_1,vv_2,\cdots,vv_t\}+\{uv_1,uv_2,\cdots,uv_t\}$*. Suppose that* $|N_{G_0}(u)|=|N_{G_0}(v)|=m$*,* $N_{G_0}(u)=\{u_1^0,u_2^0,\cdots,u_m^0\}$*,* $N_{G_0}(v)=\{v_1^0,v_2^0,\cdots,v_m^0\}$ *and* $d_{G_0}(u_i^0)=d_{G_0}(v_i^0)=d^{(i)}$ *for* $i=1,2,\cdots,m$*. Then* $\prod_{SO}(G)<\prod_{SO}(G^*)$*.*

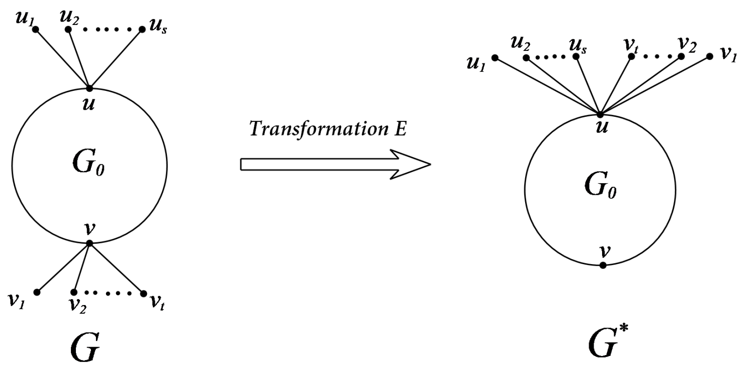

Figure 5: Transformation E.

Proof. By Lemma 2.6, we have

$$\begin{aligned}\prod\nolimits_{SO}(G^*)-\prod\nolimits_{SO}(G) \;\geq\;& ((s+t+m)^2+1)^{\frac{s+t}{2}}\cdot\prod_{i=1}^{m}\sqrt{(d^{(i)})^2+(s+t+m)^2}\cdot\prod_{i=1}^{m}\sqrt{(d^{(i)})^2+m^2}\\ &-((s+m)^2+1)^{\frac{s}{2}}\cdot((t+m)^2+1)^{\frac{t}{2}}\cdot\prod_{i=1}^{m}\sqrt{(d^{(i)})^2+(s+m)^2}\cdot\prod_{i=1}^{m}\sqrt{(d^{(i)})^2+(t+m)^2}\\ \triangleq\;& f(s+t)f(0)-f(s)f(t)>0,\end{aligned}$$

where

$$f(x)=((x+m)^2+1)^{\frac{x}{2}}\cdot\prod_{i=1}^{m}\sqrt{(x+m)^2+(d^{(i)})^2}.$$

□

3. Main results

Denote by $\mathcal{T}_n$ and $\mathcal{U}_n$, the sets of trees and unicyclic graphs with n vertices, respectively. By transformations given in Section 2, we have the following results.

Theorem 3.1. *Let $G\in\mathcal{T}_n$ and $G\ncong P_n$, $G\ncong S_n$. Then $\prod_{SO}(P_n)<\prod_{SO}(G)<\prod_{SO}(S_n)$.*

Denote by $T(a,b,c)$ the set of trees of maximum degree 3 with the unique maximum-degree vertex w such that

$$T(a,b,c)-w=P_a\cup P_b\cup P_c,$$

$a+b+c=n-1$, and $a\geq b\geq c\geq 2$. Let S_n^* be the tree obtained from star S_n by subdividing one of its edges. By transformations considered in Section 2, especially Transformation E, we have the next result.

Theorem 3.2. *Let $G\in\mathcal{T}_n$ and $G\ncong P_n,S_n,S_n^*,T(a,b,c)$. Then $\prod_{SO}(T(a,b,c))<\prod_{SO}(G)<\prod_{SO}(S_n^*)$.*

Let C_n^k be the unicyclic graph obtained from the cycle C_k by attaching $n-k$ pendent edges to one vertex of C_k.

Theorem 3.3. *Let $G\in\mathcal{U}_n$ and $G\ncong C_n$, $G\ncong C_n^3$. Then $\prod_{SO}(C_n)<\prod_{SO}(G)<\prod_{SO}(C_n^3)$.*

Denote by $\mathcal{U}_{n,g}$ the set of unicyclic graphs with n vertices and girth g. Denote by $\mathcal{F}_{n,g}$ the set of graphs with n vertices and girth g. By Lemma 2.2, we have the next theorem.

Theorem 3.4. *Let $G\in\mathcal{U}_{n,g}$. Then $\prod_{SO}(G)\geq 13\sqrt{65}\cdot 8^{\frac{n-4}{2}}$ with equality if and only if G is the unicyclic graph with girth g and one pendent path.*

Combining Theorem 3.4 and Lemma 2.1, we have the next result.

Theorem 3.5. *Let $G\in\mathcal{F}_{n,g}$. Then $\prod_{SO}(G)\geq 13\sqrt{65}\cdot 8^{\frac{n-4}{2}}$ with equality if and only if G is the unicyclic graph with girth g and one pendent path.*

A molecular tree is a tree with maximum degree $\Delta\leq 4$. We end this paper with the remark that the extremal multiplicative Sombor indices of (molecular) trees are under investigation.

Acknowledgements

I would like to gratefully thank Professor Ivan Gutman for encouraging me to consider the multiplicative Sombor index. I also would like to thank the anonymous referees for invaluable comments and suggestions. This work is supported by the Hunan Provincial Natural Science Foundation of China (Grant No. 2020JJ4423) and the Department of Education of Hunan Province (Grant No. 19A318).

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