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Research Article

Minimum distance-unbalancedness of graphs with diameter 2 and given number of edges

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Abstract

For a graph G, and for two distinct vertices u and v of G, let $n_G(u,v)$ be the number of vertices of G that are closer in G to u than to v. The distance-unbalancedness of G is the sum of $|n_G(u,v) - n_G(v,u)|$ over all unordered pairs of distinct vertices u and v of G. We determine the minimum distance-unbalancedness of 2-self-centered graphs with given number of edges. We also determine the minimum distance-unbalancedness of graphs with at least one universal vertex and given number of edges.

Keywords: distance-unbalancedness; total irregularity; graph of diameter 2.

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1. Introduction

We consider only finite, simple, and undirected graphs. Given a graph $G = (V(G), E(G))$, we let $n(G) = |V(G)|$ and $m(G) = |E(G)|$. For any graph G, we denote by $\overline{G}$ the complement of G. The set of neighbours of a vertex v in G is denoted by $N_G(v)$. The degree of $v \in V(G)$ is denoted by $d_G(v)$ or $d(v)$ for short if G is clear. A *universal vertex* is the vertex adjacent to all other vertices, and the maximum degree and the minimum degree of a graph G will be written as $\Delta(G)$ and $\delta(G)$, respectively. Denote by $\overline{d}(G)$ the average degree of a graph G and by $D(G)$ the set of the degrees of all vertices in G. Moreover, for any $a \in D(G)$, we denote by $\ell_G(a)$ the number of a in the degree sequence of G. For a graph G, and two vertices u and v of G, let $d_G(u,v)$ denote the distance in G between u and v, and let $n_G(u,v)$ be the number of vertices w of G that are closer to u than to v, that is, $d_G(u,w) < d_G(v,w)$. The *eccentricity* $e(v)$ of a vertex v is the maximum distance from v to all other vertices in G, that is, $e(v) = \max_{u \in V(G)} d(u,v)$. Moreover, the maximum eccentricity is the *diameter* of G and the minimum is the *radius* of G. A disconnected graph is said to have an infinite diameter. Clearly, any disconnected graph G has $diam(G) > k$ for any positive integer k. A graph is *self-centered* if all its vertices have the same eccentricity. And a self-centered graph G is k-self-centered if all its vertices have eccentricity k. The distance-unbalancedness [5] of a graph G is

$$uB(G) = \sum_{\{u,v\} \subseteq V(G)} |n_G(u,v) - n_G(v,u)|. \tag{1}$$

More results on uB are reported in [7, 8]. The distance-unbalancedness can be viewed as the total version of Mostar index [3]. More other distance-based topological indices can be found in [6]. A graph G is highly distance-balanced [4] if $n_G(u,v) = n_G(v,u)$ for every two distinct vertices u and v of G, that is, $uB(G) = 0$. The *total irregularity*, first introduced in [1], of a graph G is defined as

$$irr_t(G) = \sum_{\{u,v\} \subseteq V(G)} |d(u) - d(v)|.$$

Let $\mathcal{G}_n^2$ be the set of graphs of order n with diameter 2. For two vertex-disjoint graphs G and H, we denote by $G \cup H$ the union of graphs G and H. And the *join* $G \oplus H$ of graphs G and H is a new graph obtained from $G \cup H$ by joining each vertex of G with each vertex of H. In the rest of this paper we focus on the characterization of graphs from $\mathcal{G}_n^2$ with $m = \binom{n}{2} - k$ edges minimizing the distance-unbalancedness. In Section 2, some preliminary results are proven. In Section 3, we determine the minimum uB of the graphs in $\mathcal{G}_n^2$ with given number of edges with the corresponding minimal graphs.

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2. Preliminaries

Lemma 2.1. *Let $G \in \mathcal{G}_n^2$ with $n \geq 3$. Then $|n_G(u,v) - n_G(v,u)| = |d(u) - d(v)|$ for every two distinct vertices u and v of G.*

Proof. For any two distinct vertices $u, v \in V(G)$, we assume that $d(u) = k_1, d(v) = k_2$ and $|N_G(u) \cap N_G(v)| = p$. Next, we distinguish the following two cases.

Case 1. u and v are adjacent in G.
In this case we have $|N_G(u) \setminus N_G(v)| = k_1 - p$ and $|N_G(v) \setminus N_G(u)| = k_2 - p$. From the structure of G, any vertex in $N_G(u) \setminus N_G(v)$ except v is closer to u than to v in G, so is the vertex u itself. Therefore $n_G(u,v) = k_1 - p$. Similarly, $n_G(v,u) = k_2 - p$. It follows that $|n_G(u,v) - n_G(v,u)| = |d(u) - d(v)|$.

Case 2. u and v are not adjacent in G.
Similarly as above, we have $n_G(u,v) = |N_G(u) \setminus N_G(v)| + 1 = k_1 - p + 1$ and $n_G(v,u) = |N_G(v) \setminus N_G(u)| + 1 = k_2 - p + 1$. Then $|n_G(u,v) - n_G(v,u)| = |d(u) - d(v)|$ holds immediately. □

From Lemma 2.1, the following result is obvious.

Corollary 2.1. *For any $G \in \mathcal{G}_n^2$, we have $uB(G) = irr_t(G)$.*

From the definition of irr_t, the next result follows.

Proposition 2.1. *$irr_t(G) = irr_t(\overline{G})$ for any graph G of order $n > 1$.*

Lemma 2.2. *Let $G \in \mathcal{G}_n^2$ with $\{u,v,w\} \subseteq V(G)$, $uv \notin E(G)$, $vw \in E(G)$ and $G' = G + uv - vw$. If $d_G(u) < d_G(v) \leq d_G(w)$, then $irr_t(G) \geq irr_t(G')$ with equality holding if and only if $d_G(v) = d_G(w) = d_G(u) + 1$.*

Proof. Note that the degree of vertex v in G' remains unchanged with that in G. Let

$$I(x) = \sum_{y \in \{u,w\}} \Big[|d_G(y) - d(x)| - |d_{G'}(y) - d(x)|\Big]$$

for any vertex $x \in V(G) \setminus \{u,w\}$ and

$$I = \sum_{x \in V(G) \setminus \{u,w\}} I(x).$$

Observe that $I(x) \geq 0$ for any vertex $x \in V(G) \setminus \{u,w\}$. Then $I \geq 0$. By the structure of G and G' and the assumption $d_G(u) < d_G(v) \leq d_G(w)$, we have

$$\begin{aligned} irr_t(G) - irr_t(G') &= I + \sum_{\{x,y\} \subseteq \{u,v,w\}} \Big[|d_G(x) - d_G(y)| - |d_{G'}(x) - d_{G'}(y)|\Big] \\ &\geq \sum_{\{x,y\} \subseteq \{u,v,w\}} \Big[|d_G(x) - d_G(y)| - |d_{G'}(x) - d_{G'}(y)|\Big] \\ &\geq 0. \end{aligned}$$

Note that the last inequality holds if and only if $d_G(v) = d_G(w) = d_G(u) + 1$. When $d_G(v) = d_G(w) = d_G(u) + 1$, we have $I(x) = 0$ for any vertex $x \in V(G) \setminus \{u,w\}$. Hence $irr_t(G) = irr_t(G')$ if and only if $d_G(v) = d_G(w) = d_G(u) + 1$. □

Lemma 2.3. *Let $G \in \mathcal{G}_n^2$ with $\{u,v,w\} \subseteq V(G)$, $uv \notin E(G)$, $vw \in E(G)$ and $G' = G + uv - vw$. If $d_G(u) \leq d_G(v) < d_G(w)$, then $irr_t(G) \geq irr_t(G')$ with equality holding if and only if $d_G(u) = d_G(v) = d_G(w) - 1$.*

Proof. Note that the degree of vertex v in G' remains unchanged with that in G. Let

$$I(x) = \sum_{y \in \{u,w\}} \Big[|d_G(y) - d(x)| - |d_{G'}(y) - d(x)|\Big]$$

for any vertex $x \in V(G) \setminus \{u,w\}$ and

$$I = \sum_{x \in V(G) \setminus \{u,w\}} I(x).$$

Observe that $I(x) \geq 0$ for any vertex $x \in V(G) \setminus \{u,w\}$. Then $I \geq 0$.

By the structure of G and G' and the assumption $d_G(u) \leq d_G(v) < d_G(w)$, we have

$$irr_t(G) - irr_t(G') = I + \sum_{\{x,y\} \subseteq \{u,v,w\}} \Big[|d_G(x) - d_G(y)| - |d_{G'}(x) - d_{G'}(y)|\Big]$$

$$\begin{aligned} &\geq \sum_{\{x,y\}\subseteq\{u,v,w\}} \Big[|d_G(x)-d_G(y)| - |d_{G'}(x)-d_{G'}(y)|\Big] \\ &\geq 0. \end{aligned}$$

Note that the last inequality holds if and only if $d_G(u) = d_G(v) = d_G(w) - 1$. When $d_G(u) = d_G(v) = d_G(w) - 1$, we have $I(x) = 0$ for any vertex $x \in V(G) \setminus \{u, w\}$. Hence $irr_t(G) = irr_t(G')$ if and only if $d_G(u) = d_G(v) = d_G(w) - 1$. □

Lemma 2.4. *Let G be a graph of order n and size m such that $\overline{d}(G)$ is not an integer. If $irr_t(G)$ gets the minimum value, then $D(G) = \{\lfloor \overline{d}(G) \rfloor, \lceil \overline{d}(G) \rceil\}$.*

Proof. Since $\overline{d}(G)$ is not an integer, then we have $|D(G)| \geq 2$. Setting $a = \lfloor \overline{d}(G) \rfloor$, then $\lceil \overline{d}(G) \rceil = a + 1$. Thus $\Delta(G) \geq a + 1$ and $0 \leq \delta(G) \leq a$.

Let G_0 be a graph with minimum irr_t. It suffices to prove $\Delta(G_0) = a + 1$ and $\delta(G_0) = a$. For convenience, we denote $\Delta(G_0) = \Delta$ and $\delta(G_0) = \delta$. We first prove $\Delta = a + 1$. Otherwise, we have $\Delta \geq a + 2$. Then there exists a vertex w with $d_{G_0}(w) = \Delta$ and another vertex u with $d_{G_0}(u) = \delta \leq a$. So there exists a vertex $v \in N_{G_0}(w) \setminus N_{G_0}(u)$. If $d_G(v) > \delta$, then we construct a graph $G' = G_0 + uv - vw$. Then we have $irr_t(G') < irr_t(G_0)$ by Lemma 2.2, contradicting the minimality of G_0. If $d_G(v) = \delta$, then a new graph $G'' = G + uv - vw$ can be constructed with $irr_t(G'') < irr_t(G)$ by Lemma 2.3 as a contradiction, again. Therefore $\Delta = a + 1$ holds immediately.

Next we turn to the proof for $\delta = a$. If not, we have $0 < \delta \leq a - 1$. Then there exists a vertex u with $d_{G_0}(u) = \delta$ and another vertex w with $d_{G_0}(w) = \Delta \geq a + 1$. So there exists a vertex $v \in N_{G_0}(w) \setminus N_{G_0}(u)$. If $d_G(v) > \delta$, then we construct a graph $G' = G_0 + uv - vw$ with $irr_t(G') < irr_t(G)$ by Lemma 2.2, contradicting the minimality of G_0. If $d_G(v) = \delta$, then a new graph $G'' = G_0 + uv - vw$ can be constructed with $irr_t(G'') < irr_t(G)$ by Lemma 2.3 as a contradiction, again. Thus we have $\delta = a$, completing the proof. □

Note that if $\overline{d}(G)$ is an integer, then we can find a regular graph G^* such that $irr_t(G^*) = 0$. Obviously, the total irregularity of G^* is the minimum.

3. Main results

Let $k \leq h$ be positive integer(s). A graph G is called a $[k, h]$-graph if $k \leq d_G(v) \leq h$ for any $v \in V(G)$ and there are at least two vertices $x, y \in V(G)$ with $d_G(x) = k$ and $d_G(y) = h$. In particular, a $[k, k]$-graph is just a k-regular graph. Denote by $\mathcal{G}_n^2[k, h]$ with $k \leq h$ the set of all $[k, h]$-graphs in $\mathcal{G}_n^2$.

Theorem 3.1. *Let G be a 2-self-centered graph of even order n and size $m = \binom{n}{2} - k$. If $2k \equiv t \pmod{n}$ with $0 \leq t < n$, then $uB(G) \geq t(n - t)$ with equality holding if and only if $G \in \mathcal{G}_n^2[a, h]$ with $a = n - 1 - \lceil \frac{2k}{n} \rceil$ and $h = a$ if $t = 0$ or $h = a + 1$ otherwise.*

Proof. By Corollary 2.1, we only need to characterize the minimum 2-self-centered graphs with respect to irr_t. By Proposition 2.1, we can consider the complement $\overline{G}$ of graph G. Since G is connected, we have $k \leq \binom{n}{2} - n + 1$.

Since $2k \equiv t \pmod{n}$ with $0 \leq t < n$, there exists an integer c such that $2k = cn + t$. If $t = 0$, we have $irr_t(\overline{G}) \geq 0$ with equality holding if and only if $\overline{G}$ is a c-regular graph, that is, $G \in \mathcal{G}_n^2[a, a]$ with $a = n - 1 - \frac{2k}{n}$. If $t > 0$, then $\overline{d}(\overline{G}) = \frac{2k}{n}$ is not an integer with $\lfloor \frac{2k}{n} \rfloor = c$. By Lemma 2.4, $D(\overline{G}) = \{c, c + 1\}$ if $irr_t(\overline{G})$ gets the minimum value.

Assume that $\ell_{\overline{G}}(c) = x$ and $\ell_{\overline{G}}(c + 1) = y$. Then $x + y = n$ and $cx + (c + 1)y = t + cn$, which imply that $x = n - t$ and $y = t$. Therefore, $\overline{G}$ is a graph with $n - t$ vertices of degrees c and t vertices of degrees $c + 1$. Thus we have

$$uB(G) = irr_t(G) = irr_t(\overline{G}) \geq t(n - t)$$

with the last equality holds if and only if $\overline{G}$ is a $[c, c + 1]$-graph with $c = \lfloor \frac{2k}{n} \rfloor$ and $G \in \mathcal{G}_n^2$, that is, $G \in \mathcal{G}_n^2[a, a + 1]$ with $a = n - 1 - \lceil \frac{2k}{n} \rceil$, completing the proof. □

Theorem 3.2. *Let G be a 2-self-centered graph of odd order n and size $m = \binom{n}{2} - k$. If $2k \equiv t \pmod{2n}$ $(0 \leq t < 2n)$, then*

$$uB(G) \geq \begin{cases} t(n-t), & 0 \leq t \leq n-1; \\ (2n-t)(t-n), & n+1 \leq t < 2n \end{cases}$$

with equality holding if and only if $G \in \mathcal{G}_n^2[a, h]$ with $a = n - 1 - \lceil \frac{2k}{n} \rceil$ and $h = a$ if $t = 0$ or $h = a + 1$ otherwise.

Proof. By Corollary 2.1, we only need to characterize the minimum 2-self-centered graphs with respect to irr_t. By Proposition 2.1, we can consider the complement $\overline{G}$ of graph G. Since $2k \equiv t \pmod{2n}$ $(0 \le t < 2n)$, there exists an integer c such that $2k = 2cn + t$.

If $t = 0$, we have $irr_t(\overline{G}) \ge 0$ with equality holding if and only if $\overline{G}$ is a $2c$-regular graph, that is, $G \in \mathcal{G}_n^2[a, a]$ with $a = n - 1 - \frac{2k}{n}$. If $0 < t \le n - 1$, then $\overline{d}(\overline{G})$ is not an integer with $\lfloor \overline{d}(\overline{G}) \rfloor = 2c$. By Lemma 2.4, we have $D(\overline{G}) = \{2c, 2c+1\}$ if $irr_t(\overline{G})$ gets the minimum value. Assume that $\ell_{\overline{G}}(2c) = x$ and $\ell_{\overline{G}}(2c+1) = y$. Then $x + y = n$ and $2cx + (2c+1)y = t + 2cn$, which imply that $x = n - t$ and $y = t$. Therefore, $\overline{G}$ is a graph with $n - t$ vertices of degrees $2c$ and t vertices of degrees $2c+1$. Thus we have $uB(G) \ge t(n-t)$ with equality holding if and only if $\overline{G}$ is a $[2c, 2c+1]$-graph with $2c = \lfloor \frac{2k}{n} \rfloor$ and $G \in \mathcal{G}_n^2$, that is, $G \in \mathcal{G}_n^2[a, a+1]$ with $a = n - 1 - \lceil \frac{2k}{n} \rceil$.

If $n + 1 \le t < 2n$, then $\overline{d}(\overline{G})$ is not an integer with $\lfloor \overline{d}(\overline{G}) \rfloor = 2c + 1$. By Lemma 2.4, we have $D(\overline{G}) = \{2c+1, 2c+2\}$ if $irr_t(\overline{G})$ gets the minimum value. Assume that $\ell_{\overline{G}}(2c+1) = w$ and $\ell_{\overline{G}}(2c+2) = z$. Similarly as above, we have $w = 2n - t$ and $z = t - n$. Therefore, $\overline{G}$ is a graph with $2n - t$ vertices of degrees $2c + 1$ and $t - n$ vertices of degrees $2c + 2$. Thus we have $uB(G) \ge (2n - t)(t - n)$ with equality holding if and only if $\overline{G}$ is a $[2c+1, 2c+2]$-graph with $2c+1 = \lfloor \frac{2k}{n} \rfloor$ and $G \in \mathcal{G}_n^2$, that is, $G \in \mathcal{G}_n^2[a, a+1]$ with $a = n - 1 - \lceil \frac{2k}{n} \rceil$, completing the proof. □

Note that $\overline{G \cup H} = \overline{G} \oplus \overline{H}$ has diameter 2 for any two vertex-disjoint graphs G and H such that at least one of them is non-complete. Combining it with a well-known fact that the complement of connected graph G has diameter 2 if G has a finite diameter greater than 3 (see [2]), we have the following result.

Remark 3.1. *Let G be a graph with (in)finite diameter greater than 3. Then $\overline{G}$ has diameter 2.*

We give a sufficient condition on the set $\mathcal{G}_n^2[a, h]$ with $h \in \{a, a+1\}$ in terms of the complement.

Remark 3.2. *Let G be a graph of order $n \ge 3$ with m edges. If $\overline{G}$ is a $[n - 1 - \lceil \frac{2m}{n} \rceil, n - 1 - \lfloor \frac{2m}{n} \rfloor]$-graph of diameter greater than 3, then $G \in \mathcal{G}_n^2[\lfloor \frac{2m}{n} \rfloor, \lceil \frac{2m}{n} \rceil]$.*

Note that Petersen graph and its complement have diameters 2, the former is 3-regular and the latter is 6-regular. Therefore Remark 3.2 is just a sufficient but not necessary condition of set $\mathcal{G}_n^2[a, h]$ with $h \in \{a, a+1\}$.

Theorem 3.3. *Let $G \in \mathcal{G}_n^2$ of odd order n and size $m = \binom{n}{2} - k$ with $\Delta(G) = n - 1$. If $2k \equiv t \pmod{n-1}$ with $0 \le t < n - 1$, then $uB(G) \ge t(n - t - 1) + 2k$ with equality holding if and only if $G \cong H \oplus K_1$ where H is an $[a, h]$-graph with $a = n - 2 - \lceil \frac{2k}{n-1} \rceil$ and $h = a$ if $t = 0$ or $h = a + 1$ otherwise.*

Proof. Since $\Delta(G) = n - 1$, $\overline{G}$ contains at least one isolated vertex. Define G_1 to be a graph obtained from $\overline{G}$ by deleting an isolated vertex. Thus $uB(G) = irr_t(G) = irr_t(\overline{G}) = irr_t(G_1) + 2k$ by Corollary 2.1 and Proposition 2.1.

Since $2k \equiv t \pmod{n-1}$ with $0 \le t < n - 1$, there exists an integer c such that $2k = c(n-1) + t$. If $t = 0$, we have $irr_t(G_1) \ge 0$ with equality holding if and only if G_1 is a c-regular graph with $\overline{G} \cong G_1 \cup K_1$, that is, $G \cong H \oplus K_1$ where $H \cong \overline{G_1}$ is an a-regular graph with $a = n - 2 - \frac{2k}{n-1}$.

If $t > 0$, then $\overline{d}(G_1)$ is not an integer with $\lfloor \overline{d}(G_1) \rfloor = c$. By Lemma 2.4, we have $D(G_1) = \{c, c+1\}$ if $irr_t(G_1)$ gets the minimum value. Assume that $\ell_{G_1}(c) = x$ and $\ell_{G_1}(c+1) = y$. Then $x + y = n - 1$ and $cx + (c+1)y = c(n-1) + t$, which imply that $x = n - t - 1$ and $y = t$. Therefore G_1 is a graph with $n - t - 1$ vertices of degrees c and t vertices of degrees $c + 1$. Thus $uB(G) = irr_t(G_1) + 2k \ge t(n - t - 1) + 2k$ with equality holding if and only if G_1 is a $[c, c+1]$-graph with $c = \lfloor \frac{2k}{n-1} \rfloor$, that is, $G \cong H \oplus K_1$ where $H \cong \overline{G_1}$ is an $[a, a+1]$-graph with $a = n - 2 - \lceil \frac{2k}{n-1} \rceil$. □

Theorem 3.4. *Let $G \in \mathcal{G}_n^2$ of even order n and size $m = \binom{n}{2} - k$ with $\Delta(G) = n - 1$. If $2k \equiv t \pmod{2(n-1)}$ with $0 \le t < 2n - 2$, then*

$$uB(G) \ge \begin{cases} t(n - t - 1) + 2k, & 0 \le t \le n - 2; \\ (2n - t - 2)(t - n + 1) + 2k, & n \le t < 2n - 2 \end{cases}$$

with equality holding if and only if $G \cong H \oplus K_1$ where H is an $[a, h]$-graph with $a = n - 2 - \lceil \frac{2k}{n-1} \rceil$ and $h = a$ if $t = 0$ or $h = a + 1$ otherwise.

Proof. Since $\Delta(G) = n - 1$, $\overline{G}$ contains at least one isolated vertex. Let G_1 to be a graph obtained from $\overline{G}$ by deleting an isolated vertex. Thus $uB(G) = irr_t(G) = irr_t(\overline{G}) = irr_t(G_1) + 2k$ by Corollary 2.1 and Proposition 2.1.

Since $2k \equiv t \pmod{2(n-1)}$ with $0 \le t < 2n - 2$, there exists an integer c such that $2k = 2c(n-1) + t$. If $t = 0$, we have $irr_t(G_1) \ge 0$ with equality holding if and only if G_1 is a $2c$-regular graph, that is, $G \cong H \oplus K_1$ where $H \cong \overline{G_1}$ is an a-regular graph with $a = n - 2 - \frac{2k}{n-1}$.

If $0 < t \le n - 2$, then $\overline{d}(G_1)$ is not an integer with $\lfloor \overline{d}(G_1) \rfloor = 2c$. By Lemma 2.4, we have $D(G_1) = \{2c, 2c+1\}$ if $irr_t(G_1)$ gets the minimum value. Assume that $\ell_{G_1}(2c) = x$ and $\ell_{G_1}(2c+1) = y$. Then $x + y = n - 1$ and $2cx + (2c+1)y = 2c(n-1) + t$, which imply that $x = n - t - 1$ and $y = t$. Therefore G_1 is a graph with $n - t - 1$ vertices of degrees $2c$ and t vertices of degrees $2c + 1$. Thus $uB(G) = irr_t(G_1) + 2k \geq t(n - t - 1) + 2k$ with equality holding if and only if G_1 is a $[2c, 2c + 1]$-graph with $2c = \lfloor \frac{2k}{n-1} \rfloor$, that is, $G \cong H \oplus K_1$ where $H \cong \overline{G_1}$ is an $[a, a + 1]$-graph with $a = n - 2 - \lceil \frac{2k}{n-1} \rceil$.

If $n \leq t < 2n - 2$, then $\overline{d}(G_1)$ is not an integer with $\lfloor \overline{d}(G_1) \rfloor = 2c + 1$. By Lemma 2.4, we have $D(G_1) = \{2c + 1, 2c + 2\}$ if $irr_t(G_1)$ gets the minimum value. Assume that $\ell_{G_1}(2c+1) = w$ and $\ell_{G_1}(2c+2) = z$. Similarly as above, we have $w = 2n-t-2$ and $z = t-n+1$. Therefore, G_1 is a graph with $2n-t-2$ vertices of degree $2c+1$ and $t-n+1$ vertices of degree $2c+2$. Thus we have $uB(G) = irr_t(G_1) + 2k \geq (2n - t - 2)(t - n + 1) + 2k$ with equality holding if and only if G_1 is a $[2c + 1, 2c + 2]$-graph with $2c + 1 = \lfloor \frac{2k}{n-1} \rfloor$, that is, $G \cong H \oplus K_1$ where $H \cong \overline{G_1}$ is an $[a, a + 1]$-graph with $a = n - 2 - \lceil \frac{2k}{n-1} \rceil$. □

In Theorems 3.1, 3.2, 3.3 and 3.4 we determine the minimal graphs from $\mathcal{G}_n^2$ with given number of edges with respect to uB. We would like to end this paper with the following problem.

Problem 3.1. *Determine the maximal graphs from $\mathcal{G}_n^2$ with given number of edges with respect to uB.*

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