## Research Article

# Some new families of compositions based on big part restrictions 

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#### Abstract

In this paper, we enumerate four new families of compositions whose members satisfy certain conditions on the sizes of the big (i.e., $>1$ ) parts and/or lengths of the 1 -strings. In particular, we consider various classes of compositions whose members do not contain two consecutive big parts. We make use of combinatorial arguments, mainly direct enumeration and bijections, in determining the cardinalities of these classes. As a consequence of our results, new combinatorial interpretations in terms of restricted compositions are obtained for some well-known sequences, including the Padovan, Narayana and $m$-step Fibonacci sequences.


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## 1. Introduction

A composition of a positive integer $n$ is a representation of $n$ as a sequence of positive integers, called parts, which sum to $n$. For example, the compositions of $n=4$ are given by

$$
(4),(1,3),(2,2),(3,1),(1,1,2),(1,2,1),(2,1,1),(1,1,1,1)
$$

Denote the set of all compositions of $n$ by $Z(n)$ and let $z(n)=|Z(n)|$. Considering whether the last part within a member of $Z(n+1)$ for $n \geq 1$ is 1 or greater than 1 leads to the recurrence relation:

$$
\begin{equation*}
z(n+1)=2 z(n), \quad z(1)=1 \tag{1}
\end{equation*}
$$

which is equivalent to the well-known formula $z(n)=2^{n-1}$. By convention, $Z(0)$ denotes the set consisting of the empty composition with no parts, with $z(0)=1$. We refer the reader to the classic text by MacMahon [9] and to the more recent one by Heubach and Mansour [6] for an introduction to the study of compositions and related structures.

The problem of enumerating compositions based on various restrictions of the part sizes has been an object of ongoing research in combinatorics. See, e.g., [1-5, 7, 8, 11] and references contained therein. A notable result (see Sills [11]) states that the number of compositions of $n$ with all parts odd is the same as the number of compositions of $n+1$ with no 1's in analogy to Euler's celebrated theorem on partitions with odd parts. Combinatorial proofs have been given for this equivalence and extensions to an arbitrary modulus $m$ [10] have been found. Related results were shown for the number of compositions containing no part of size $k$ for any $k$ [3] or whose parts belong to a particular set [5]. Here, we consider subsets of $Z(n)$ in which the part restrictions applying to the first and last parts differ from those applying to all other parts. These restrictions will vary according to an arbitrary positive integer parameter $m$ and yield elegant results for small $m$.

In what follows, it will be convenient to write compositions, symbolically, by representing a maximal string of 1's of length $x$ by $1^{x}$, where two adjacent big parts (i.e., parts $>1$ ) are assumed to be separated by $1^{0}$. A general composition then has one of the following two forms:

$$
\begin{gather*}
\pi=\left(1^{a_{1}}, b_{1}, 1^{a_{2}}, b_{2}, \ldots\right), \quad a_{1} \geq 1, a_{i} \geq 0, i>1, b_{i} \geq 2 \quad \forall i ;  \tag{2}\\
\pi=\left(b_{1}, 1^{a_{1}}, b_{2}, 1^{a_{2}}, \ldots\right), \quad a_{i} \geq 0, b_{i} \geq 2, \tag{3}
\end{gather*}
$$

where it is understood in each case that $\pi$ either ends in $1^{a_{r}}$ or $b_{r}$ for some $r \geq 1$ with $a_{r} \geq 1$ and $b_{r} \geq 2$. Throughout, we will write $\pi \vdash n$ to indicate that $\pi$ is a composition of $n$. By an interior part of $\pi \vdash n$, we will mean one that is neither the

[^0]first nor the last part of $\pi$. A part that is not interior will be referred to as a boundary part. In the one-part composition $(n) \vdash n$, the $n$ is a boundary, but not an interior, part.

In this paper, we enumerate four families of compositions of $n$ based on various bounds for the exponents $a_{i}$ and parts $b_{i}$ in (2) and (3). In particular, we count classes of $\pi \vdash n$ in which the $a_{i}$ and $b_{i}$ are either bounded from below, from above or both, along with another class in which all big parts must be greater than some fixed number. Further, the bounds on the $b_{i}$ will be dictated in three of the families based on whether a big part is an interior or a boundary part. Our proofs are combinatorial in nature and make use of various bijections defined on certain subsets of the compositions in question. As special cases of our results, we obtain new combinatorial interpretations in terms of restricted compositions for several well-known counting sequences, among them the Padovan and Narayana numbers and A005251 in OEIS [12], in addition to various $m$-step Fibonacci sequences.

## 2. Enumeration of some restricted classes of compositions

## The class $G(n, m)$

Given $m \geq 1$, let $G(n, m)$ denote the subset of $Z(n)$ consisting of those compositions in which there are no two adjacent big parts and all big parts are at least $m+1$ and let $g(n, m)=|G(n, m)|$. For example, we have $g(6,2)=11$, the enumerated compositions being

$$
(6),(1,5),(5,1),\left(1^{2}, 4\right),(1,4,1),\left(4,1^{2}\right),\left(1^{3}, 3\right),\left(1^{2}, 3,1\right),\left(1,3,1^{2}\right),\left(3,1^{3}\right),\left(1^{6}\right) .
$$

Let $a_{n}=g(n, m)$ for a fixed $m$. The $a_{n}$ are given recursively in the following theorem.
Theorem 2.1. We have

$$
\begin{equation*}
a_{n}=a_{n-1}+a_{n-m-1}+\sum_{i=2}^{m+1} a_{n-m-i-1}, \quad n \geq 2 m+2, \tag{4}
\end{equation*}
$$

with $a_{0}=\cdots=a_{m}=1$ and $a_{m+i}=\binom{i+1}{2}+1$ for $1 \leq i \leq m+1$.
Proof. As for the initial values, note first that $\mathcal{A}_{n}=G(n, m)$ is a singleton set if $0 \leq n \leq m$. If $n=m+i$ where $i \in$ $[m+1]=\{1,2, \ldots, m+1\}$, then the members of $\mathcal{A}_{n}-\left\{\left(1^{n}\right)\right\}$ must contain exactly one big part $b$, namely, $b=m+j$ for some $j \in[i]$. Then there are $i-j+1$ compositions consisting of $i-j$ 1's and a single part $b$ and summing over all $j$ gives $\sum_{j=1}^{i}(i-j+1)=\binom{i+1}{2}$ members of $\mathcal{A}_{n}-\left\{1^{n}\right\}$.

To show (4), we enumerate members $\pi \in \mathcal{A}_{n}$ by considering the first letter of $\pi$. If $\pi$ starts with 1 , then there are clearly $a_{n-1}$ possibilities. If the first two parts of $\pi$ are $m+i, 1$ where $i \in[2, m+1]$, then there are $a_{n-m-i-1}$ such members of $\mathcal{A}_{n}$. Note that the members of $\mathcal{A}_{n}$ that have not been enumerated to this point are of the form $\pi=(m+1) 1 \pi^{\prime}$ or $\pi=b \pi^{\prime}$, where $b \geq 2 m+2$. Let $\mathcal{A}_{n}^{*}$ denote this subset of $\mathcal{A}_{n}$ and to complete the proof of (4), we define a bijection $f$ between $\mathcal{A}_{n-m-1}$ and $\mathcal{A}_{n}^{*}$. To do so, suppose $\lambda=\ell \lambda^{\prime} \in \mathcal{A}_{n-m-1}$, where $\ell \geq 1$. If $\ell=1$, then let $f(\lambda)=(m+1) 1 \lambda^{\prime}$. If $\ell \geq m+1$, then let $f(\lambda)=(\ell+m+1) \lambda^{\prime}$. Note that $\ell \geq m+1$ in the latter case implies $\lambda^{\prime}$ starts with a 1 if nonempty. One may verify that $f$ is a bijection, which completes the proof of (4).

Let $G(n)=G(n, 1)$ and $g(n)=g(n, 1)$. Note that $G(n)$ is the set of all compositions of $n$ in which no two big parts are adjacent. We have the following recurrence formula for $g(n)$.
Corollary 2.1. The sequence $g(n)$ satisfies

$$
g(n)=g(n-1)+g(n-2)+g(n-4), \quad n \geq 4,
$$

with $g(0)=g(1)=1, g(2)=2$ and $g(3)=4$. Moreover, $g(n)$ is also the number of compositions of $n+1$ without 2 's.
Proof. The first statement follows from the $m=1$ case of Theorem 2.1. For the second, let $G_{2}(n)$ denote the set of compositions of $n$ without 2's and we define a bijection $f$ between $G(n)$ and $G_{2}(n+1)$ as follows. Given $\pi=\pi_{1} \cdots \pi_{m} \in G(n)$, where each $\pi_{i}$ represents a part, we append 1 to obtain $\pi^{\prime}$, i.e., let $\pi_{m+1}=1$. Note that each big part $b$ of $\pi^{\prime}$ is followed by one or more l's. Consider replacing $b, 1$ in $\pi^{\prime}$ for each $b$ with the single part $b+1$ to obtain $f(\pi) \in G_{2}(n+1)$. One may verify that $f$ is a bijection.

To illustrate the bijection from the preceding proof, let $n=5$ and $\left(1^{2}, 3\right),(2,1,2) \in G(5)$. Then we have

$$
\left(1^{2}, 3\right) \rightarrow\left(1^{2}, 3,1\right) \rightarrow\left(1^{2}, 4\right) \in G_{2}(6) \quad \text { and } \quad(2,1,2) \rightarrow(2,1,2,1) \rightarrow(3,3) \in G_{2}(6) .
$$

Note that $g(n)=A 005251(n+2)$ for $n \geq 0$, where A005251 denotes the respective entry in [12].
We now consider further some special subsets of $G(n)$.

## The class $R(n, m)$

Given $m \geq 1$, let $R(n, m)$ denote the set of compositions $\pi \vdash n$ which, in their symbolic form (2) or (3), satisfy
(i) every string $1^{x}$ satisfies $x \geq m$,
(ii) each boundary part $b>1$ satisfies $b \geq m+1$,
(iii) each interior part $b>1$ satisfies $b \geq m+2$.

Let $r(n, m)$ denote the cardinality of $R(n, m)$.

## Example 2.1.

$$
r(6,1)=16:(6),(1,5),(5,1),\left(1^{2}, 4\right),(1,4,1),(2,1,3),(3,1,2),\left(4,1^{2}\right),\left(1^{3}, 3\right),\left(1^{2}, 3,1\right),\left(1,3,1^{2}\right),\left(2,1^{2}, 2\right),\left(3,1^{3}\right),\left(1^{4}, 2\right),\left(2,1^{4}\right),\left(1^{6}\right)
$$

Note that $R(n, 1)$ consists of compositions in $G(n)$ without interior 2's.

$$
\begin{aligned}
& r(6,2)=6:(6),\left(1^{2}, 4\right),\left(4,1^{2}\right),\left(1^{3}, 3\right),\left(3,1^{3}\right),\left(1^{6}\right) \\
& r(6,3)=r(6,4)=r(6,5)=2:(6),\left(1^{6}\right) \\
& r(6,6)=1:\left(1^{6}\right)
\end{aligned}
$$

Let $b_{n}=r(n, m)$ for a fixed $m$. The $b_{n}$ satisfy the recursion given in the next theorem.
Theorem 2.2. We have

$$
\begin{equation*}
b_{n}=b_{n-1}+b_{n-m-1}, \quad n \geq 2 m+2, \tag{5}
\end{equation*}
$$

with initial values $b_{0}=1, b_{n}=0$ for $1 \leq n<m, b_{m}=1, b_{n}=2$ for $m+1 \leq n \leq 2 m$ and $b_{2 m+1}=4$.
Proof. We enumerate members of $\mathcal{B}_{n}=R(n, m)$ where $n \geq 2 m+2$ as follows. First consider $\lambda \in \mathcal{B}_{n}$ having one of the following forms: (i) $\lambda=1^{\ell} \lambda^{\prime}$, where $\ell \geq m+1$ and $\lambda^{\prime}$ starts with a big part if nonempty, or (ii) $\lambda=b \lambda^{\prime}$, where $b \geq m+2$ and $\lambda^{\prime}$ starts with 1 if nonempty. We define a bijection $f$ between the subset of $\mathcal{B}_{n}$ comprising those members satisfying (i) or (ii) and $\mathcal{B}_{n-1}$ by setting $f(\lambda)=1^{\ell-1} \lambda^{\prime}$ if (i) holds or setting $f(\lambda)=(b-1) \lambda^{\prime}$ if (ii). On the other hand, suppose $\lambda \in \mathcal{B}_{n}$ is expressible either as (a) $\lambda=1^{m} b \lambda^{\prime}$, where $b \geq m+2$, or (b) $\lambda=(m+1) \lambda^{\prime}$. Note that $n \geq 2 m+2$ implies $b \geq m+2$ is indeed required in part (a). We then define a bijection $g$ between members of $\mathcal{B}_{n}$ satisfying (a) or (b) and $\mathcal{B}_{n-m-1}$ by letting $g(\lambda)=(b-1) \lambda^{\prime}$ if (a) applies or letting $g(\lambda)=\lambda^{\prime}$ if (b). Since all members of $\mathcal{B}_{n}$ are of the form (i), (ii), (a) or (b) above, the proof of (5) is complete. The initial values for $n<2 m+2$ may be verified directly using the definitions.

Remark 2.1. Note that $r(n, m)$ satisfies the same recurrence (with m replaced by $m-1$ ) as the three classes of compositions considered in [10, Theorem 1.2], though the initial values are different.

The case $m=2$ of Theorem 2.2 gives the Narayana's cows sequence [12, A000930].
Corollary 2.2. The numbers $r(n):=r(n, 2)$ satisfy the recurrence

$$
r(n)=r(n-1)+r(n-3), \quad n \geq 6
$$

with $r(0)=1, r(1)=0, r(2)=1, r(3)=r(4)=2$ and $r(5)=4$. Thus, we have $r(n)=2 \cdot A 000930(n-2)$ for $n \geq 3$. Moreover, $r(n+3)$ is twice the number of compositions of $n$ into parts 1 and 2 with no adjacent 2 's.

Proof. The first two statements follow from the $m=2$ case of Theorem 2.2 and a comparison with the OEIS entry. The third statement may be shown by comparing recurrences and initial values, but we find the following bijective proof more instructive. Let $R(n)=R(n, 2)$ and let $R^{\prime}(n)$ denote the subset of $R(n)$ consisting of those members that start with 1 . We first define a bijection $\alpha$ between $R^{\prime}(n)$ and $R(n)-R^{\prime}(n)$ for $n \geq 3$.

Note $\pi \in R^{\prime}(n)$ implies either
(i) $\pi=1^{u_{1}} v_{1} \cdots 1^{u_{m}} v_{m}$, where $m \geq 1$ and $u_{i} \geq 2$ for all $i$, with $v_{1}, \ldots, v_{m-1} \geq 4, v_{m} \geq 3$,
or
(ii) $\pi=1^{u_{1}} v_{1} \cdots 1^{u_{m-1}} v_{m-1} 1^{u_{m}}$, with the same restrictions on the $u_{i}$ and $v_{i}$.

If (i) holds, then let

$$
\alpha(\pi)=\left(u_{1}+1\right) 1^{v_{1}-2} \delta_{2} \cdots \delta_{m-1}\left(u_{m}+2\right) 1^{v_{m}-1}
$$

if $m \geq 2$ where $\delta_{i}=\left(u_{i}+2\right) 1^{v_{i}-2}$ for $2 \leq i \leq m-1$, with $\alpha\left(1^{u_{1}} v_{1}\right)=\left(u_{1}+1\right) 1^{v_{1}-1}$. If (ii), then let

$$
\alpha(\pi)=\left(u_{1}+1\right) 1^{v_{1}-2} \delta_{2} \cdots \delta_{m-1}\left(u_{m}+1\right), \quad m \geq 2
$$

with $\alpha\left(1^{n}\right)=n$. One may verify that $\alpha$ yields the desired bijection.
Now let $K(n)$ denote the compositions of $n$ with parts in $\{1,2\}$ with no 2's adjacent. In light of $\alpha$, to prove the third statement, it is enough to define a bijection $\beta$ between $R^{\prime}(n+3)$ and $K(n)$ for $n \geq 0$. We again consider the cases (i) and (ii) for $\pi \in R^{\prime}(n+3)$. If (i) holds, then let

$$
\beta(\pi)=1^{u_{1}-2} \gamma_{1} \cdots \gamma_{m-1} 21^{v_{m}-3}
$$

where $\gamma_{i}=21^{v_{i}-3} 21^{u_{i+1}-1}$ for $1 \leq i \leq m-1$. For (ii), let

$$
\beta(\pi)=1^{u_{1}-2} \gamma_{1} \cdots \gamma_{m-2} 21^{v_{m-1}-3} 21^{u_{m}-2}, \quad m \geq 2
$$

with $\beta\left(1^{n+3}\right)=1^{n}$. Note that $\beta(\pi)$ has an odd or even number of 2's depending on whether $\pi$ is of form (i) or (ii), from which it is possible to construct $\beta^{-1}$.

## The class $S(n, m)$

In this subsection, we consider the class of compositions obtained by reversing the inequalities in the restrictions for membership in $R(n, m)$. Consider the set $S(n, m)$ of compositions $\pi \vdash n$ which, in symbolic form, satisfy
(i) every string $1^{x}$ satisfies $0 \leq x \leq m$,
(ii) each boundary part $b>1$ satisfies $b \leq m+1$,
(iii) each interior part $b>1$ satisfies $b \leq m+2$.

By convention, we have $S(n, m)=Z(n)$ if $m \geq n$. Let $s(n, m)=|S(n, m)|$.

## Example 2.2.

$$
\begin{aligned}
s(6,1)=6: & (1,3,2),(2,2,2),(2,3,1),(1,2,1,2),(1,2,2,1),(2,1,2,1) \\
s(6,2)= & 18:(3,3),(1,2,3),(1,3,2),(1,4,1),(2,1,3),(2,2,2),(2,3,1),(3,1,2),(3,2,1),\left(1^{2}, 2,2\right),\left(1^{2}, 3,1\right),(1,2,1,2),(1,2,2,1), \\
& \left(1,3,1^{2}\right),\left(2,1^{2}, 2\right),(2,1,2,1),\left(2,2,1^{2}\right),\left(1^{2}, 2,1^{2}\right) \\
s(6,3)= & 26: Z(6) \backslash\left\{(6),(1,5),(5,1),\left(1^{4}, 2\right),\left(2,1^{4}\right),\left(1^{6}\right)\right\} \\
s(6,4)= & 30: Z(6) \backslash\left\{(6),\left(1^{6}\right)\right\} \\
s(6,5)= & 31: Z(6) \backslash\left\{\left(1^{6}\right)\right\} .
\end{aligned}
$$

Let $c_{n}=s(n, m)$, where $m \geq 1$ is fixed. Then the $c_{n}$ satisfy the linear recurrence given in the following theorem.
Theorem 2.3. We have

$$
\begin{equation*}
c_{n}=c_{n-1}+c_{n-2}+\cdots+c_{n-m-1}, \quad n \geq 2 m+3 \tag{6}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{n}=c_{n-1}+c_{n-2}+\cdots+c_{n-m-1}-1, \quad m+3 \leq n \leq 2 m+2, \tag{7}
\end{equation*}
$$

where $c_{0}=1, c_{n}=2^{n-1}$ for $1 \leq n \leq m, c_{m+1}=2^{m}-1$ and $c_{m+2}=2^{m+1}-2$.

Proof. We first show (6), where we may assume $m \geq 2$ since the case $m=1$ will be seen to follow in a similar manner. Let $\mathcal{C}_{n}=S(n, m)$. We now enumerate members of $\mathcal{C}_{n}$ as follows where $n \geq 2 m+3$. First suppose $\lambda \in \mathcal{C}_{n}$ is of the form $\lambda=i \lambda^{\prime}$, where $2 \leq i \leq m+1$ and $\lambda^{\prime}$ does not start with $m+2$. Upon deleting $i$, it is seen that there are $\sum_{i=2}^{m+1} c_{n-i}$ possibilities for such $\lambda$. So assume $\lambda$ does not have the form specified above and let $\mathcal{C}_{n}^{*}$ denote the subset of $\mathcal{C}_{n}$ consisting of such $\lambda$. Note that members of $\mathcal{C}_{n}^{*}$ either start with 1 or are of the form $\alpha=i(m+2) \alpha^{\prime}$ where $i>1$.

To complete the proof of (6), it then suffices to define a bijection $f$ between $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n}^{*}$ as follows. Let $\rho \in \mathcal{C}_{n-1}$. If $\rho$ does not start with 1 , then let $f(\rho)=1 \rho$, which yields (uniquely) all members of $\mathcal{C}_{n}^{*}$ starting with 1 and whose second part belongs to $[2, m+1]$. So assume $\rho=1^{\ell} j \rho^{\prime}$, where $1 \leq \ell \leq m$ and $2 \leq j \leq m+2$. Note that $n \geq 2 m+3$ implies $\rho^{\prime} \neq \varnothing$ for all possible $\ell$ and $j$, upon considering separately the case when $j=m+2$. To define $f$ for such $\rho$, we consider cases on $\ell$ and $j$. If $2 \leq \ell \leq m$ and $2 \leq j \leq m+1$, then replace $j$ with $j+1$ within $\rho$ to obtain $f(\rho)$ in this case, which is seen to yield all members of $\mathcal{C}_{n}^{*}$ starting with two or more 1's and whose first big part is not 2 . Note that $\rho^{\prime} \neq \varnothing$ implies $f(\rho)$ does not end in $m+2$ in the case when $j=m+1$. If $1 \leq \ell \leq m$ and $j=m+2$, then replace the string $1^{\ell}$ in $\rho$ with the single part $\ell+1$ to obtain $f(\rho)$, which gives members of $\mathcal{C}_{n}^{*}$ whose first part is not 1 . If $\ell=1$ and $2 \leq j \leq m$, then replace the initial section $1 j$ of $\rho$ with $1^{j} 2$ to obtain $f(\rho)$, whereas if $\ell=1$ and $j=m+1$, then replace $1(m+1)$ with $1(m+2)$ to obtain $f(\rho)$. Note that the last two cases yield all members of $\mathcal{C}_{n}^{*}$ that start with a string of two or more 1's followed by 2 or that start $1, m+2$. Checking separately each of the cases described above, one may verify that $f$ is reversible in each case and hence yields the desired bijection between $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n}^{*}$.

Concerning the initial values, clearly we have $c_{n}=2^{n-1}$ for $1 \leq n \leq m$ since there are no restrictions on any of the run lengths or part sizes in this case. Further, we have $c_{m+1}=2^{m}-1$ and $c_{m+2}=2^{m+1}-2$ since the composition consisting of all 1's must be excluded in both cases with the single-part composition also being excluded in the latter case. If $m+3 \leq n \leq 2 m+2$, then recurrence (7) is seen to hold, as the bijection $f$ above cannot be defined when $\ell=n-m-2$ and $j=m+1$ since the resulting composition would end in $m+2$ in that case, which completes the proof.

Taking $m=1$ in Theorems 2.2 and 2.3 yields the following result.

## Corollary 2.3. The following sets of compositions are equinumerous for $n \geq 2$ :

(i) $R(n, 1)$,
(ii) $S(n+2,1)$.

The common count is $2 F_{n}$.
Note that a bijective proof similar to that given above for the final statement in Corollary 2.2 above may be given to show $|R(n, 1)|=2 F_{n}$, though we were unable to find a direct bijective proof establishing the cardinality of $S(n+2,1)$.

## The class $T\left(n, m_{1}, m_{2}\right)$

Here, we consider a class of compositions where there are both upper and lower bounds on the lengths of the 1 -strings and the sizes of big parts. Given positive integers $m_{1}<m_{2}$, let $T\left(n, m_{1}, m_{2}\right)$ denote the set of compositions $\pi \vdash n$ which, in symbolic form, satisfy
(i) every string $1^{x}$ satisfies $m_{1} \leq x \leq m_{2}$,
(ii) each boundary part $b>1$ satisfies $m_{1}+1 \leq b \leq m_{2}+1$,
(iii) each interior part $b>1$ satisfies $m_{1}+2 \leq b \leq m_{2}+2$.

Let $t\left(n, m_{1}, m_{2}\right)=\left|T\left(n, m_{1}, m_{2}\right)\right|$.

## Example 2.3.

(i) $t(9,1,2)=14:(1,4,1,3),(3,1,4,1),\left(1^{2}, 3,1,3\right),\left(1^{2}, 4,1,2\right),\left(1,3,1^{2}, 3\right),(1,3,1,3,1),\left(1,4,1^{2}, 2\right),\left(2,1^{2}, 4,1\right),(2,1,3,1,2)$,

$$
\left(2,1,4,1^{2}\right),\left(3,1^{2}, 3,1\right),\left(3,1,3,1^{2}\right),\left(1^{2}, 3,1^{2}, 2\right),\left(2,1^{2}, 3,1^{2}\right)
$$

(ii) $t(9,2,3)=6:\left(3,1^{2}, 4\right),\left(4,1^{2}, 3\right),\left(1^{2}, 5,1^{2}\right),\left(3,1^{3}, 3\right),\left(1^{3}, 4,1^{2}\right),\left(1^{2}, 4,1^{3}\right)$.
(iii) $t(9,2,4)=8:\left(3,1^{2}, 4\right),\left(4,1^{2}, 3\right),\left(1^{4}, 5\right),\left(1^{2}, 5,1^{2}\right),\left(3,1^{3}, 3\right),\left(5,1^{4}\right),\left(1^{3}, 4,1^{2}\right),\left(1^{2}, 4,1^{3}\right)$.

Let $d_{n}=t\left(n, m_{1}, m_{2}\right)$ for fixed $m_{2}>m_{1} \geq 1$. The $d_{n}$ are given recursively in the next theorem, where $\chi(P)=1$ or 0 depending upon the truth or falsity of the statement $P$.

## Theorem 2.4. We have

$$
\begin{equation*}
d_{n}=d_{n-m_{1}-1}+d_{n-m_{1}-2}+\cdots+d_{n-m_{2}-1}-\chi\left(m_{1}+m_{2}+2 \leq n \leq 2 m_{2}+2\right), \quad n \geq m_{1}+m_{2}+2 \tag{8}
\end{equation*}
$$

with $d_{0}=1=d_{m_{1}}, d_{n}=0$ if $1 \leq n \leq m_{1}-1$ and

$$
\begin{equation*}
d_{n}=2-\delta_{n, m_{2}+1}-2 \chi\left(m_{2}+2 \leq n \leq m_{1}+m_{2}+1\right)+2 \sum_{j=2}^{\left\lfloor\frac{n+1}{m_{1}+1}\right\rfloor}\binom{n-j m_{1}}{j-1} \tag{9}
\end{equation*}
$$

for $m_{1}+1 \leq n \leq m_{1}+m_{2}+1$.
Proof. First suppose $n \geq 2 m_{2}+3$. Let $\mathcal{D}_{n}=T\left(n, m_{1}, m_{2}\right)$ and $\mathcal{D}_{n}^{(i)}$ for $i \in\left[m_{2}-m_{1}+1\right]$ denote the subset of $\mathcal{D}_{n}$ consisting of those $\pi$ that are expressible either as (i) $\pi=\left(m_{1}+i\right) \pi^{\prime}$ or (ii) $\pi=1^{m_{1}+i-1} \pi^{\prime}$. Note that $\mathcal{D}_{n}$ is a disjoint union of the $\mathcal{D}_{n}^{(i)}$. For each $i$, we define a bijection $f_{i}$ between $\mathcal{D}_{n-m_{1}-i}$ and $\mathcal{D}_{n}^{(i)}$ as follows. If $\lambda \in \mathcal{D}_{n-m_{1}-i}$ starts with a 1 , then let $f_{i}(\lambda)=\left(m_{1}+i\right) \lambda$. If $\lambda \in \mathcal{D}_{n-m_{1}-i}$ is of the form $\lambda=b \lambda^{\prime}$ where $m_{1}+1 \leq b \leq m_{2}+1$, then let $f_{i}(\lambda)=1^{m_{1}+i-1}(b+1) \lambda^{\prime}$. Note that $n \geq 2 m_{2}+3$ implies $\lambda^{\prime}$ is always nonempty regardless of the values of $b$ and $i$, and hence $b+1$ is an interior part of $f_{i}(\lambda)$ in all cases, in particular when $b=m_{2}+1$ where it is required. Therefore, one may verify for each $i$ that $f_{i}$ is reversible and hence a bijection. Combining all of the $f_{i}$ then yields a bijection $f$ between $\mathcal{D}_{n}$ and $\cup_{i=1}^{m_{2}-m_{1}+1} \mathcal{D}_{n-m_{1}-i}$, which implies (8) when $n \geq 2 m_{2}+3$.

Now suppose $m_{1}+m_{2}+2 \leq n \leq 2 m_{2}+2$. Then $f_{i}(\lambda)$ is not defined when $\lambda=b \lambda^{\prime}$ with $b=m_{2}+1$ and $i=n-m_{1}-m_{2}-1$, for in this case $\lambda^{\prime}$ would be empty and hence $b+1$ would not be an interior part of $f_{i}(\lambda)$. In all other cases, the mapping $f$ is defined, which implies we must subtract 1 in (8) as indicated.

To prove (9), suppose $\pi \in \mathcal{D}_{n}$, where $m_{1}+1 \leq n \leq m_{1}+m_{2}+1$ and $\pi \neq 1^{n}, n$. Let $s(\pi)$ denote the sum of the number of nonempty 1 -runs and the number of big parts in $\pi$. Note that $\pi \neq 1^{n}, n$ implies $s(\pi) \geq 2$. We now count members of $\mathcal{D}_{n}$ according to the value of $s(\pi)$. To do so, it is convenient to consider first a composition $\alpha$ of minimal length for which $s(\alpha)=j$. If $j$ is even, then we have

$$
\alpha=1^{m_{1}}\left(m_{1}+2\right) \cdots 1^{m_{1}}\left(m_{1}+2\right) 1^{m_{1}}\left(m_{1}+1\right) \text { or } \alpha=\left(m_{1}+1\right) 1^{m_{1}}\left(m_{1}+2\right) 1^{m_{1}} \cdots\left(m_{1}+2\right) 1^{m_{1}}
$$

where in either case it is understood that there are $\frac{j}{2}$ runs $1^{m_{1}}$. Note that the length of $\alpha$ is given by $j m_{1}+j-1$ in each case. Similarly, if $j \geq 3$ is odd, then there are either $\frac{j+1}{2}$ or $\frac{j-1}{2}$ runs $1^{m_{1}}$ implying

$$
\alpha=1^{m_{1}}\left(m_{1}+2\right) \cdots 1^{m_{1}}\left(m_{1}+2\right) 1^{m_{1}} \text { or } \alpha=\left(m_{1}+1\right) 1^{m_{1}}\left(m_{1}+2\right) \cdots 1^{m_{1}}\left(m_{1}+2\right) 1^{m_{1}}\left(m_{1}+1\right)
$$

both of which again have length $j m_{1}+j-1$.
Now imagine creating from $\alpha$ members $\pi \in \mathcal{D}_{n}$ for which $s(\pi)=j$ by increasing the lengths of the 1-runs of $\alpha$ by nonnegative amounts and/or increasing the size of the big parts of $\alpha$. Note that the sum of all of the increases of the run lengths and big part sizes of $\alpha$ must be $n-j m_{1}-j+1$ in order to obtain a member of $\mathcal{D}_{n}$. Moreover, there is no further restriction on the amount a particular run length or big part can be increased since $n \leq m_{1}+m_{2}+1$. Thus, the number of $\pi \in \mathcal{D}_{n}$ where $s(\pi)=j$ that can be obtained from a given $\alpha$ is the same as the number of non-negative integer solutions to the equation $x_{1}+x_{2}+\cdots+x_{j}=n-j m_{1}-j+1$, which is given by $\binom{n-j m_{1}}{j-1}$. Doubling this formula to account for the two possible $\alpha$ then gives the total number of $\pi$ for which $s(\pi)=j$. Considering all possible $j \geq 2$ then yields all members of $\mathcal{D}_{n}-\left\{1^{n}, n\right\}$. To complete the enumeration of $\mathcal{D}_{n}$, we then must add 2 to account for the compositions $1^{n}$ and $n$ if $m_{1}+1 \leq n \leq m_{2}$ and add 1 to account for the composition $n$ if $n=m_{2}+1$. If $m_{2}+2 \leq n \leq m_{1}+m_{2}+1$, then neither of these compositions would be allowed. Combining these observations then accounts for the quantity $2-\delta_{n, m_{2}+1}-2 \chi\left(m_{2}+2 \leq n \leq m_{1}+m_{2}+1\right)$ and completes the proof of (9). Finally, the values of $d_{n}$ for $0 \leq n \leq m_{1}$ are apparent from the definitions.
Remark 2.2. Note that the generating function of the quantity $p_{n}=2+2 \sum_{j=2}^{\left\lfloor\frac{n+1}{m+1}\right\rfloor}\binom{n-j m}{j-1}$ if $n \geq m+1$, with $p_{0}=1=p_{m}$ and $p_{1}=\cdots=p_{m-1}=0$, is given by

$$
\sum_{n \geq 0} p_{n} x^{n}=\frac{1-x+x^{m}+x^{2 m+1}}{1-x-x^{m+1}}
$$

If $m_{2} \leq 2 m_{1}+1$, then the values of $d_{n}$ for $m_{1}+1 \leq n \leq m_{1}+m_{2}+1$ are readily found using the explicit formula (9). On the other hand, if $m_{2} \geq 2 m_{1}+2$, then it is easier to observe that $d_{n}=p_{n}$ for $2 m_{1}+2 \leq n \leq m_{2}$, where $p_{n}$ is given recursively by

$$
\begin{equation*}
p_{n}=p_{n-1}+p_{n-m_{1}-1}, \quad n \geq 2 m_{1}+2 \tag{10}
\end{equation*}
$$

which follows from the generating function formula found above for $p_{n}$ (with $m_{1}$ in place of $m$ ). Further, it is seen from (9) and (10) that $d_{n}=p_{n}-2$ for $m_{2}+2 \leq n \leq m_{1}+m_{2}+1$, with $d_{n}=p_{n}-1$ if $n=m_{2}+1$. Note that the initial values for (10) are $p_{m_{1}+1}=\cdots=p_{2 m_{1}}=2$ and $p_{2 m_{1}+1}=4$.

Taking $m_{1}=1, m_{2}=2$ in Theorem 2.4 yields the following result for $t(n)=t(n, 1,2)$.

## Corollary 2.4. We have

$$
t(n)=2 P_{n+4}, \quad n \geq 4
$$

with initial values $t(0)=t(1)=1, t(2)=2$ and $t(3)=3$, where $P_{n}$ is the Padovan number sequence defined by

$$
P_{n}=P_{n-2}+P_{n-3} \quad \text { for } \quad n \geq 3, \quad \text { with } \quad P_{0}=1, P_{1}=P_{2}=0
$$

## References

[1] K. Alladi, V. E. Hoggatt, Compositions with ones and twos, Fibonacci Quart. 13 (1975) 233-239.
[2] L. Carlitz, Restricted compositions, Fibonacci Quart. 14 (1976) 361-371.
[3] P. Chinn, S. Heubach, Compositions of $n$ with no occurences of $k$, Congr. Numer. 164 (2003) 33-51.
[4] R. P. Grimaldi, Compositions with odd summands, Congr. Numer. 142 (2000) 113-127.
[5] S. Heubach, T. Mansour, Compositions of $n$ with parts in a set, Congr. Numer. 164 (2004) 127-143.
[6] S. Heubach, T. Mansour, Combinatorics of Compositions and Words, CRC Press, Boca Raton, 2010.
[7] V. E. Hoggatt, M. Bicknell, Palindromic compositions, Fibonacci Quart. 13 (1975) 350-356.
[8] G. Jaklič, V. Vitrih, E. Žagar, Closed form formula for the number of restricted compositions, Bull. Aust. Math. Soc. 81 (2010) $289-297$.
[9] P. A. MacMahon, Combinatory Analysis, Vol. 1, Cambridge University Press, Cambridge, 1915.
[10] A. O. Munagi, Euler-type identities for integer compositions via zig-zag graphs, Integers 12 (2012) \#A60.
[11] A. V. Sills, Compositions, partitions and Fibonacci numbers, Fibonacci Quart. 49 (2011) 348-354.
[12] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, Available at https://oeis.org.


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