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Research Article

An improved lower bound for the degree Kirchhoff index of bipartite graphs

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Abstract

For a connected graph G with n vertices and m edges, the degree Kirchhoff index of G is defined as $Kf^*(G) = 2m\sum_{i=1}^{n-1}(\gamma_i)^{-1}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0$ are the normalized Laplacian eigenvalues of G. In this paper, a lower bound on the degree Kirchhoff index of bipartite graphs is established. Also, it is proved that the obtained bound is stronger than a lower bound derived by Zhou and Trinajstić in [*J. Math. Chem.* **46** (2009) 283–289].

Keywords: topological indices; degree Kirchhoff index.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with n vertices and m edges, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. The degree of a vertex $v_i \in V(G)$ is denoted by d_i, where $i = 1, 2, \ldots, n$. If v_i and v_j are two adjacent vertices of G, then it is written as $i \sim j$.

Denote by $A(G)$ and $D(G) = diag(d_1, d_2, \ldots, d_n)$ the adjacency and the diagonal degree matrix of G, respectively. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$ (see [16]). Since G is assumed to be a connected graph, the matrix $D(G)^{-1/2}$ exists. The normalized Laplacian matrix of G is the matrix defined [8] by

$$\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}.$$

The eigenvalues $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0$ of $\mathcal{L}(G)$ represent the normalized Laplacian eigenvalues of G. Details on the spectra of $\mathcal{L}(G)$ can be found in [8].

Chen and Zhang [7] introduced the degree Kirchhoff index of a connected graph G as

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}, \tag{1}$$

where r_{ij} is the effective resistance distance between the vertices v_i and v_j of G. In [7], it was also demonstrated that the degree Kirchhoff index can be expressed in terms of normalized Laplacian eigenvalues as follows:

$$Kf^*(G) = 2m\sum_{i=1}^{n-1}\frac{1}{\gamma_i}. \tag{2}$$

Both of the definitions of the graph invariant $Kf^*(G)$ given by (1) and (2) are much studied in the chemical and mathematical literature. For survey and details, see [1,2,4,5,10–12,14,15,17,18,20,21].

In this paper, we present a lower bound on the degree Kirchhoff index of bipartite graphs. In addition, we show that our lower bound improves the lower bound obtained by Zhou and Trinajstić [21].

2. Lemmas

In this section, we recall a few well-known properties of the normalized Laplacian eigenvalues of graphs.

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Lemma 2.1. [8] *Let G be a connected graph with $n \geq 2$ vertices. Then, the following properties regarding the normalized Laplacian eigenvalues are valid:*

1. $\sum_{i=1}^{n} \gamma_i = n$.
2. $\gamma_1 \leq 2$ *with equality if and only if G is a bipartite graph.*
3. *For each $1 \leq i \leq n$, $\gamma_i \in [0, 2]$, $\gamma_n = 0$ and $\gamma_{n-1} \neq 0$.*

Lemma 2.2. [9] *Let G be a connected graph with n vertices and m edges. Then,*

$$\prod_{i=1}^{n-1} \gamma_i = \frac{2m\, t(G)}{\prod_{i=1}^{n} d_i},$$

where $t(G)$ is the total number of spanning trees of G.

Lemma 2.3. [13] *Let G be a connected graph of order n. Then, $\gamma_2 \geq 1$ with equality if and only if G is a complete bipartite graph.*

3. A lower bound for the degree Kirchhoff index of bipartite graphs

We now give an improved lower bound on the degree Kirchhoff index of bipartite graphs.

Theorem 3.1. *Let G be a connected bipartite graph with $n \geq 2$ vertices, m edges and $t(G)$ spanning trees. Then, for any real α, $\gamma_2 \geq \alpha \geq 1$*

$$Kf^*(G) \geq 2m \left(\frac{1}{2} + \frac{1}{\alpha} + n - 3 - \ln \left(\frac{m\, t(G)}{\prod_{i=1}^{n} d_i} \right) + \ln \alpha \right). \tag{3}$$

Equality in (3) holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$ $(p + q = n)$.

Proof. For $x > 0$, the following inequality can be found in the monograph [19]

$$x \leq 1 + x \ln x,$$

where the equality holds if and only if $x = 1$. For $x > 0$, the above inequality can be considered as

$$\frac{1}{x} \geq 1 - \ln x$$

with equality if and only if $x = 1$. By Lemma 2.1, $\gamma_1 = 2$ and $\gamma_i > 0$, $i = 1, 2, \ldots, n-1$, since G is a connected bipartite graph. Then, using these results and Lemma 2.2, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{1}{\gamma_i} &= \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \sum_{i=3}^{n-1} \frac{1}{\gamma_i} \\
&= \frac{1}{2} + \frac{1}{\gamma_2} + \sum_{i=3}^{n-1} \frac{1}{\gamma_i} \\
&\geq \frac{1}{2} + \frac{1}{\gamma_2} + \sum_{i=3}^{n-1} (1 - \ln \gamma_i) \\
&= \frac{1}{2} + \frac{1}{\gamma_2} + n - 3 - \ln \prod_{i=3}^{n-1} \gamma_i \\
&= \frac{1}{2} + \frac{1}{\gamma_2} + n - 3 - \ln \left(\frac{m\, t(G)}{\prod_{i=1}^{n} d_i} \right) + \ln \gamma_2. \qquad (4)
\end{aligned}$$

Now, consider the function $f(x) = \frac{1}{x} + \ln x$. It can be easily seen that this function is increasing in the interval $1 \leq x \leq 2$. Then for any real α, $\gamma_2 \geq \alpha \geq 1$, we have that

$$f(\gamma_2) \geq f(\alpha) = \frac{1}{\alpha} + \ln \alpha.$$

Bearing this fact in mind and using (2) and (4), we obtain that

$$Kf^*(G) \geq 2m \left(\frac{1}{2} + \frac{1}{\alpha} + n - 3 - \ln \left(\frac{m\, t(G)}{\prod_{i=1}^{n} d_i} \right) + \ln \alpha \right)$$

which is the required inequality (3). Now, assume that the equality holds in (3). Then

$$\gamma_2 = \alpha \text{ and } \gamma_3 = \cdots = \gamma_{n-1} = 1.$$

Since G is bipartite, by Lemma 2.1, $\sum_{i=2}^{n-1} \gamma_i = n-2$. Considering this with the above conditions, we get that $\gamma_2 = \alpha = 1$, which implies that $G \cong K_{p,q}$.

Conversely, it is not difficult to show that the equality holds in (3) for the complete bipartite graph $K_{p,q}$. Hence, the proof is completed. □

By Theorem 3.1 and Lemma 2.3, we have the following corollary.

Corollary 3.1. *Let G be a connected bipartite graph with $n \geq 2$ vertices, m edges and $t(G)$ spanning trees. Then,*

$$Kf^*(G) \geq m(2n-3) - 2m \ln\left(\frac{m\, t(G)}{\prod_{i=1}^{n} d_i}\right). \tag{5}$$

Equality in (5) holds if and only if $G \cong K_{p,q}$ $(p+q=n)$.

Remark 3.1. *For a connected bipartite graph G with $n \geq 2$ vertices and m edges, Zhou and Trinajstić [21] obtained that*

$$Kf^*(G) \geq m(2n-3) \tag{6}$$

with equality if and only if G is a complete bipartite graph. Furthermore, for connected bipartite graphs, the following inequality can be obtained from Theorem 3 of [3]:

$$0 < \frac{m\, t(G)}{\prod_{i=1}^{n} d_i} \leq 1.$$

From the above and (5), we conclude that

$$\begin{aligned} Kf^*(G) &\geq m(2n-3) - 2m \ln\left(\frac{m\, t(G)}{\prod_{i=1}^{n} d_i}\right) \\ &\geq m(2n-3). \end{aligned}$$

This implies that the lower bound (5) improves the lower bound (6).

Recall that the general Randić index of a graph G is one of the graph topological indices defined by $R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}$ (see [6]). The following lower bound was found in Theorem 3.2 of [5]

$$\gamma_2 \geq 1 + \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}.$$

Remark 3.2. *Notice that the lower bound (5) can be improved by taking $\alpha = 1 + \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}$ in Theorem 3.1.*

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