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Research Article

When the Cartesian product of directed cycles is hyper-Hamiltonian

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Abstract

Sufficient conditions for the Cartesian product of directed cycles to be hyper-Hamiltonian are given, and it is proved that only the Cartesian product of at least three directed cycles can be hyper-Hamiltonian.

Keywords: hyper-Hamiltonian graph; hypo-Hamiltonian graph; Cartesian product of directed cycles; Hamiltonian digraph.

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1. Introduction

A Hamiltonian (a non-Hamiltonian, respectively) graph G is *hyper-Hamiltonian* (*hypo-Hamiltonian*, respectively) if every vertex-deleted subgraph $G\backslash\{v\}$ is Hamiltonian. A Cartesian product $G = C_{n_1}\square C_{n_2}\square\cdots\square C_{n_k}$ of k directed cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ is the graph such that the vertex set $V(G)$ equals the Cartesian product $V(C_{n_1}) \times V(C_{n_2}) \times \cdots \times V(C_{n_k})$ and there is an arc in G from vertex $u = (u_1, u_2, \ldots, u_k)$ to vertex $v = (v_1, v_2, \ldots, v_k)$ if and only if there exists $1 \le r \le k$ such that there is an arc (u_r, v_r) in C_{n_r} and $u_i = v_i$ for all $i \ne r$. In this paper we study when the Cartesian product of directed cycles is hyper-Hamiltonian. Hence, throughout the rest of this paper C_{n_i} denotes a directed simple cycle of length $n_i \ge 2$. This topic has long related history in the literature.

In 1948, Rankin [7] gave an implicit necessary and sufficient condition for the existence of a Hamilton cycle in $C_{n_1}\square C_{n_2}$ based on embedding it on a torus. Later, Trotter and Erdős proved independently the result, that turned out to be a special case of Rankin's result, for the Cartesian product of two directed cycles as follows:

Theorem 1.1. [9] *The Cartesian product of two directed cycles $C_{n_1}\square C_{n_2}$ contains a Hamilton cycle if and only if there exist positive integers t_1 and t_2 such that $\gcd(n_1, n_2) = t_1 + t_2$ and $\gcd(n_1, t_1) = \gcd(n_2, t_2) = 1$.*

Subsequently, Thomassen [8] gave a sufficient condition for $C_{n_1}\square C_{n_2}$ to be hypo-Hamiltonian,

Theorem 1.2. [8] *The Cartesian product of two directed cycles $C_{n_1}\square C_{n_2}$ is hypo-Hamiltonian if there exists positive integer t such that $n_2 = t \cdot n_1 - 1$.*

In 1983, Penn and Witte extended Thomassen's result to a necessary and sufficient condition as follows:

Theorem 1.3. [6] *The Cartesian product of two directed cycles $C_{n_1}\square C_{n_2}$ is hypo-Hamiltonian if and only if there is a pair of relatively prime positive integers x and y with $xn_1 + yn_2 = n_1n_2 - 1$.*

In addition, Curran and Witte extended the result of Trotter and Erdős to the Cartesian product of k directed cycles for $k \ge 3$.

Theorem 1.4. [3] *For all $r \ge 3$ and all $n_1, n_2, \ldots, n_r > 1$, there is a Hamilton cycle in $C_{n_1}\square C_{n_2}\square\cdots\square C_{n_r}$.*

Finally, more focus in recent years has been on when the special families of graphs are hyper-Hamiltonian [1,2,4,5].

In Section 2, we leverage Theorem 1.3 and we prove that $C_{n_1}\square C_{n_2}$ is not hyper-Hamiltonian. In Section 3, we leverage Theorems 1.1, 1.2, 1.3, and 1.4, and we give the sufficient conditions for $C_{n_1}\square C_{n_2}\square\cdots\square C_{n_k}$ to be hyper-Hamiltonian if $k \ge 3$.

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2. Cartesian product of two directed cycles

In the following theorem we utilize the theory of torus knots, as was done in [6,7]. In particular, Penn and Witte [6] also used the torus knots approach in the proof of Theorem 1.3.

Theorem 2.1. *$C_{n_1}\Box C_{n_2}$ is not hyper-Hamiltonian.*

Proof. Suppose $G = C_{n_1}\Box C_{n_2}$ is hyper-Hamiltonian. Let C_G be a Hamilton cycle in G and C_H be a Hamilton cycle in $H = G\backslash\{v\}$. Let a_i be an arc in G induced by C_{n_i}, where $i \in \{1, 2\}$. Let $(s,t) \in Z \times Z$ be the knot class of C_H considered as a knot on the torus. Clearly, there must be sn_1 occurrences of a_1 and tn_2 occurrences of a_2 in C_H. Because the length of C_H is $n_1n_2 - 1$ then $sn_1 + tn_2 = n_1n_2 - 1$ must be satisfied. Hence, this excludes the case where n_1, n_2 are even. Furthermore, since (s,t) is the knot class of a circuit C_H, $\gcd(s,t) = 1$ must be satisfied too. So, by Theorem 1.3 G is hypo-Hamiltonian and C_G does not exist – a contradiction that proves this theorem. □

In addition, we obtain the following lemma that will be useful in Section 3.

Lemma 2.1. *Let $G = C_{n_1}\Box C_{n_2}$. $G\backslash\{v\}$ has a Hamilton cycle if and only if there is a pair of relatively prime positive integers s and t with $sn_1 + tn_2 = n_1n_2 - 1$.*

Proof. Based on Theorems 1.3 and 2.1, $G\backslash\{v\}$ has a Hamilton cycle if and only if it is hypo-Hamiltonian. Hence, this lemma follows directly from Theorem 1.3. □

3. Cartesian product of at least three directed cycles

First, we characterize when the Cartesian product of three directed cycles is hyper-Hamiltonian.

Theorem 3.1. *$C_{n_1}\Box C_{n_2}\Box C_{n_3}$ is hyper-Hamiltonian if there is a permutation (n'_1, n'_2, n'_3) of (n_1, n_2, n_3) and there exist positive integers t_1, t_2, t_3, t_4 so that $t_1+t_2 = \gcd(n'_2, n'_3)$, $\gcd(n'_2, t_1) = \gcd(n'_3, t_2) = 1$, $\gcd(t_3, t_4) = 1$, and $t_3\cdot n'_1+t_4\cdot n'_2\cdot n'_3 = n'_1\cdot n'_2\cdot n'_3-1$.*

Proof. If $t_1 + t_2 = \gcd(n'_2, n'_3)$ and $\gcd(n'_2, t_1) = \gcd(n'_3, t_2) = 1$ for some positive integers t_1, t_2 then according to Theorem 1.1 $C_{n'_2}\Box C_{n'_3}$ has a Hamilton cycle. Hence, $C_{n'_1}\Box C_{n'_2}\Box C_{n'_3}$ contains a spanning Cartesian product of directed cycles $H = C_{n'_1}\Box C_{n'_2\cdot n'_3}$. By Lemma 2.1 $H\backslash\{v\}$ has a Hamilton cycle if $\gcd(t_3, t_4) = 1$ with $t_3\cdot n'_1+t_4\cdot n'_2\cdot n'_3 = n'_1\cdot n'_2\cdot n'_3-1$. Furthermore, $C_{n'_1}\Box C_{n'_2}\Box C_{n'_3} \cong C_{n_1}\Box C_{n_2}\Box C_{n_3}$ and according to Theorem 1.4 $C_{n'_1}\Box C_{n'_2}\Box C_{n'_3}$ contains a Hamilton cycle, which completes the proof. □

By extension and similar approach, we can also give a sufficient condition for the Cartesian product of $k \geq 4$ directed cycles to be hyper-Hamiltonian as follows.

Theorem 3.2. *Let $G = C_{n_1}\Box C_{n_2}\Box\cdots\Box C_{n_k}$ be a Cartesian product of $k \geq 4$ directed cycles. Let j be a positive integer such that $2 \neq j < \frac{k}{2}$. G is hyper-Hamiltonian if there is a permutation $(n'_1, n'_2, \ldots, n'_k)$ of $(n_1, n_2, \ldots, n_k)$ and there exist positive integers t_1, t_2 that satisfy $\gcd(t_1, t_2) = 1$, and $t_1\cdot n'_1\cdot n'_2\cdots n'_j + t_2\cdot n'_{j+1}\cdot n'_{j+2}\cdots n'_k = n'_1\cdot n'_2\cdots n'_k - 1$.*

Proof. Since $j \neq 2$ and $j < \frac{k}{2}$ then according to Theorem 1.4 $C_{n'_1}\Box C_{n'_2}\Box\cdots\Box C_{n'_j}$ and $C_{n'_{j+1}}\Box C_{n'_{j+2}}\Box\cdots\Box C_{n'_k}$ are Hamiltonian. Hence, $G' = C_{n'_1}\Box C_{n'_2}\Box\cdots\Box C_{n'_k}$ contains a spanning Cartesian product of cycles $H = C_{n'_1\cdot n'_2\cdots n'_j}\Box C_{n'_{j+1}\cdot n'_{j+2}\cdots n'_k}$. By Lemma 2.1 $H\backslash\{v\}$ has a Hamilton cycle if $\gcd(t_1, t_2) = 1$ with $t_1\cdot n'_1\cdot n'_2\cdots n'_j + t_2\cdot n'_{j+1}\cdot n'_{j+2}\cdots n'_k = n'_1\cdot n'_2\cdots n'_k - 1$. Furthermore, $G' \cong G$ and according to Theorem 1.4 G' contains a Hamilton cycle, which completes the proof. □

While Theorems 3.1 and 3.2 give sufficient conditions for the Cartesian product of directed cycles G to be hyper-Hamiltonian based on computing relatively prime integers, which would require executing Euclid's algorithm in nontrivial cases, the following special case of Theorem 3.2 can determine if G is hyper-Hamiltonian more efficiently.

Corollary 3.1. *Let $G = C_{n_1}\Box C_{n_2}\Box\cdots\Box C_{n_k}$ be a Cartesian product of $k \geq 4$ directed cycles, where $n_1 = 2$ and $n_2, n_3, \ldots, n_k$ odd. Then G is hyper-Hamiltonian.*

Proof. By the same argument as in proof of Theorem 3.2, G contains a spanning Cartesian product of two directed cycles $H = C_2\Box C_{n_2\cdot n_3\cdots n_k}$. Furthermore, there is a positive integer t such that $2t - 1 = n_2\cdot n_3\cdots n_k$. By Theorem 1.2 H is hypo-Hamiltonian. So, $G\backslash\{v\}$ has a Hamilton cycle. Hence, according to Theorem 1.4 G is hyper-Hamiltonian. □

Finally, consider special cases when for $k \geq 4$ there exist positive integers t_1 and t_2 such that $\gcd(n'_1, n'_2) = t_1 + t_2$ and $\gcd(n'_1, t_1) = \gcd(n'_2, t_2) = 1$, where $(n'_1, n'_2, \ldots, n'_k)$ is a permutation of $(n_1, n_2, \ldots, n_k)$. First, for $k = 4$.

Theorem 3.3. *$C_{n_1} \square C_{n_2} \square C_{n_3} \square C_{n_4}$ is hyper-Hamiltonian if there is a permutation (n'_1, n'_2, n'_3, n'_4) of (n_1, n_2, n_3, n_4) and there exist positive integers $t_1, t_2, t_3,$ t_4, t_5, t_6 so that $t_1 + t_2 = \gcd(n'_1, n'_2)$, $\gcd(n'_1, t_1) = \gcd(n'_2, t_2) = 1$, $t_3 + t_4 = \gcd(n'_3, n'_4)$, $\gcd(n'_3, t_3) = \gcd(n'_4, t_4) = 1$, $\gcd(t_5, t_6) = 1$, and $t_5 \cdot n'_1 \cdot n'_2 + t_6 \cdot n'_3 \cdot n'_4 = n'_1 \cdot n'_2 \cdot n'_3 \cdot n'_4 - 1$.*

Proof. If $t_1 + t_2 = \gcd(n'_1, n'_2)$, $\gcd(n'_1, t_1) = \gcd(n'_2, t_2) = 1$, $t_3 + t_4 = \gcd(n'_3, n'_4)$, $\gcd(n'_3, t_3) = \gcd(n'_4, t_4) = 1$ for some positive integers t_1, t_2, t_3, t_4 then according to Theorem 1.1 $C_{n'_1} \square C_{n'_2}$ and $C_{n'_3} \square C_{n'_4}$ have Hamilton cycles. Hence, $G = C_{n'_1} \square C_{n'_2} \square C_{n'_3} \square C_{n'_4}$ contains a spanning Cartesian product of directed cycles $H = C_{n'_1 \cdot n'_2} \square C_{n'_3 \cdot n'_4}$. By Lemma 2.1 $H \backslash \{v\}$ has a Hamilton cycle if $\gcd(t_5, t_6) = 1$ with $t_5 \cdot n'_1 \cdot n'_2 + t_6 \cdot n'_3 \cdot n'_4 = n'_1 \cdot n'_2 \cdot n'_3 \cdot n'_4 - 1$. Furthermore, $G \cong C_{n_1} \square C_{n_2} \square C_{n_3} \square C_{n_4}$ and according to Theorem 1.4 G contains a Hamilton cycle, which completes the proof.

□

Now, for $k > 4$, one has the following theorem.

Theorem 3.4. *Let $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ be a Cartesian product of $k \geq 5$ directed cycles. G is hyper-Hamiltonian if there is a permutation $(n'_1, n'_2, \cdots, n'_k)$ of $(n_1, n_2, \cdots, n_k)$ and there exist positive integers t_1, t_2, t_3, t_4 so that $t_1 + t_2 = \gcd(n'_1, n'_2)$, $\gcd(n'_1, t_1) = \gcd(n'_2, t_2) = 1$, $\gcd(t_3, t_4) = 1$, and $t_3 \cdot n'_1 \cdot n'_2 + t_4 \cdot n'_3 \cdot n'_4 \cdots n'_k = n'_1 \cdot n'_2 \cdots n'_k - 1$.*

Proof. If $t_1 + t_2 = \gcd(n'_1, n'_2)$ and $\gcd(n'_1, t_1) = \gcd(n'_2, t_2) = 1$ for some positive integers t_1, t_2 then according to Theorem 1.1 $C_{n'_1} \square C_{n'_2}$ has a Hamilton cycle. Furthermore, since $k \geq 5$ then according to Theorem 1.4 $C_{n'_3} \square C_{n'_4} \square \cdots \square C_{n'_k}$ also has a Hamilton cycle. Hence, $G' = C_{n'_1} \square C_{n'_2} \square \cdots \square C_{n'_k}$ contains a spanning Cartesian product of directed cycles $H = C_{n'_1 \cdot n'_2} \square C_{n'_3 \cdot n'_4 \cdots n'_k}$. By Lemma 2.1 $H \backslash \{v\}$ has a Hamilton cycle if $\gcd(t_3, t_4) = 1$ with $t_3 \cdot n'_1 \cdot n'_2 + t_4 \cdot n'_3 \cdot n'_4 \cdots n'_k = n'_1 \cdot n'_2 \cdots n'_k - 1$. Furthermore, $G' \cong C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ and according to Theorem 1.4 G contains a Hamilton cycle, which completes the proof.

□

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