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*Research Article*

# A note on rainbow mean indexes of paths

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**Abstract**

For an edge coloring $c$ of a connected graph $G$ of order 3 or more with positive integers, the chromatic mean of a vertex $v$ of $G$ is defined as that vertex color which is the average of the colors of the edges incident with $v$. Only those edge colorings $c$ for which the chromatic mean of every vertex is a positive integer are considered. If distinct vertices have distinct chromatic means, then $c$ is called a rainbow mean coloring of $G$. The maximum vertex color in a rainbow mean coloring $c$ of $G$ is the rainbow mean index of $c$, while the rainbow mean index of $G$ is the minimum rainbow mean index among all rainbow mean colorings of $G$. In this note, we prove that every path $P_n$ of order $n \geq 3$ has rainbow mean index $n$ except $P_4$ which has rainbow mean index 5.

**Keywords:** chromatic mean; rainbow mean colorings; rainbow mean index; path.

**2020 Mathematics Subject Classification:** 05C07, 05C15, 05C78.

## 1. Introduction

An edge coloring $c$ of a connected graph $G$ of order 3 or more with positive integers is called a *mean coloring* of $G$ if the *chromatic mean* $\mathrm{cm}(v)$ of each vertex $v$ of $G$, defined by

$$\mathrm{cm}(v) = \frac{\sum_{e \in E_v} c(e)}{\deg v}, \text{ where } E_v \text{ is the set of edges incident with } v,$$

is an integer. If distinct vertices have distinct chromatic means, then the edge coloring $c$ is called a *rainbow mean coloring* of $G$. This concept was introduced and studied in [2] and more information on this topic has been presented in [1, 3]. It was shown in [2] that every connected graph of order 3 or more has a rainbow mean coloring.

For a rainbow mean coloring $c$ of a connected graph $G$ of order 3 or more, the maximum vertex color is the *rainbow chromatic mean index* (or simply, the *rainbow mean index*) $\mathrm{rm}(c)$ of $c$. That is,

$$\mathrm{rm}(c) = \max\{\mathrm{cm}(v) : \ v \in V(G)\}.$$

The *rainbow chromatic mean index* (or the *rainbow mean index*) $\mathrm{rm}(G)$ of the graph $G$ itself is defined as

$$\mathrm{rm}(G) = \min\{\mathrm{rm}(c) : \ c \text{ is a rainbow mean coloring of } G\}.$$

Consequently, if $G$ is a connected graph of order $n \geq 3$, then $\mathrm{rm}(G) \geq n$.

It was stated in [2] that every path $P_n$ of order $n \geq 3$ has rainbow mean index $n$ except for $P_4$ which has rainbow mean index 5. Because of the page limit of the journal, the proof of this fact was not included in [2]. The proof is given here because one of the authors recently received a note for review in which the main result of the present note was erroneously disproved. Therefore, we write this note to provide a proof of this fact.

## 2. Main result

First, we present the rainbow mean index of $P_4$, which appears in [2]. We include a proof of this result here for completeness.

**Proposition 2.1.** $\mathrm{rm}(P_4) = 5$.

*Proof.* The edge coloring in Figure 1 shows that $\mathrm{rm}(P_4) \leq 5$.

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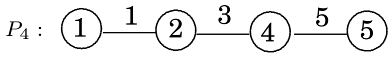


Figure 1: A rainbow mean coloring of $P_4$.

Next, we show that $\mathrm{rm}(P_4) \geq 5$. Assume, to the contrary, that there is a rainbow mean coloring $c$ of $P_4$ such that $\mathrm{rm}(c) = 4$. Let $P_4 = (v_1, v_2, v_3, v_4)$. Since $\{\mathrm{cm}(v_i) : 1 \leq i \leq 4\} = [4]$, no two edges can be colored the same. Consequently, since only one vertex is colored 1, this implies that $\mathrm{cm}(v_1) = 1$ or $\mathrm{cm}(v_4) = 1$. We may assume that $\mathrm{cm}(v_1) = 1$ and so $c(v_1v_2) = 1$. Hence, the edges of $P_4$ are colored with distinct odd integers. If some edge of $P_4$ is colored 7 or more, then some vertex of $P_4$ is colored 5 or more, which is impossible. Consequently, $\{c(v_iv_{i+1}) : i = 1, 2, 3\} = \{1, 3, 5\}$ and so $\{c(v_2v_3), c(v_3v_4)\} = \{3, 5\}$. In either case, it follows that $\{\mathrm{cm}(v_i) : 1 \leq i \leq 4\} \neq [4]$, which is a contradiction. Therefore, $\mathrm{rm}(P_4) \geq 5$ and so $\mathrm{rm}(P_4) = 5$. □

By Proposition 2.1, if $n = 4$, then $\mathrm{rm}(P_n) = n + 1$. Next, we show that $P_4$ is the only exception for all paths $P_n$ of order $n \geq 3$.

**Theorem 2.1.** *For each integer $n \geq 3$ and $n \neq 4$, $\mathrm{rm}(P_n) = n$.*

*Proof.* Since $\mathrm{rm}(P_n) \geq n$ for all integers $n \geq 3$, it remains to show that there is a rainbow mean coloring $c$ of $P_n$ such that $\mathrm{rm}(c) = n$. Let $P_n = (v_1, v_2, \ldots, v_n)$ and let $e_i = v_iv_{i+1}$ for $1 \leq i \leq n-1$. First, suppose that $n \geq 3$ is odd. Define the edge coloring $c : E(P_n) \to [n]$ of $P_n$ by $c(e) = i$ if $e$ is incident with $v_i$ where $1 \leq i \leq n$ and $i$ is odd. Figure 2 shows such an edge coloring of $P_n$ for $n = 3, 5, 7$. Since $\mathrm{cm}(v_i) = i$ for $1 \leq i \leq n$, it follows that $c$ is a rainbow mean coloring of $P_n$ with $\mathrm{rm}(c) = n$. Therefore, $\mathrm{rm}(P_n) = n$ for each odd integer $n \geq 3$.

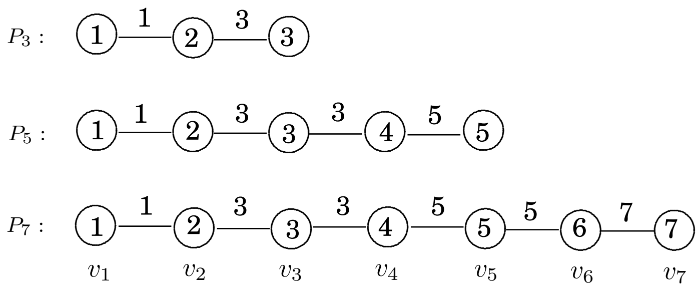


Figure 2: Rainbow mean colorings of $P_3$, $P_5$, and $P_7$.

We may therefore assume that $n \geq 6$ is even. Since $n \geq 6$ is even, it follows that either $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. We proceed by induction to prove the following statements.

- If $n \equiv 2 \pmod 4$, then there exists a rainbow mean coloring $c_n$ of $P_n$ such that $c_n(e_{n-1}) = 3$ and $\mathrm{rm}(c_n) = n$.
- If $n \equiv 0 \pmod 4$ and $n \geq 8$, then there exists a rainbow mean coloring $c_n$ of $P_n$ such that $c_n(e_{n-1}) = 5$ and $\mathrm{rm}(c_n) = n$.

The edge colorings of $P_6$ and $P_8$ in Figure 3 show that the statements are true for $n = 6, 8$. Suppose that the statement is true for an arbitrary even integer $n \geq 6$. Next, we show that the statement is true for $n + 4$ by considering two cases, according to whether $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. We use $\mathrm{cm}_t(v)$ to denote the chromatic mean of a vertex $v$ with respect to an edge coloring $c_t$ of the path $P_t$ of order $t$.

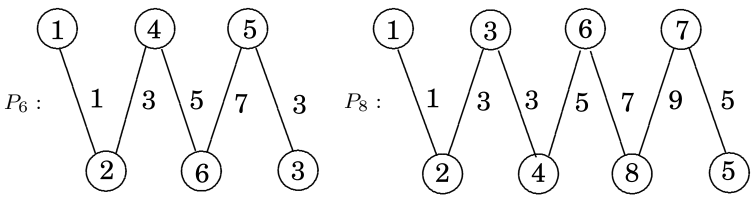


Figure 3: Rainbow mean colorings of $P_6$ and $P_8$.

*Case* 1. $n \equiv 2 \pmod 4$. By the induction hypothesis, there is a rainbow mean coloring $c_n$ of $P_n$ such that $c_n(e_{n-1}) = \mathrm{cm}_n(v_n) = 3$ and $\{\mathrm{cm}_n(v_i) : 1 \leq i \leq n\} = [n]$. We now extend $c_n$ to an edge coloring $c_{n+4}$ of $P_{n+4}$ by defining $c_{n+4}(e_n) = 2n+1$, $c_{n+4}(e_{n+1}) = 1$, $c_{n+4}(e_{n+2}) = 2n+5$, and $c_{n+4}(e_{n+3}) = 3$. Then $\mathrm{cm}_{n+4}(v_i) = \mathrm{cm}_n(v_i)$ for $1 \leq i \leq n-1$ and $\mathrm{cm}_{n+4}(v_n) = n+2$, $\mathrm{cm}_{n+4}(v_{n+1}) = n+1$, $\mathrm{cm}_{n+4}(v_{n+2}) = n+3$, $\mathrm{cm}_{n+4}(v_{n+3}) = n+4$, and $\mathrm{cm}_{n+4}(v_{n+4}) = 3$. It follows that

$$\{\text{cm}_{n+4}(v_i) : 1 \le i \le n+4\} = [n+4].$$

Figure 4 illustrates the construction of such an edge coloring for $n = 6$, where a rainbow mean coloring $c_{10}$ of $P_{10}$ is constructed from the given rainbow mean coloring $c_6$ of $P_6$.

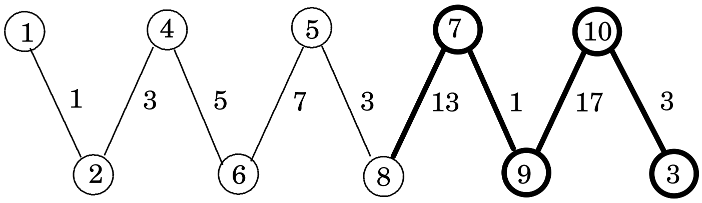


Figure 4: The construction of the rainbow mean coloring $c_{10}$ of $P_{10}$ in Case 1 in the proof of Theorem 2.1.

*Case* 2. $n \equiv 0 \pmod 4$ *and* $n \ge 8$. By the induction hypothesis, there is a rainbow mean coloring $c_n$ of $P_n$ such that $c_n(e_{n-1}) = \text{cm}_n(v_n) = 5$ and $\{\text{cm}_n(v_i) : 1 \le i \le n\} = [n]$. We now extend $c_n$ to an edge coloring $c_{n+4}$ of $P_{n+4}$ by defining $c_{n+4}(e_n) = 2n-3$, $c_{n+4}(e_{n+1}) = 7$, $c_{n+4}(e_{n+2}) = 2n+1$, and $c_{n+4}(e_{n+3}) = 5$. Then $\text{cm}_{n+4}(v_i) = \text{cm}_n(v_i)$ for $1 \le i \le n-1$ and $\text{cm}_{n+4}(v_n) = n+1$, $\text{cm}_{n+4}(v_{n+1}) = n+2$, $\text{cm}_{n+4}(v_{n+2}) = n+4$, $\text{cm}_{n+4}(v_{n+3}) = n+3$, and $\text{cm}_{n+4}(v_{n+4}) = 5$. Thus, $\{\text{cm}_{n+4}(v_i) : 1 \le i \le n+4\} = [n+4]$. Figure 5 illustrates the construction of such an edge coloring for $n = 8$, where a rainbow mean coloring $c_{12}$ of $P_{12}$ is constructed from the given rainbow mean coloring $c_8$ of $P_8$. □

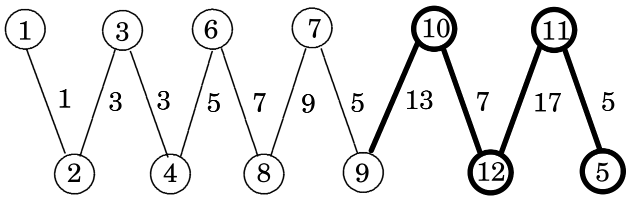


Figure 5: The construction of a rainbow mean coloring $c_{12}$ of $P_{12}$ in Case 2 in the proof of Theorem 2.1.

By Proposition 2.1 and Theorem 2.1, we have the following result.

**Corollary 2.1.** *For each integer* $n \ge 3$,

$$\text{rm}(P_n) = \begin{cases} n+1 & \text{if } n = 4 \\ n & \text{if } n \ne 4. \end{cases}$$

## References


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