Research Article

Binomial tribonacci sums

Kunle Adegoke1,*, Robert Frontczak2, Taras Goy3

1Department of Physics and Engineering Physics, Obafemi Awolowo University, Ile-Ife, Nigeria
2Landesbank Baden-Württemberg, Stuttgart, Germany
3Faculty of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine

(Received: 20 August 2021. Received in revised form: 28 September 2021. Accepted: 6 October 2021. Published online: 13 October 2021.)

© 2021 the authors. This is an open access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

We derive expressions for several binomials sums involving a generalized tribonacci sequence. We also study double binomial sums involving this sequence. Several explicit examples involving tribonacci and tribonacci–Lucas numbers are stated to highlight the results.

Keywords: generalized tribonacci sequence; tribonacci number; tribonacci–Lucas number; binomial transform.

2020 Mathematics Subject Classification: 11B37, 11B39.

1. Introduction

There is a dearth of tribonacci summation identities including binomial coefficients. Our goal in this paper is to derive several new binomial tribonacci sums such as

\[
\sum_{k=0}^{n} \binom{n}{k} G_{4k+s} = 2^n G_{3n+s},
\]

\[
\sum_{k=1}^{n} \binom{n}{k} G_{4k+s} = \sum_{m=1}^{n} \frac{2^m G_{3m+s} - G_s}{m},
\]

\[
\sum_{k=0}^{[3n/2]} \binom{3n}{2k} G_{2k+s} = 2^{n-1} \left( G_{4n+s} + (-1)^n G_{s-2n} \right),
\]

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \frac{4^k}{n+k} G_{4n+2k+s} = \frac{G_{8n+s} + G_s}{2n},
\]

and double binomial tribonacci summation identities such as

\[
\sum_{k=0}^{n} (-1)^{k+p} \binom{n}{k} \binom{5k+p+s}{k} G_{5k+p+s} = 3^n G_{3n+s},
\]

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{k}{p} \frac{G_{k+5p+s}}{3^k} = \left( \frac{7}{3} \right)^n G_{3n+s}.
\]

In the above identities, \( n \) denotes a non-negative integer, \( s \) and \( p \) are arbitrary integers and \( G_n \) is a generalized tribonacci number.

The generalized tribonacci sequence \( G_n = G_n(c_0, c_1, c_2) \), \( n \geq 0 \), is defined recursively by

\[
G_n = G_{n-1} + G_{n-2} + G_{n-3}, \quad n \geq 3,
\]

with initial values \( G_0 = c_0, G_1 = c_1, G_2 = c_2 \) not all being zero. Extension of the definition of \( G_n \) to negative subscripts is provided by writing the recurrence relation as

\[
G_{-n} = G_{-(n-3)} - G_{-(n-2)} - G_{-(n-1)},
\]

so that \( G_n \) is defined for all integers \( n \).

The most prominent representatives of \( G_n \) and widely studied in the literature are \( G_n(0, 1, 1) = T_n \) the sequence of tribonacci numbers and \( G_n(3, 1, 3) = K_n \) the sequence of tribonacci-Lucas numbers (sequences A000073 and A001644 in [19], respectively).

The first few tribonacci numbers and tribonacci-Lucas numbers with positive and negative subscripts are given in Table 1.

*Corresponding author (adegoke00@gmail.com).
Table 1: Tribonacci and tribonacci–Lucas numbers.

<table>
<thead>
<tr>
<th>n</th>
<th>T_n</th>
<th>T_{-n}</th>
<th>K_n</th>
<th>K_{-n}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0</td>
<td>21</td>
<td>44</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>2</td>
<td>44</td>
<td>92</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>-3</td>
<td>71</td>
<td>81</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>1</td>
<td>131</td>
<td>149</td>
</tr>
<tr>
<td>8</td>
<td>24</td>
<td>4</td>
<td>241</td>
<td>504</td>
</tr>
<tr>
<td>9</td>
<td>44</td>
<td>-8</td>
<td>443</td>
<td>504</td>
</tr>
<tr>
<td>10</td>
<td>81</td>
<td>5</td>
<td>43</td>
<td>274</td>
</tr>
<tr>
<td>11</td>
<td>149</td>
<td>7</td>
<td>149</td>
<td>274</td>
</tr>
<tr>
<td>12</td>
<td>274</td>
<td>-20</td>
<td>149</td>
<td>274</td>
</tr>
<tr>
<td>13</td>
<td>504</td>
<td>18</td>
<td>274</td>
<td>504</td>
</tr>
<tr>
<td>14</td>
<td>927</td>
<td>9</td>
<td>274</td>
<td>927</td>
</tr>
</tbody>
</table>

Properties of (generalized) tribonacci sequences were investigated in the recent articles [1–4, 7, 8, 10, 12–18, 20, 21], among others. For instance, Janjić [16] found the remarkable combinatorial identity

$$T_n = 1 + \sum_{k=1}^{n-1} \sum_{i=0}^{n-k} \binom{k}{i} \binom{j-i-1}{n-k-2j}.$$  

A generalized tribonacci number $G_n(c_0, c_1, c_2)$ is given by the Binet formula

$$G_n(c_0, c_1, c_2) = A\alpha^n + B\beta^n + C\gamma^n,$$  \hspace{1cm} (1)

where $\alpha$, $\beta$ and $\gamma$ are the distinct roots of the equation $x^3 - x^2 - x - 1 = 0$. The coefficients $A$, $B$ and $C$ depend on the initial values and are determined by the system

\[
\begin{align*}
A + B + C &= c_0, \\
A\alpha + B\beta + C\gamma &= c_1, \\
A\alpha^2 + B\beta^2 + C\gamma^2 &= c_2.
\end{align*}
\]

The Binet formulas for $T_n$ and $K_n$ are

$$T_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n,$$

where

$$\alpha = 1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}, \quad \beta = 1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}},$$

$$\gamma = 1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}},$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

Tribonacci and tribonacci-Lucas numbers with negative indices can be accessed directly, using the following result.

**Lemma 1.1.** For integer $n$,

$$T_{-n} = T_{n-1}^2 - T_{n-2}T_n,$$  \hspace{1cm} (2)

$$K_{-n} = \frac{K_n^2 - K_{2n}}{2}.$$  \hspace{1cm} (3)

For a proof of (2), see, for example, [8, Theorem 2.2]. The proof of (3) one can find in [6, Formula (9)].

In this article, we study binomial and double binomial sums with terms being a generalized tribonacci sequence. We derive closed forms for several such sums. We also prove a general binomial identity characterizing $G_{an+b}$ for $a \geq 1$ and $b$ an arbitrary integer.

### 2. Some auxiliary results

In this section we present some results that we will use in the sequel.

**Lemma 2.1.** Let $\phi \in \{\alpha, \beta, \gamma\}$. Then, for all $n \geq 0$, we have

$$\phi^{n+1} = \phi^2 T_n + \phi(T_{n-1} + T_{n-2}) + T_{n-1}.$$  \hspace{1cm} (4)

For a proof of (4), see [7, Formula (6)].
Lemma 2.2. We have

\[
\begin{align*}
(\alpha - 1)^3 &= 2\alpha^{-2}, \\
(\alpha + 1)^3 &= 2\alpha^4, \\
(\alpha^2 + 1)^3 &= 4\alpha^5, \\
(\alpha^3 - 1)^3 &= 2\alpha^7, \\
\alpha^4 + 1 &= 2\alpha^3,
\end{align*}
\]

with identical relations for \(\beta\) and \(\gamma\).

Proof. Since
\[
1 + \alpha + \alpha^2 = \alpha^3,
\]
we have
\[
\frac{\alpha^2 + 1}{\alpha^2 - 1} = \alpha
\]
and
\[
\frac{\alpha + 1}{\alpha - 1} = \alpha^2.
\]
Addition of (11) and (12) gives
\[
(\alpha + 1)^2(\alpha - 1) = 2\alpha^2,
\]
while their subtraction produces
\[
(\alpha - 1)^2(\alpha + 1) = 2.
\]
Eliminating \(\alpha + 1\) between (13) and (14) gives identity (5), while the elimination of \(\alpha - 1\) yields (6).

Cubing identity \(\alpha^2 + 1 = 2\alpha\alpha^{-1}\) and making use of (5) gives (7). Subtracting (10) from \(\alpha + \alpha^2 + \alpha^3 = \alpha^4\) produces identity (8). Identity (9) follows from \(\alpha^4 + 1 = \alpha^4 + \alpha^3 + \alpha^2 + \alpha = (\alpha^2 + 1)(\alpha + 1)\) with the help of (6) and (7). \(\square\)

Lemma 2.3. Let \(a, b, c\) and \(d\) be rational numbers and \(\lambda\) an irrational number. Then
\[
a + \lambda b = c + \lambda d \iff a = c, \quad b = d.
\]

3. Identities from the binomial theorem and binomial transform

The next lemma will be the key ingredient to derive many results in this paper. For a proof and some applications to Horadam numbers, see [11].

Lemma 3.1. Let \(n\) and \(j\) be integers with \(0 \leq j \leq n\). Then, for each \(x, y \in \mathbb{C}\), we have
\[
\sum_{k=j}^{n} \binom{n}{k} (\pm 1)^{k-j} \binom{k}{j} y^{k-n-k} x^{n-k} = \binom{n}{j} y^j (x \pm y)^{n-j}.
\]

We also mention the standard fact about sequences and their binomial transforms [5]: Let \((a_n)_{n \geq 0}\) be a sequence of numbers and \((b_n)_{n \geq 0}\) be its binomial transform. Then we have the following relations:
\[
b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \iff a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k.
\]

Furthermore, if \(a_0 = 0\) (so that \(b_0 = 0\) too) the binomial pair exhibits the following properties:
\[
\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k} = \frac{b_n}{m}
\]
and
\[
\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k+1} = \frac{1}{n+1} \sum_{m=1}^{n} b_m.
\]

Theorem 3.1. Let \(j\) and \(s\) be integers such that \(s\) is arbitrary and \(j \geq 0\). Then
\[
\sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} G_{4k+s} = \binom{n}{j} 2^{n-j} G_{3n+j+s}.
\]
Proof. Use identity (9) in Lemma 3.1 with \( x = 1 \) and \( y = \alpha^4 \), taking note of Lemma 2.3.

Corollary 3.1. For \( n \) a non-negative integer and \( s \) any integer,

\[
\sum_{k=0}^{n} \binom{n}{k} G_{4k+s} = 2^n G_{3n+s},
\]

(19)

\[
\sum_{k=0}^{n} \binom{n}{k} (-2)^k G_{3k+s} = (-1)^n G_{4n+s},
\]

(20)

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{G_{4k+s}}{k} = \sum_{m=1}^{n} \frac{2^m G_{3m+s} - G_s}{m}
\]

(21)

and

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{G_{4k+s}}{k+1} = \frac{1}{n+1} \left( \sum_{m=1}^{n} 2^m G_{3m+s} - nG_s \right).
\]

(22)

Proof. To obtain (19) set \( j = 0 \) in (18). Identities (20), (21) and (22) follow from (15), (16) and (17), respectively.

From (19) and (20) we immediately obtain the following binomial tribonacci and tribonacci–Lucas relations.

Corollary 3.2. For \( n \geq 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} T_{4k} = 2^n T_{3n}, \quad \sum_{k=0}^{n} \binom{n}{k} K_{4k} = 2^n K_{3n},
\]

\[
\sum_{k=0}^{n} \binom{n}{k} T_{4k-3n+1} = 2^n, \quad \sum_{k=0}^{n} \binom{n}{k} K_{4k-3n+1} = 2^n,
\]

\[
\sum_{k=0}^{n} \binom{n}{k} T_{4k-3n} = 0, \quad \sum_{k=0}^{n} \binom{n}{k} K_{4k-3n} = 3 \cdot 2^n.
\]

Theorem 3.2. For non-negative integer \( n \), any integer \( s \) we have

\[
\sum_{k=0}^{3n} \delta^k \binom{3n}{k} G_{pk+s} = \delta^n 2^n G_{rn+s},
\]

where the values of \( \delta, p, q \) and \( r \) as given in each column in Table 2.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( -1 )</th>
<th>( 1 )</th>
<th>( 1 )</th>
<th>( -1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
<td>( 3 )</td>
</tr>
<tr>
<td>( q )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( r )</td>
<td>( -2 )</td>
<td>( 4 )</td>
<td>( 5 )</td>
<td>( 7 )</td>
</tr>
</tbody>
</table>

Table 2: Values of \( \delta, p, q \) and \( r \) from Theorem 3.2.

Proof. Each of the identities (5)–(8) can be written as \((\alpha^p + \delta)^3 = 2^q \alpha^r\), where the values of \( \delta, p, q \) and \( r \) in each case are as given in each column in Table 2. The identity of the theorem then follows from the binomial theorem and Lemma 2.3.

Lemma 3.2. For non-negative integer \( n \) and real or complex \( z \),

\[
2 \sum_{k=0}^{[3n/2]} \binom{3n}{2k} z^{2k} = (1 + z)^{3n} + (1 - z)^{3n},
\]

\[
2 \sum_{k=1}^{[3n/2]} \binom{3n}{2k-1} z^{2k-1} = (1 + z)^{3n} - (1 - z)^{3n}.
\]

Theorem 3.3. For non-negative integer \( n \) and any integer \( s \),

\[
\sum_{k=0}^{[3n/2]} \binom{3n}{2k} G_{2k+s} = 2^{n-1} \left( G_{4n+s} + (-1)^n G_{s-2n} \right),
\]

\[
\sum_{k=1}^{[3n/2]} \binom{3n}{2k-1} G_{2k+s-1} = 2^{n-1} \left( G_{4n+s} - (-1)^n G_{s-2n} \right).
\]
Proof. Set \( z = \alpha \) in Lemma 3.2, make use of identities (5) and (6), noting Lemma 2.3 with \( \lambda = \alpha \).

Setting \( s = 0 \) in Theorem 3.3, we immediately obtain the following.

**Corollary 3.3.** For non-negative integer \( n \),

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} T_{2k} = 2^{n-1} \left( T_{4n} + (-1)^n \left( T_{2n-1}^2 - T_{2n-2} T_{2n} \right) \right),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} T_{2k-1} = 2^{n-1} \left( T_{4n} - (-1)^n \left( T_{2n-1}^2 - T_{2n-2} T_{2n} \right) \right)
\]

and

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} K_{2k} = 2^{n-2} \left( 2K_{4n} + (-1)^n \left( K_{2n}^2 - K_{4n} \right) \right),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} K_{2k-1} = 2^{n-2} \left( 2K_{4n} - (-1)^n \left( K_{2n}^2 - K_{4n} \right) \right).
\]

**Theorem 3.4.** For non-negative integer \( n \) and any integer \( s \),

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} G_{4k+s} = 2^{2n-1} \left( G_{5n+s} + (-1)^n G_{2n+s} \right),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} G_{4k+s-2} = 2^{2n-1} \left( G_{5n+s} - (-1)^n G_{2n+s} \right).
\]

Proof. Combining (5) with (6) yields

\[(\alpha^2 - 1)^3 = 4n^2.\]  \hspace{1cm} (23)

Now set \( z = \alpha^2 \) in Lemma 3.2 and make use of identities (7) and (23), noting Lemma 2.3 with \( \lambda = \alpha \).

**Theorem 3.5.** For non-negative integer \( n \) and any integer \( s \),

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} G_{8k+s} = 2^{3n-1} \left( G_{9n+s} + (-2)^n G_{7n+s} \right),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} G_{8k+s-4} = 2^{3n-1} \left( G_{9n+s} - (-2)^n G_{7n+s} \right).
\]

Proof. Combining (7) and (23) we have

\[(\alpha^4 - 1)^3 = 16\alpha^2.\]  \hspace{1cm} (24)

Set \( z = \alpha^4 \) in Lemma 3.2 and make use of identities (9) and (24), noting Lemma 2.3 with \( \lambda = \alpha \).
4. Identities from the Waring formulas

Our next result provides two combinatorial identities for generalized tribonacci numbers involving binomial coefficients.

**Lemma 4.1.** The following identities hold for \( n \geq 0 \) and real or complex \( x \) and \( y \):

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x-y} \tag{25}
\]

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \frac{n}{n-k} (xy)^k (x+y)^{n-2k} = x^n + y^n. \tag{26}
\]

Formulas (25) and (26) are well-known in combinatorics and called Waring (sometimes Girard-Waring) formulas. The proof of these formulas can be found, for example, in [9].

**Theorem 4.1.** Let \( n \) be a non-negative integer and \( s \) any integer. Then

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} (G_{3n-2k+s+4} - G_{3n-2k+s}) = \frac{G_{4n+s+4} - G_s}{2^n}
\]

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} \frac{G_{3n-2k+s}}{n-k} = \frac{G_{4n+s} + G_s}{2^n n}. \tag{27}
\]

**Proof.** Set \((x, y) = (1, \alpha^4)\) in (25) and (26), respectively, Lemma 4.1 and use identity (8) and Lemma 2.3.

**Corollary 4.1.** For \( n \geq 0 \),

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} (G_{n-2k+4} - G_{n-2k}) = \frac{G_{2n+4} - G_{-2n}}{2^n},
\]

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} G_{n-2k} = \frac{G_{2n} + G_{-2n}}{2^n n},
\]

In particular,

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} (T_{n-2k+4} - T_{n-2k}) = \frac{T_{2n+4} - T_{2n-1}^2 + T_{2n-2} T_{2n}}{2^n},
\]

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} (K_{n-2k+4} - K_{n-2k}) = \frac{2K_{2n+4} - K_{2n}^2 + K_{4n}}{2^n n+1},
\]

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} \frac{T_{n-2k}}{n-k} = \frac{T_{2n-1}^2 + T_{2n}(1-T_{2n-2})}{n2^n},
\]

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{4} \right)^k \binom{n-k}{k} \frac{K_{n-2k}}{n-k} = \frac{K_{2n}^2 + 2K_{2n} - K_{4n}}{n2^n+1}.
\]

5. Double binomial tribonacci sums

**Theorem 5.1.** Let \( n, j \) and \( s \) be integers with \( s \) arbitrary and \( j \geq 0 \). Then,

\[
\sum_{k=j}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{k}{j} \binom{n}{k} G_{5k+p+s} = 3^{n-j} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} G_{3n+2j+p+s}. \tag{27}
\]

**Proof.** The identity can be derived from Lemma 3.1 using \( 3\phi^3 = \phi^6 - \phi^5 + 1 \).

**Corollary 5.1.** Let \( n \) and \( s \) be integers. Then,

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{n}{k} \binom{k}{p} G_{5k+p+s} = 3^n G_{3n+s},
\]

\[
\sum_{k=1}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{n}{k} \binom{k}{p} kG_{5k+p+s} = n3^{n-1}(G_{3n+s} + G_{3n+s+1}).
\]
Proof. Set $j = 0$ and $j = 1$ in (27), respectively.

**Theorem 5.2.** Let $j$ and $s$ be integers with $s$ arbitrary and $j \geq 0$. Then

\[
\sum_{k=p}^{n} \sum_{j=0}^{k} \binom{k}{j} \binom{n}{j} \frac{G_{k+4p+s}}{2^k} = 2^{n-j} \sum_{m=0}^{j} \binom{j}{m} G_{n-2j+4m+s}, \tag{28}
\]

\[
\sum_{k=j}^{n} \sum_{k=p}^{n} \binom{k}{j} \binom{n}{j} \frac{G_{k+5p+s}}{3^k} = 7^{n-j} \sum_{m=0}^{j} \binom{j}{m} G_{n-2j+5m+s}. \tag{29}
\]

Proof. Use Lemma 3.1 in conjunction with $4\phi^3 = \phi^5 + \phi + 2$ and $7\phi^3 = \phi^6 + \phi + 3$, respectively.

**Corollary 5.2.** Let $n$ and $s$ be integers. Then,

\[
\sum_{k=0}^{n} \sum_{k=p}^{n} \binom{n}{k} \binom{k}{j} \frac{G_{k+4p+s}}{2^k} = 2^n G_{n+s},
\]

\[
\sum_{k=0}^{n} \sum_{k=p}^{n} \binom{n}{k} \binom{k}{j} \frac{G_{k+4p+s}}{2^k} = 2^{n-2} n (G_{n+s+2} + G_{n+s-2})
\]

and

\[
\sum_{k=0}^{n} \sum_{k=p}^{n} \binom{n}{k} \binom{k}{j} \frac{G_{k+5p+s}}{3^k} = \left(\frac{7}{3}\right)^n G_{n+s},
\]

\[
7 \sum_{k=0}^{n} \sum_{k=p}^{n} \binom{n}{k} \binom{k}{j} \frac{G_{k+5p+s}}{3^k} = \left(\frac{7}{3}\right)^n n (G_{n+s+3} + G_{n+s-2}).
\]

Proof. Set $j = 0$ and $j = 1$ in (28) and (29), respectively.

6. A general binomial sum identity

**Theorem 6.1.** Let $j$, $s$ and $v$ be integers with $j, v \geq 0$, $v \neq 0$, $v \neq 1$. Then,

\[
\binom{n}{j} \sum_{m=0}^{j-m} (-1)^{j+m+q} \binom{j}{j-m} \left(\frac{T_v}{T_{v-1}}\right)^m G_{vn-j(v-1)+q+s} = \frac{T_{v-2}}{T_{v-1}} \sum_{k=j}^{n} \sum_{k=p}^{n} \sum_{w=0}^{k-p} (-1)^{k+w+p} \binom{k}{j} \binom{n}{k} \binom{k-p}{w} \left(\frac{T_{v-1}}{T_{v-2}}\right)^k \left(\frac{T_v}{T_{v-1}}\right)^p G_{k+w+s}.
\]

Proof. For $v \geq 1$ and $\phi = \alpha$ write (4) in the form

\[
\alpha^v = \alpha (T_v + T_{v-1} (\alpha - 1)) + T_{v-2}.
\]

Now, identify $x = \alpha (T_v + T_{v-1} (\alpha - 1))$ and $y = T_{v-2}$ and use Lemma 3.1 and the binomial theorem to get

\[
\sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} (-1)^k T_{v-2-k} \sum_{p=0}^{k} \binom{k-p}{w} \left(\frac{T_{v-1}}{T_{v-2}}\right)^{k-w} \alpha^{k+w} = \binom{n}{j} \sum_{m=0}^{j-m} (-1)^{j+m+q} \alpha^{j-(m+q)} G_{vn-j(v-1)+q+s}.
\]

Multiply both sides by $\alpha^s$ and combine the similar results for $\beta$ and $\gamma$ according to the Binet formula (1).

**Corollary 6.1.** We have

\[
\sum_{k=0}^{n-1} \sum_{k=p}^{n-1} (-1)^{k+w+p} \binom{n}{k} \binom{k-p}{w} \left(\frac{T_{v-1}}{T_{v-2}}\right)^{k-w} \left(\frac{T_v}{T_{v-1}}\right)^p G_{k+w+s} = \frac{T_n}{T_{v-2}} G_{n+s}.
\]

For $v = 1$ the left-hand side collapses and we end with $G_{n+s}$ on both sides of the equality sign. The special values for $v = 2$ and $v = 3$ are given by

\[
\sum_{k=0}^{n-2} (-1)^{w+p} \binom{n}{w} G_{n+w+s} = (-1)^n G_{2n+s}
\]

and

\[
\sum_{k=0}^{n-1} \sum_{k=p}^{n-1} (-1)^{w+p+k} \binom{n}{k} \binom{k-p}{w} 2^p G_{k+w+s} = G_{3n+s}.
\]
References