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Research Article

Plethystic exponential calculus and characteristic polynomials of permutations

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Abstract

We prove a family of identities that express generating functions of powers of characteristic polynomials of permutations as finite or infinite products. These identities generalize some existing formulae, first obtained in a study of the geometry/topology of symmetric products of real/algebraic tori. The proof uses formal power series expansions of plethystic exponentials and has been motivated by some recent applications of these combinatorial tools in supersymmetric gauge and string theories. Since the methods are elementary, we try to be self-contained, and relate to other topics such as the q-binomial theorem, and the cycle index and Molien series for the symmetric group.

Keywords: plethystic calculus; permutations; symmetric functions; product-sum identities; cycle index.

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1. Introduction

The study of infinite products is a classical theme and such objects arise in many parts of mathematics since the early works of Leonhard Euler. They are fundamental in the theory of holomorphic functions of a complex variable and some notable infinite products of simple fractions are generating functions for counting various types of combinatorial objects such as partitions of natural integers.

Power series expansions of infinite products are also ubiquitous and some have become famous such as Euler's pentagonal number theorem or Jacobi's triple product identity; these have a wide range of applications in arithmetic geometry, complex function theory, and number theory, *et cetera*. However, it is not so often that the power series expansion of a given infinite product can be written only in terms of elementary functions and concepts.

The purpose of this article is two-fold. Firstly, we want to provide a completely elementary proof of the following family of simple identities, parametrized by an integer $r \in \mathbb{Z}$. Let $\mathbb{Z}[\![x,y]\!]$ be the ring of formal power series in the variables x and y with integer coefficients. We denote by S_n the symmetric group on n letters, $n \in \mathbb{N}$.

Theorem 1.1. *Fix $r \in \mathbb{Z}$. Then, the following are equalities in $\mathbb{Z}[\![x,y]\!]$*

$$1 + \sum_{n\geq 1} \sum_{\sigma\in S_n} \frac{y^n}{n!} \det(I_n - xM_\sigma)^r = \prod_{k\geq 0} (1 - yx^k)^{-\binom{k-r-1}{k}}, \tag{1}$$

where M_σ is the $n \times n$ permutation matrix of $\sigma \in S_n$ and I_n is the identity matrix of the same size.

The identities (1) are actually valid in $\mathbb{R}[\![x,y]\!]$ (and also in $\mathbb{C}[\![x,y]\!]$, with appropriate use) upon letting r be an arbitrary real number, with the binomial numbers defined by $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$, for all $k \geq 0$ (and $\binom{0}{0} = 1$). In the important cases when r is a *non-negative integer*, the product on the right hand side of (1) becomes finite:

$$1 + \sum_{n\geq 1} \sum_{\sigma\in S_n} \frac{y^n}{n!} \det(I_n - xM_\sigma)^r = \prod_{k=0}^{r} (1 - yx^k)^{(-1)^{k+1}\binom{r}{k}}, \tag{2}$$

given that $\binom{k-r-1}{k} = (-1)^k\binom{r}{k}$, for $r \in \mathbb{N}$; and the series converges absolutely for $x, y \in \mathbb{C}$, with $|x| < 1$, $|y| < 2^{-r}$ (and uniformly in compact subsets).

For a given $\sigma \in S_n$, the polynomials $q_\sigma(x) := \det(I_n - xM_\sigma)$ will be called *characteristic polynomials of permutations* (the more common polynomials $\pm\det(I_n\lambda - M_\sigma)$ are obtained by an obvious substitution). Hence, the left hand side of the identities above are (ordinary) *generating functions for the average rth powers of characteristic polynomials of permutations*.

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The proof of Theorem 1.1 is presented in Section 3 using only a few simple and well-known interesting tools in combinatorics, borrowed from the theory of symmetric functions: the so-called plethystic exponential calculus. Hence, our second purpose is to provide a concise introduction, in Section 2, to the aspects of plethystic calculus showing up in the proof – the plethystic exponential in a ring of formal power series – to help making them better known. We believe that this exponential is bound to play an increasing role in many topics, as it is shown by several recent and diverse applications in so many areas of mathematics and mathematical physics (see, for example, [1,2,5,6]).

Actually, to obtain our main identities, we do not need the full power of plethysms from the theory of symmetric functions or lambda rings, and we only need to establish product-sum identities for certain plethystic exponentials. For $r \geq 1$, denote by $\mathsf{PE}(f(\underline{x}, y))$ the plethystic exponential of a formal power series f, with $\mathbb{R}$ coefficients, in $r+1$ variables $\underline{x} = (x_1, \cdots, x_r)$ and y, without constant term (see Definition 2.2 below). Let us use the multi-index notation for the $\underline{x}$ variables: $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{N}_0^r$, $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ and write $\mathbf{x}^{\mathbf{k}} = x_1^k \cdots x_r^k$.

Let $g(\underline{x}) = \sum_{\mathbf{k} \in \mathbb{N}_0^r} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{R}[\![\underline{x}]\!]$, so that $a_{\mathbf{k}} = a_{k_1,\cdots,k_r} \in \mathbb{R}$. The source of our product-sum identities, is the following evaluation of plethystic exponentials, both as an infinite product, and as a sum over *partitions* of the natural number n:

$$\mathsf{PE}\left(g(\underline{x})\, y\right) = \prod_{\mathbf{k} \in \mathbb{N}_0^r} (1 - y\,\mathbf{x}^{\mathbf{k}})^{-a_{\mathbf{k}}} = 1 + \sum_{n \geq 1} \left(\sum_{\lambda \vdash n} \prod_{j=1}^{n} \frac{g(\underline{x}^j)^{\lambda(j)}}{\lambda(j)!\, j^{\lambda(j)}} \right) y^n \in \mathbb{R}[\![\underline{x}, y]\!]. \tag{3}$$

Above, we write $\lambda \vdash n$ when λ is a partition of n, and $\lambda(j) \geq 0$ denotes the number of parts of λ of size equal to $j \in \{1, \cdots, n\}$. We also used the notation: $\underline{x}^j := (x_1^j, \cdots, x_r^j)$, for $j \in \mathbb{N}$.

We will apply Equation (3) to several polynomials/formal power series $g(\underline{x})$. By using appropriate rational functions in $r+s$ variables, now denoted $x_1, \cdots, x_r$ and $q_1, \cdots, q_s$ (given the interesting relations to q-series, see ğ3.2) we obtain the following, returning now to $\mathbb{Z}$ coefficients.

Theorem 1.2. *Fix $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. With the same notations as in Theorem 1.1, we have:*

$$1 + \sum_{n \geq 1} \frac{y^n}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{r} \prod_{k=1}^{s} \frac{\det(I_n - x_j M_\sigma)}{\det(I_n - q_k M_\sigma)} = \prod_{\mathbf{j} \in \{0,1\}^r} \prod_{\mathbf{k} \in \mathbb{N}_0^s} (1 - y\,\mathbf{x}^{\mathbf{j}}\,\mathbf{q}^{\mathbf{k}})^{(-1)^{|\mathbf{j}|+1}}, \tag{4}$$

as identities in $\mathbb{Z}[\![x_1, \cdots, x_r, q_1, \cdots, q_s, y]\!]$, with $|\mathbf{j}| = \sum_{i=1}^r j_i$.

Note that Theorem 1.1 follows from Theorem 1.2, by letting s or r equal zero, and identifying all the remaining variables. We now present a couple of interesting consequences of these identities. First note that, from Theorem 1.1, our basic generating function: $\Phi^r(x, y) := 1 + \sum_{n \geq 1} \sum_{\sigma \in S_n} \frac{y^n}{n!} \det(I_n - x M_\sigma)^r$, has always a factor $1/(1-y)$, corresponding to a pole of order 1 at $y = 1$. Its residue is again expressed as a product, and has the following enumerative interpretation.

Corollary 1.1. *For every $r \in \mathbb{Z}$, we have: $\mathrm{Res}_{y=1} \Phi^r(x, y) = -\prod_{k \geq 1} (1 - x^k)^{-\binom{k-r-1}{k}}$. In particular, when $c = -r > 0$, $\mathrm{Res}_{y=1} \Phi^{-c}(x, y)$ is the (the negative of) the generating function for partitions of a natural number n, with parts colored with c distinct colors.*

We also obtain, in Section 3, a recursion formula for the *average* r power of characteristic polynomials of permutations:

$$Q_n^r(x) := \frac{1}{n!} \sum_{\sigma \in S_n} \det(I_n - x M_\sigma)^r, \tag{5}$$

and, more generally, with $\mathbf{r} = (r_1, \cdots, r_s) \in \mathbb{R}^s$, for any formal power series of the form:

$$Q_n^{\mathbf{r}}(\underline{x}) := \frac{1}{n!} \sum_{\sigma \in S_n} \det(I_n - x_1 M_\sigma)^{r_1} \cdots \det(I_n - x_s M_\sigma)^{r_s} \quad \in \mathbb{R}[\![\underline{x}]\!].$$

Corollary 1.2. *Fix $\mathbf{r} = (r_1, \cdots, r_s) \in \mathbb{R}^s$. Then, as elements in $\mathbb{R}[\![\underline{x}]\!]$, we have: $Q_n^{\mathbf{r}}(\underline{x}) = \frac{1}{n} \sum_{k=1}^n h(\underline{x}^k)\, Q_{n-k}^{\mathbf{r}}(\underline{x})$, with $h(\underline{x}) := (1 - x_1)^{r_1} \cdots (1 - x_s)^{r_s}$.*

This recursion relation is an immediate consequence of the interpretation of such formal power series $Q_n^{\mathbf{r}}(\underline{x})$ as evaluations of the cycle index of the symmetric group S_n on $h(\underline{x}) \in \mathbb{R}[\![\underline{x}]\!]$ (see ğ3.3). We then explore, mainly for the one variable case ($s = 1$) and with $r \in \mathbb{Z}$, some of the properties of the rational functions/polynomials $Q_n^r(x)$, and list a few of them for low r, n.

We end the introduction with some comments on existing literature. The finite product cases ($r \in \mathbb{N}_0$) of the identities in Theorem 1.1 appeared in [9, Theorem 5.27], where they are seen as consequences of the Macdonald-Cheah formula for the Poincaré/mixed Hodge polynomials of symmetric products of finite CW complexes/algebraic varieties. This is recalled in Example 3.4. On the other hand, the present approach is much more elementary, places all $r \in \mathbb{R}$ in the same setting, and applies to several variables.

2. Plethystic exponentials

In this section, we recall the definitions and fundamental properties of plethystic exponentials in multivariable formal power series rings, with the goal of proving the fundamental product-sum identity in Equation (3). Usually, the plethystic exponential and logarithmic functions are defined in the context of so-called λ-rings, and related to Adams operations in algebra/algebraic topology. In turn, these stem from the so-called plethysms, which have been widely used in the theory of symmetric polynomials (see, for instance, [13, 14, 20]).

As an application to theoretical physics, plethystic exponentials are a fundamental ingredient of the famous DMVV formula for the orbifold elliptic genus of symmetric products ([5]). More recently, they were used for counting BPS gauge invariant operators in supersymmetric gauge theories of D-branes probing Calabi-Yau singularities: the *plethystic program* of Feng, Hanany and He ([2, 6]). Here, these exponentials are introduced in elementary terms.

2.1. One variable plethystic exponentials

We start by defining plethystic exponentials in a simple case: on the ring $\mathbb{Q}[\![x]\!]$ of formal power series in the variable x, with *rational coefficients*, written as $f(x)=\sum_{n\geq 0} a_n x^n$, $a_n\in\mathbb{Q}$. Recall that $\mathbb{Q}[\![x]\!]$ has an obvious addition, and the multiplication of such $f(x)$ with $g(x)=\sum_{n\geq 0} b_n x^n$ is given by the Cauchy product: $f(x)\cdot g(x)=\sum_{n\geq 0}(\sum_{k=0}^{n} a_k b_{n-k})x^n\in\mathbb{Q}[\![x]\!]$. One can also compose formal power series $(f\circ g)(x)$ provided $g(x)=\sum_{n\geq 1} b_n x^n$, i.e., $g(x)$ has *zero constant term* ($b_0=0$):

$$(f\circ g)(x)=a_0+a_1(\sum_{n\geq 1} b_n x^n)+a_2(\sum_{n\geq 1} b_n x^n)^2+\cdots, \tag{6}$$

(this is well defined, as only a finite sum is involved in getting the coefficient of a given x^n). Some important examples of formal power series are the geometric and the (usual) exponential:

$$\begin{aligned}\frac{1}{1-x} &= 1+x+x^2+\cdots+x^n+\cdots,\\ \exp(x) &= 1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\cdots.\end{aligned}$$

For cases like these, which are actually convergent in some region, we adopt the usual notations for analytic functions. Denote by $\mathbb{Q}^0[\![x]\!]:=x\mathbb{Q}[\![x]\!]$ the ideal consisting of series with zero constant term in $\mathbb{Q}[\![x]\!]$.

Definition 2.1. *Let* $f(x)=\sum_{k\geq 1} a_k x^k\in\mathbb{Q}^0[\![x]\!]$. *The* harmonic operator $\Psi:\mathbb{Q}^0[\![x]\!]\to\mathbb{Q}^0[\![x]\!]$, *is the linear map defined by:* $\Psi[f](x)=\sum_{m\geq 1}\frac{f(x^m)}{m}\in\mathbb{Q}^0[\![x]\!]$. *The* plethystic exponential *of* f*, denoted by* $\mathsf{PE}[f]$ *is the composition:*

$$\mathsf{PE}[f](x)=(\exp\circ\Psi[f])(x). \tag{7}$$

Remark 2.1. *(1) Observe that we cannot define the harmonic operator on* $\mathbb{Q}[\![x]\!]$*, since there is no meaning for the divergent* harmonic series*:* $\sum_{m\geq 1}\frac{1}{m}$. *Also note that* $\mathsf{PE}[0]=1$.
(2) Often, for simplicity of notation, we write $\mathsf{PE}(f(x))$ *for* $\mathsf{PE}[f](x)$. *For example, if* $f(x)=x+2x^3$*, we write* $\mathsf{PE}[f](x)$ *as* $PE(x+2x^3)$.
(3) Naturally, we can define PE *in* $R[\![x]\!]$ *for any ring* R *containing* $\mathbb{Q}$. *We will use* real coefficients *in Subsection 2.2 and make occasional comments when using other coefficients.*

Proposition 2.1. *Let* $f,g\in\mathbb{Q}^0[\![x]\!]$. *The plethystic exponential verifies:*

(i) $\mathsf{PE}[f](x)$ *has constant term 1,*
(ii) If $n\in\mathbb{N}$, $\mathsf{PE}(x^n)=\frac{1}{1-x^n}=1+x^n+x^{2n}+\cdots$,
(iii) $\mathsf{PE}[f+g]=\mathsf{PE}[f]\cdot\mathsf{PE}[g]$,
(iv) $\mathsf{PE}[-f]=(\mathsf{PE}[f])^{-1}$.

Proof. (i) Follows from Eq. (6), as the constant term of exp is 1. (ii) We evaluate the harmonic operator on $f(x)=x^n\in\mathbb{Q}^0[\![x]\!]$, using the series expansion of the logarithm:

$$\Psi[f](x)=\sum_{m\geq 1}\frac{f(x^m)}{m}=\sum_{m\geq 1}\frac{x^{mn}}{m}=-\log(1-x^n)=\log\left(\frac{1}{1-x^n}\right).$$

Then, $\mathsf{PE}[f](x)=\exp(\Psi[f](x))=1/(1-x^n)$ as wanted. (iii) Follows from the linearity of Ψ, and the usual properties of exp. (iv) Immediate from (iii) and from $\mathsf{PE}[0]=1$. □

Example 2.1. *Using the rules above, some simple examples follow, such as:*
(i) With $n > 0$, $\mathsf{PE}(x^n - x^{2n}) = \mathsf{PE}(x^n)/\mathsf{PE}(x^{2n}) = \frac{1-x^{2n}}{1-x^n} = 1 + x^n$,
(ii) $\mathsf{PE}(\frac{x}{1-x}) = \mathsf{PE}(x + x^2 + x^3 + \cdots) = \prod_{k\geq 1} \frac{1}{1-x^k} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots$,
(iii) $\mathsf{PE}\left(\frac{x(2-x)}{(1-x)^2}\right) = \prod_{k\geq 1} \frac{1}{(1-x^k)^{k+1}} = 1 + 2x + 6x^2 + 14x^3 + 33x^4 + 70x^5 + 149x^6 + \cdots$.

The second and third examples above, which will show up later on, relate to the famous Euler and Macmahon functions, respectively. They are the generating functions for the sequence of partitions of natural numbers $1, 2, 3, 5, 7, 11, \cdots$, and the sequence $2, 6, 14, 33, 70, \cdots$ of partitions of $n \geq 1$ objects colored using 2 colors (see ğ3.2 below).

These examples illustrate why the plethystic exponential plays such an important role in the study of integer sequences, in the spirit described in [3]: indeed, when all the coefficients of f are integers, $\mathsf{PE}[f]$ corresponds precisely to the operator S defined there, and is sometimes called the *Euler transform* ([18]). On the other hand, the proofs of Theorems 1.1 and 1.2 require the generalization to the multi-variable plethystic exponential.

2.2. Multi-variable plethystic exponentials, partitions and the cycle index of S_n

Consider the ring of formal power series $R := \mathbb{R}[\![x_1, \cdots, x_s, y]\!]$ with *real coefficients* in $s + 1$ variables $x_1, \cdots, x_s$ and y, and let $R^0 \subset R$ be the ideal of series without constant term. For example, when $s = 1$, elements of R^0 can be written alternatively as: $f(x, y) = \sum_{j,k\geq 0} a_{j,k}\, x^j\, y^k = \sum_{k\geq 0} b_k(x)\, y^k$. In the first expansion we have $a_{j,k} \in \mathbb{R}$ and $a_{0,0} = 0$, and in the second we have $b_k(x) \in \mathbb{R}[\![x]\!]$ and $b_0(x) \in \mathbb{R}^0[\![x]\!]$. When $s > 1$, to simplify the notation, we write $\underline{x} := (x_1, \cdots, x_s)$, $\mathbb{R}[\![\underline{x}, y]\!] = \mathbb{R}[\![x_1, \cdots, x_s, y]\!]$, and $\underline{x}^m$ means the s-tuple $(x_1^m, \cdots, x_s^m)$, when $m \in \mathbb{N}$.

Definition 2.2. *Let* $f(\underline{x}, y) \in R^0$. *The* plethystic exponential *of* f*, denoted by* $\mathsf{PE}[f]$ *is the composition:* $\mathsf{PE}[f] = \exp \circ \Psi[f] \in R$, *with the harmonic operator* $\Psi : R^0 \to R^0$*, now given by* $\Psi[f](\underline{x}, y) = \sum_{m\geq 1} \frac{f(\underline{x}^m, y^m)}{m} \in R^0$.

We now show a fundamental *product-sum formula* for $\mathsf{PE}[f]$, when f is of the form: $f(\underline{x}, y) = g(\underline{x})\, y$, $g(\underline{x}) \in \mathbb{R}[\![\underline{x}]\!]$, which can be expressed either in terms of partitions of natural numbers or in terms of certain *cycle indices* for symmetric groups (see also ğ3.3). In [6], these are called "y-inserted" plethystic exponentials.

A partition of $n \in \mathbb{N}$, is given by the sum $n = n_1 + \cdots + n_m$ where each n_j is a positive integer. Alternatively, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, a partition λ of n is a map $\lambda : \{1, \cdots, n\} \mapsto \mathbb{N}_0$ such that $\lambda(j)$ is the number of parts of size j (zero, if there are no parts of size j). In this notation, we have $\sum_{j=1}^n j\lambda(j) = n$. As usual, we write $\lambda \vdash n$ when λ is a partition of n.

Let S_n be the symmetric group on n letters, that is, the group of permutations of the set $\{1, \cdots, n\}$. Every permutation $\sigma \in S_n$ can be written as a product $\sigma = \gamma_1\gamma_2 \cdots \gamma_m$ where each γ_i is a cycle, say of length n_i, and all such cycles are disjoint. If we also include in this product all cycles of length 1 (elements fixed by σ), then $n = n_1 + n_2 + \cdots$ is a partition of n. Conversely, any partition of n determines a unique conjugacy class of elements $\sigma \in S_n$.

Given n variables $p_1, \cdots, p_n$, the *cycle index of* S_n is defined as the degree n polynomial, with $\mathbb{Q}$ coefficients:

$$Z_n(p_1, \cdots, p_n) = \frac{1}{n!} \sum_{\sigma\in S_n} \prod_{j=1}^{n} p_j^{\sigma_j} \quad \in \mathbb{Q}[p_1, \cdots, p_n],$$

where $\sigma_j \geq 0$ is the number of cycles of $\sigma \in S_n$ of each size $j \in \{1, 2, \cdots, n\}$. Now, if $g(\underline{x}) \in \mathbb{R}[\![\underline{x}]\!]$ is a formal power series in $\underline{x} = (x_1, \cdots, x_n)$, we denote by $Z_n[g(\underline{x})]$ the substitution $Z_n(g(\underline{x}), g(\underline{x}^2), \cdots, g(\underline{x}^n))$, and call it the *evaluation of* Z_n *at* $g(\underline{x})$; this is generally *not a polynomial* in $x_1, \cdots, x_s$, but another formal power series in $\mathbb{R}[\![\underline{x}]\!]$. Recall the notations: $\underline{x}^j = (x_1^j, \cdots, x_s^j)$, if $j \in \mathbb{N}$, and $\mathbf{x}^{\mathbf{k}} = x_1^k \cdots x_s^k$ when $\mathbf{k} = (k_1, \cdots, k_s) \in \mathbb{N}_0^s$.

Proposition 2.2. *Let* $g(\underline{x}) = \sum_{\mathbf{k}\in\mathbb{N}_0^r} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} \in \mathbb{R}[\![\underline{x}]\!]$. *We have the equalities in* $R = \mathbb{R}[\![\underline{x}, y]\!]$

$$\mathsf{PE}\,(g(\underline{x})\, y) = \prod_{\mathbf{k}\in\mathbb{N}_0^s} (1 - y\,\mathbf{x}^{\mathbf{k}})^{-a_{\mathbf{k}}} = 1 + \sum_{n\geq 1} \left(\sum_{\lambda\vdash n} \prod_{j=1}^{n} \frac{g(\underline{x}^j)^{\lambda(j)}}{\lambda(j)!\, j^{\lambda(j)}} \right) y^n = 1 + \sum_{n\geq 1} Z_n[g(\underline{x})]\, y^n, \tag{8}$$

where $\lambda(j)$ *is the number of parts of* $\lambda \vdash n$ *of each size* $j \in \{1, 2, \cdots, n\}$.

Proof. Starting from the definition, we perform the following computation (cf. also [6, 8]):

$$\mathsf{PE}[f](\underline{x}, y) = \exp\left(\sum_{m\geq 1} \frac{g(\underline{x}^m)\, y^m}{m} \right) = \prod_{m\geq 1} \exp\left(\frac{g(\underline{x}^m)}{m}\, y^m \right) = \sum_{k_1\geq 0} \sum_{k_2\geq 0} \cdots \prod_{m\geq 1} \frac{1}{k_m!} \left(\frac{g(\underline{x}^m)}{m}\, y^m \right)^{k_m}.$$

To get the coefficient of y^n, write $n = \sum_{k_m\geq 0} mk_m$; so, we are dealing with partitions λ of n with $k_j = \lambda(j)$ being the number of parts of size j, and we get:

$$\mathsf{PE}[f](\underline{x},y)=1+\sum_{n\geq 1}\left(\sum_{\lambda\vdash n}\prod_{j=1}^{n}\frac{g(\underline{x}^{j})^{\lambda(j)}}{\lambda(j)!\,j^{\lambda(j)}}\right)y^{n}=1+\sum_{n\geq 1}\frac{1}{n!}\sum_{\sigma\in S_n}g(\underline{x})^{\sigma_1}\cdots g(\underline{x}^{n})^{\sigma_n}\,y^{n},$$

which are the last two expressions in Equation (8). Above, we used the fact that the number of permutations with the cycle type of the partition $\lambda\vdash n$ equals exactly $n!\prod_{j=1}^{n}\frac{1}{\lambda(j)!\,j^{\lambda(j)}}$ (see, eg, [20, 1.3.2]). Finally, to obtain the product form, note that an analogous computation as in the proof of Proposition 2.1(ii) gives $\mathsf{PE}(y\,x_1^{k_1}\cdots x_s^{k_s})=\frac{1}{1-y\,\mathbf{x}^{\mathbf{k}}}$, for every $k_1,\cdots,k_s,n\geq 0$ not all zero. Then, Proposition 2.1(iii) transforms the sum into a product: $\mathsf{PE}(\sum_{\mathbf{k}\in\mathbb{N}_0^r}a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}\,y)=\prod_{\mathbf{k}\in\mathbb{N}_0^s}(1-y\,\mathbf{x}^{\mathbf{k}})^{-a_{\mathbf{k}}}\in\mathbb{Z}[\![\underline{x},y]\!]$, as wanted. Note that, if $-a_{\mathbf{k}}\in\mathbb{N}$, then $\mathsf{PE}(a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}\,y)=(1-y\,\mathbf{x}^{\mathbf{k}})^{-a_{\mathbf{k}}}$ is actually a polynomial. $\square$

Remark 2.2. *(1) It is clear that the subset $1+\mathbb{R}^0[\![\underline{x}]\!]\subset\mathbb{R}[\![\underline{x}]\!]$ (series with 1 as constant term) is a commutative group with multiplication of formal power series, and 1 as the identity. If we consider $(\mathbb{R}[\![\underline{x}]\!],+,0)$ also as an abelian group one sees that* $\mathsf{PE}:\mathbb{R}^0[\![\underline{x}]\!]\to 1+\mathbb{R}^0[\![\underline{x}]\!]$ *is an* isomorphism of abelian groups. *Its inverse, is called naturally the* plethystic logarithm *and it is also important in enumerative combinatorics. For example, it allows the enumeration of connected simple graphs, from the knowledge of the generating function of all graphs (see [12]).*
(2) Even though Equations (8) appear to be make sense only in $\mathbb{Q}[\![\underline{x},y]\!]$ or $\mathbb{R}[\![\underline{x},y]\!]$, Proposition 2.2 shows that if $f\in\mathbb{Z}^0[\![\underline{x},y]\!]\subset\mathbb{Q}^0[\![\underline{x},y]\!]$ (series with zero constant term), then $\mathsf{PE}[f]$ is in $\mathbb{Z}[\![\underline{x},y]\!]$: plethystic exponentials of series with integer coefficients, *still have integer coefficients.*
(3) We thank M. Wildon for suggesting to consider the evaluation of Z_n in Proposition 2.2 ([22]), which also led to the treatment of other results in Section 3.3.

3. Plethystic exponentials of rational functions

We come now to the proof of our main identities (1) and (4). They follow from Theorem 2.2, using the simple *generalized* rational function: $g(\underline{x})=(1-x_1)^{r_1}\cdots(1-x_s)^{r_s}\in\mathbb{R}[\![\underline{x}]\!]$, for certain choices of *real* numbers $r_1,\cdots,r_s$. We then explore some consequences of the identities, their relation to colored partitions and the Euler and Macmahon functions, and to the Macdonald/Cheah formula for symmetric products. We also observe some of their consequences when used in conjunction with the so-called q-binomial theorem and Heine's summation formula. We end with a study of some simple properties of the polynomials $Q_n^r(x)$ in Equation (5), for $r\in\mathbb{Z}$, which are evaluations of the cycle index of S_n, and explicitly list some of them.

3.1. The main identities

The symmetric group S_n acts linearly on a n dimensional vector space, such as $\mathbb{R}^n$, by permuting the elements of a fixed standard basis. This way, we identify every element $\sigma\in S_n$ with an element $M_\sigma\in GL(\mathbb{R}^n)$ (the group of linear automorphisms of $\mathbb{R}^n$ with identity I_n).

Theorem 3.1. *Fix $s\in\mathbb{N}$ and $r_1,\cdots,r_s\in\mathbb{R}$. Then, the following are equalities in $\mathbb{R}[\![\mathbf{x},y]\!]$:*

$$\mathsf{PE}((1-x_1)^{r_1}\cdots(1-x_s)^{r_s}y)\;=1+\sum_{n\geq 1}\sum_{\sigma\in S_n}\frac{y^n}{n!}\prod_{i=1}^{s}\det(I_n-x_jM_\sigma)^{r_i}.$$

Proof. Given a permutation $\sigma\in S_n$, written as a product of disjoint cycles $\sigma=\gamma_1\gamma_2\cdots\gamma_m$, the endomorphism I_n-xM_σ is a direct sum:

$$I_n-xM_\sigma=(I_{n_1}-xM_{\gamma_1})\oplus\cdots\oplus(I_{n_m}-xM_{\gamma_m}).$$

(with M_{γ_i} considered as acting on a n_i-dimensional subspace of $\mathbb{R}^n$). For a cycle γ of length k, we readily compute $\det(I_k-xM_\gamma)=1-x^k$, so we see that

$$\det(I_n-xM_\sigma)^r=\prod_{j=1}^{n}(1-x^j)^{r\sigma_j},$$

where σ_j is the number of cycles of σ with length $j\in\{1,\cdots,n\}$, and this determinant is invariant under conjugation of σ in S_n. Hence, in the multivariable case, we get:

$$\begin{aligned}1+\sum_{n\geq 1}\frac{y^n}{n!}\sum_{\sigma\in S_n}\prod_{i=1}^{s}\det(I_n-x_iM_\sigma)^{r_i}\;&=\;1+\sum_{n\geq 1}\frac{y^n}{n!}\sum_{\sigma\in S_n}\prod_{j=1}^{n}\left[(1-x_1^j)^{r_1}\cdots(1-x_s^j)^{r_s}\right]^{\sigma_j}\\&=\;\mathsf{PE}(g(\underline{x})\,y),\end{aligned}$$

where $g(\underline{x})=(1-x_1)^{r_1}\cdots(1-x_s)^{r_s}$, by Proposition 2.2. $\square$

We now complete the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Setting $x_1 = \cdots = x_s$ and $r_1+\cdots+r_s = r$ in Theorem 3.1 we are computing the plethystic exponential of $f(x,y) = (1-x)^r y$. By the product formula of Proposition 2.2, this amounts to $\prod_{k\geq 0}(1-yx^k)^{-a_k}$ with a_k being the coefficients of the series expansion of $g(x) = (1-x)^r = \sum_{k\geq 0}\binom{k-r-1}{k}x^k$, valid for every real r. When $r\in\mathbb{N}$, from $\binom{k-r-1}{k} = (-1)^k\binom{r}{k}$ we obtain Equation (2), and the convergence of the series follows from the estimate: $\sum_{\sigma\in S_n}|\det(I_n - xM_\sigma)|^r \leq \sum_{\sigma\in S_n} 2^{nr} = n!2^{nr}$, valid when $|x|<1$. □

For the proof of Equation (4), recall that we restrict to integer coefficients.

Proof of Theorem 1.2. From Theorem 3.1, the summation on the left hand side of Equation (4) equals $\mathsf{PE}(\frac{(1-x_1)\cdots(1-x_r)}{(1-q_1)\cdots(1-q_s)}y)$. Given the series expansion:

$$\frac{(1-x_1)\cdots(1-x_r)}{(1-q_1)\cdots(1-q_s)} = \sum_{j_1=0}^{1}\cdots\sum_{k_1=0}^{\infty}\cdots(-x_1)^{j_1}\cdots(-x_r)^{j_r}q_1^{k_1}\cdots q_s^{k_s},$$

the result follows immediately from the product formula, Proposition 2.2. □

Remark 3.1. *We should point out that a special case of Theorem 3.1 had already been obtained in the study of symmetric products of algebraic groups (see [17, Lemma 4.2]); namely, for a sequence $r_1,\cdots,r_s\in\mathbb{N}_0$:* $\mathsf{PE}\left[y\prod_{j=1}^{s}(1-x^{2j-1})^{r_j}\right] = 1+\sum_{n\geq 1}\sum_{\sigma\in S_n}\frac{y^n}{n!}\prod_{j=1}^{s}\det(I_n - x^{2j-1}M_\sigma)^{r_j}$.

3.2. Applications and relations to other identities

We now present some relations of the main identities with other subjects, and explore some consequences of these connections.

Example 3.1. *(Generating function for colored partitions) We now prove Corollary 1.1, which determines the residue of our basic generating function $\Phi^r(x,y) := 1+\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^n}{n!}\det(I_n - xM_\sigma)^r$, $r\in\mathbb{Z}$, at $y=1$. In fact, this is an immediate consequence of our main identity:* $\mathrm{Res}_{y=1}\Phi^r(x,y) = \mathrm{Res}_{y=1}\left((1-y)^{-1}\prod_{k\geq 1}(1-yx^k)^{-\binom{k-r-1}{k}}\right) = -\prod_{k\geq 1}(1-x^k)^{-\binom{k-r-1}{k}}$.

Two special cases of this formula are noteworthy, namely when $r=-1$ or $r=-2$ ($r=0$ is the geometric series $\Phi^0(x,y) = \frac{1}{1-y}$ with residue -1): $-\mathrm{Res}_{y=1}\Phi^{-1}(x,y) = \frac{1}{\phi(x)} = \sum_{n\geq 0}p(n)x^n = 1+x+2x^2+3x^3+5x^4+\cdots$, $-\mathrm{Res}_{y=1}\Phi^{-2}(x,y) = \frac{M(x)}{\phi(x)} = 1+2x+6x^2+14x^3+33x^4+70x^5+149x^6+\cdots$, where $\phi(x) := \prod_{k\geq 1}(1-x^k)$ and $M(x) = \prod_{k\geq 1}(1-x^k)^{-k}$ are the famous Euler function and Macmahon function, respectively, which are well known to be the generating functions for *partitions* and for *plane partitions*, respectively. These are the sequences A000041 and A000219 in [16]. Note also that $\frac{M(x)}{\phi(x)} = \prod_{k\geq 1}(1-x^k)^{-k-1}$ is actually the generating function for the number of *colored partitions of n with 2 colors* (sequence A005380 in [16]). More generally, a colored partition of $n\in\mathbb{N}$ with $c\in\mathbb{N}$ colors is defined as follows. Take a partition $n_1\geq n_2\geq\cdots\geq n_m\geq 1$ of n (so that $n = n_1+\cdots+n_m$) and consider each part n_j as n_j identical objects which are then colored with one of c possible colors; we disregard the order of the colored objects. For example, with $c = 3 = |\{b,w,r\}|$, for black, white and red colors, colored partitions of $n=2$ are the 12 elements: bb, bw, br, ww, wr, rr, $b+b$, $b+w$, $b+r$, $w+w$, $w+r$, $r+r$. Hence, Corollary 1.1 follows from the well-known formula for the generating function for colored partitions of n with c colors is (see [16], sequence A217093, and [21, exercise 7.99]): $\prod_{k\geq 1}(1-x^k)^{-\binom{k-r-1}{k}}$.

Example 3.2. *(the q-binomial theorem). In Theorem 1.2, consider one variable x, and one q. We then get the infinite product of the famous q-*binomial theorem*:*

$$1+\sum_{n\geq 0}\frac{y^n}{n!}\sum_{\sigma\in S_n}\frac{\det(I_n - xM_\sigma)}{\det(I_n - qM_\sigma)} = \prod_{k\in\mathbb{N}_0}\frac{1-xyq^k}{1-yq^k} = \sum_{n\geq 0}\frac{(x;q)_n}{(q;q)_n}y^n, \tag{9}$$

where $(a;q)_n$ denote the (finite) Pochhammer symbols: $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$, $(a;q)_0 := 1$.

The right hand side of (9) is also the *basic hypergeometric series* denoted by ${}_1\phi_0$. We conclude the interesting formula:

$$\frac{1}{n!}\sum_{\sigma\in S_n}\frac{\det(I_n - xM_\sigma)}{\det(I_n - qM_\sigma)} = \frac{(x;q)_n}{(q;q)_n} = \prod_{k=1}^{n}\frac{1-xq^{k-1}}{1-q^k}. \tag{10}$$

In turn, by letting $x=0$ in Equation (10) we get Molien's formula for the Hilbert series that captures the dimensions of the graded components of the natural action of S_n on the polynomial ring in n variables, $\mathbb{C}[t_1,\cdots,t_n]$: $\frac{1}{n!}\sum_{\sigma\in S_n}\frac{1}{\det(I_n - qM_\sigma)} = \prod_{k=1}^{n}\frac{1}{1-q^k}$. We thank Jimmy He for suggesting to look for analogies between our identities and other q-series identities.

Example 3.3. *(Heine's summation) Consider now two variables x, and one q. Theorem 1.2 then gives:*

$$1+\sum_{n\geq 0}\frac{y^n}{n!}\sum_{\sigma\in S_n}\frac{\det[(I_n-x_1M_\sigma)(I-x_2M_\sigma)]}{\det(I_n-qM_\sigma)}=\mathsf{PE}\left(\frac{(1-x_1)(1-x_2)}{1-q}y\right)$$

which is precisely the infinite product in the so-called Heine's summation theorem, *in the form:*

$$\sum_{n\geq 0}\prod_{k=1}^{n}\frac{(1-x_1^{-1}q^{k-1})(1-x_2^{-1}q^{k-1})}{(1-q^k)(1-yq^{k-1})}(x_1x_2y)^n=\prod_{k\in\mathbb{N}_0}\frac{(1-yx_1q^k)(1-yx_2q^k)}{(1-yq^k)(1-yx_1x_2q^k)}$$

since this is the basic hypergeometric series ${}_2\phi_1(\frac{1}{x_1},\frac{1}{x_2};\,y;\,q;\,yx_1x_2)$ (see, e.g., [11, (1.5.1)]). The generalization of these identities to other cases, and their interesting relation with more general basic hypergeometric series will be dealt in forthcoming work ([10]).

Example 3.4. *(Relation with Macdonald/Cheah formula for symmetric products). Let X be a d-dimensional finite CW complex, with Poincaré polynomial $P_X(x)=\sum_{k=0}^{d}b_k(X)\,x^k$, where $b_k(X)$ are its Betti numbers. The nth symmetric product $\mathsf{Sym}^nX:=X^n/S_n$ is the quotient of the n-fold cartesian product under the action of S_n by permutation of the variables. The famous Macdonald formula reads (see [15, Page 563]):*

$$1+\sum_{n\geq 1}P_{\mathsf{Sym}^nX}(-x)\,y^n=\prod_{k=0}^{d}\left(1-x^ky\right)^{(-1)^{k+1}b_k(X)},\tag{11}$$

and we see that the right hand side is naturally a plethystic exponential: from Proposition 2.2, it is precisely $\mathsf{PE}(P_X(-x)y)$.

This formula was generalized for the mixed Hodge polynomials μ_X of symmetric products of complex algebraic varieties by J. Cheah in [4], which reads, in our notation: $1+\sum_{n\geq 1}\mu_{\mathsf{Sym}^nX}(-t,u,v)\,y^n=\mathsf{PE}(\mu_X(-t,u,v)\,y)$. More recently, by studying the space of commuting elements in a compact Lie group G (see [19]), it was shown that:

$$P_{T^r/W}(x)=\frac{1}{|W|}\sum_{\sigma\in W}\det(I+x\sigma)^r,\tag{12}$$

where W is the Weyl group of G acting by reflections on the dual of the Lie algebra of a maximal torus T, and I is the identity automorphism. An analogous study for mixed Hodge structures on character varieties of abelian groups led to a proof of the positive integer case ($r\geq 1$) of Theorem 1.1 in [9], by combining equations (11) and (12), in the case of the unitary group $G=U(n)$, with maximal torus $T=(S^1)^n$ and Weyl group S_n (and using the deformation retraction from these varieties to the quotient T^r/W, see [7]).

3.3. The cycle index and average powers of characteristic polynomials of permutations

Recall Proposition 2.2 and, in particular, the identity:

$$\mathsf{PE}(g(\underline{x})y)=1+\sum_{n\geq 1}Z_n[g(\underline{x})]\,y^n,\tag{13}$$

for *any* formal power series $g(\underline{x})\in\mathbb{R}[\![\underline{x}]\!]$ in an arbitrary number of variables $\underline{x}=(x_1,\cdots,x_s)$. In the language of the *Plethystic Program* of [6], this says that "y-inserted" plethystic exponentials are just the generating functions of the evaluations of cycle indices of S_n. Now fix a vector $\mathbf{r}=(r_1,\cdots,r_s)\in\mathbb{R}^s$ and consider the formal power series: $Q_n^{\mathbf{r}}(\underline{x}):=\frac{1}{n!}\sum_{\sigma\in S_n}\det(I_n-x_1M_\sigma)^{r_1}\cdots\det(I_n-x_sM_\sigma)^{r_s}=Z_n[(1-x_1)^{r_1}\cdots(1-x_s)^{r_s}]$. To prove Corollary 1.2, we just need to use the analogous property of the cycle index of the symmetric group Z_n. For the benefit of the reader, we reproduce the argument, that boils down to taking the derivative of Equation (13) with respect to y:

$$\sum_{n\geq 1}n\,Z_n[g(\underline{x})]\,y^{n-1}=\frac{\partial}{\partial y}\left(\exp\left[\sum_{k\geq 1}g(\underline{x}^k)\frac{y^k}{k}\right]\right)=\left(1+\sum_{m\geq 1}Z_m[g(\underline{x})]\,y^m\right)\left(\sum_{k\geq 1}g(\underline{x}^k)y^{k-1}\right),$$

which implies, upon picking the y^{n-1} term: $Z_n[g(\underline{x})]=\frac{1}{n}\sum_{k=1}^{n}g(\underline{x}^k)\,Z_{n-k}[g(\underline{x})]$. In particular,

$$Q_n^{\mathbf{r}}(\underline{x})=\frac{1}{n}\sum_{k=1}^{n}(1-x_1^k)^{r_1}\cdots(1-x_s^k)^{r_s}\,Q_{n-k}^{\mathbf{r}}(\underline{x}),\tag{14}$$

which shows Corollary 1.2. Consider now the one variable case, $s=1$, and let $Q_n^r(x):=\frac{1}{n!}\sum_{\sigma\in S_n}\det(I_n-xM_\sigma)^r$, be the average rth power of permutation polynomials for a given matrix size $n\in\mathbb{N}$. Then, the recurrence relation (14) becomes $Q_n^r(x)=\frac{1}{n}\sum_{k=1}^{n}(1-x^k)^rQ_{n-k}^r(x)$, for any $r\in\mathbb{R}$, from which we obtain the following explicit expressions.

Proposition 3.1. *For $n = 1, 2, 3$ and 4, and all $r \in \mathbb{R}$, we have:*

(i) $Q_1^r(x) = (1-x)^r$

(ii) $Q_2^r(x) = \frac{1}{2}\left((1-x)^{2r} + (1-x^2)^r\right)$

(iii) $Q_3^r(x) = \frac{1}{6}(1-x)^{3r} + \frac{1}{2}(1-x^2)^r(1-x)^r + \frac{1}{3}(1-x^3)^r$

(iv) $Q_4^r(x) = \frac{1}{24}(1-x)^{4r} + \frac{1}{4}(1-x^4)^r + \frac{1}{8}(1-x^2)^{2r} + \frac{1}{3}(1-x)^r(1-x^3)^{3r} + \frac{1}{4}(1-x^2)^r(1-x)^{2r}$.

In the positive integer case $r \in \mathbb{N}$, the functions $Q_n^r(x)$ are one variable *polynomials*, and we derive some of their general properties as follows.

Proposition 3.2. *For $r, n \geq 1$, $Q_n^r(0) = 1$ and the degree of $Q_n^r(x)$ is $\leq nr$. Moreover:*

(i) $Q_n^r(x)$ is divisible by $Q_1^r(x) = (1-x)^r$, in particular $Q_n^r(1) = 0$.

(ii) $Q_n^r(x)$ has integer coefficients.

(iii) When r is even, $Q_n^r(x)$ is palindromic *(hence monic of degree nr).*

Proof. Since the degree of the permutation polynomial $\det(I_n - xM_\sigma)$ is n, and $|S_n| = n!$, the first sentence is clear. (i) For any permutation $\sigma \in S_n$, the matrix $M_\sigma \in GL(n, \mathbb{C})$ always has an eigenvalue equal to 1; hence, for $r \geq 1$, $Q_1^r(x) = (1-x)^r$ divides $\det(I_n - xM_\sigma)^r$. (ii) This is an immediate consequence of the product expansion of plethystic exponentials and Remark 2.2(2). (iii) A polynomial $p(x)$ is palindromic if and only if $p(x) = x^{\deg(p)}p(\frac{1}{x})$. Since every cycle $\delta \in S_n$ of length k satisfies $\det(I_k - xM_\delta)^r = (1-x^k)^r$, which is a palindromic polynomial when $r \in 2\mathbb{N}$, and every permutation is a product of cycles, the result follows from the fact that the product of palindromic polynomials is palindromic. □

Finally, for low values of $r \in \mathbb{N}$ and arbitrary n, the following is obtained by simple computations.

Proposition 3.3. *Let $r = 1, 2$ or 3. We have, for all $n \geq 1$:*

(i) $Q_n^1(x) = (1-x)$.

(ii) $Q_n^2(x) = (1-x)^2 \sum_{k=0}^{n-1} x^{2k}$.

(iii) $Q_n^3(x) = (1-x)^3 \left(\sum_{k=0}^{n-1} \binom{k+2}{2} x^{2k} - \sum_{k=1}^{n-2} \binom{k+1}{2} x^{2k+1}\right)$.

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