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Research Article

Upper domatic number of regular graphs

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Abstract

A partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V(G)$ of a graph $G = (V, E)$ is an *upper domatic partition* if V_i dominates V_j or V_j dominates V_i or both for all $V_i, V_j \in \pi$. The maximum order of an upper domatic partition of G is called the *upper domatic number* $D(G)$ of G. In this article, we determine the upper domatic number of 4-regular graphs. We also find the upper domatic number of 5-regular graphs with girth at least five and determine the upper domatic number of the complements of cycles.

Keywords: domatic number; r-regular graphs; upper domatic number.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

Graph colouring and domination are two well-studied concepts in graph theory. Connecting these two concepts, Cockayne and Hedetniemi initiated the study of the domatic number in 1977 [1]. The *domatic number*, denoted by $d(G)$, is the maximum number of sets in a partition of the vertex set into mutually disjoint dominating sets. For two disjoint sets A and B of vertices, set A dominates set B, denoted by $A \to B$, if each vertex in B is adjacent to at least one vertex in A, else we write $A \not\to B$. A subset S of $V(G)$ is a *dominating set* of G if S dominates $V(G)\backslash S$. Exploring the relation between sets in a vertex partition, Haynes et al. [3] introduced the concept of *upper domatic number* $D(G)$, which is the maximum cardinality of a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of a graph G such that for each pair of sets $V_i, V_j \in \pi$, $V_i \to V_j$ or $V_j \to V_i$ or both. An upper domatic partition of G with cardinality $D(G)$ is referred to as a D-partition of G.

Throughout this paper, we consider only finite, simple and undirected graphs $G = (V, E)$ of order $n = |V|$ and size $m = |E|$. For the notations and terminologies not defined here, we refer the reader to [2,4,6]. The *degree* of a vertex $v \in V$, denoted by deg(v), is the number of vertices adjacent to v. The maximum (minimum) degree of a vertex in a graph G is denoted by $\Delta(G)$ ($\delta(G)$). The set of vertices at distance equal to k from the vertex u is represented by $N^k(u)$. Paths, cycles and complete graphs of order n are denoted by P_n, C_n and K_n, respectively. A maximal complete subgraph of a graph is called *clique*. The maximum order of a clique of G is called the *clique number* $\omega(G)$.

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a vertex partition of a graph G. A set $V_i \in \pi$ is a *source set* if V_i is a dominating set of G and a *sink set* if $V_j \to V_i$, for all $1 \le j \le k$. For a subset S of $V(G)$, $G[S]$ represents the subgraph of G induced by S. An r-regular graph is a graph with every vertex having degree r.

In this article, we study the upper domatic number of r-regular graphs and settle the problem in the special case when $r = 4$. The upper domatic number of 5-regular graphs with girth at least five also is determined. Further we find the upper domatic number of complements of cycles.

2. Main results

For small values of r, the upper domatic number of an r-regular graph is determined in [3].

Theorem 2.1. [3] *For any r-regular graph G where $r \in \{0, 1, 2, 3\}$, $D(G) = r + 1$.*

Haynes et al. also gave an upper bound for the upper domatic number of any graph in terms of its maximum degree.

Theorem 2.2. [3] *For a graph G with maximum degree $\Delta(G)$, $D(G) \le \Delta(G) + 1$.*

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Another upper bound for the upper domatic number in terms of the order and clique number of the graph is presented in [5].

Theorem 2.3. [5] *For any graph G with clique number $\omega(G)$, $D(G) \leq \dfrac{n+\omega(G)}{2}$.*

We next present the following sufficiency condition for a 4-regular graph to have an upper domatic number of five.

Theorem 2.4. *For any 4-regular graph G, $D(G) = 5$ if G contains a cubic graph as an induced subgraph.*

Proof. Consider a 4-regular graph G having a 3-regular induced subgraph, say H. By Theorem 2.1, $D(H) = 4$. Moreover, H being a 3-regular graph, every vertex in $V(H)$ is adjacent to exactly one vertex in $V(G)\backslash V(H)$, implying that $V(G)\backslash V(H) \to V(H)$. Hence, an upper domatic partition of order five can be obtained by adding the set $V(G)\backslash V(H)$ to a D-partition of H. Therefore, by Theorem 2.2, $D(G) = 5$. □

Corollary 2.1. *If a graph G contains a cubic induced subgraph and $\delta(G) \geq 4$, then $D(G) \geq 5$.*

Corollary 2.2. *For any 4-regular graph G, $D(G) = 5$ if G contains K_4.*

The result in Corollary 2.2 can be generalised for any r-regular graph containing K_r.

Theorem 2.5. *For any r-regular graph G, $D(G) = r + 1$ if G contains K_r.*

It is evident from the literature that for a 4-regular graph G, $4 \leq D(G) \leq 5$. Next, we characterise 4-regular graphs having $D(G) = 4$. Recall that the complete multipartite graph with k vertices each in t partite sets is denoted by $G_{t,k}$ in [3], for example the complete tripartite graph $G_{3,2}$ with three independent partite sets $\{u_1, u_4\}$, $\{u_2, u_5\}$ and $\{u_3, u_6\}$ in Figure 1.

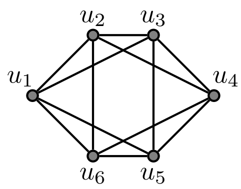

Figure 1: The graph $G_{3,2}$.

Theorem 2.6. *For a 4-regular connected graph G, $D(G) = 4$ if and only if $G = G_{3,2}$.*

Proof. Consider the 4-regular graph $G_{3,2}$ with vertex labelling as shown in Figure 1. The partition $\pi = \{\{u_4, u_5, u_6\}, \{u_1\}, \{u_2\}, \{u_3\}\}$ is an upper domatic partition, implying that $D(G_{3,2}) \geq 4$. If $D(G_{3,2}) > 4$, then any D-partition of $G_{3,2}$ contains at least four singleton sets, which is impossible as $\omega(G_{3,2}) = 3$. Hence, $D(G_{3,2}) = 4$.

Now assume that G is a connected 4-regular graph with $D(G) = 4$. Assume that the graph G is different from $G_{3,2}$. We will prove that $D(G) \geq 5$. Consider a vertex, say $u_1 \in V(G)$. Since G is 4-regular, there are 11 possible induced subgraphs $G[N[u_1]]$, as shown in Figure 2.

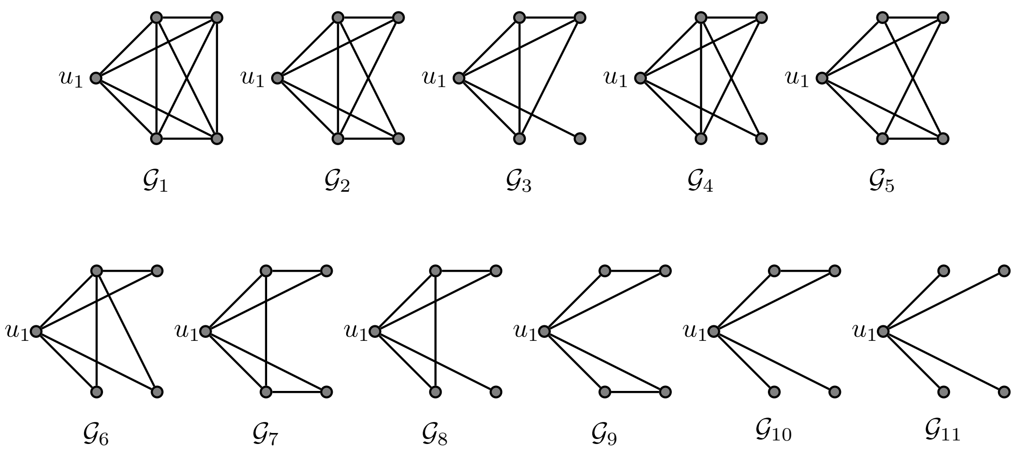

Figure 2: Graphs of order five with a dominating vertex.

Case 1: $G[N[u_1]] \in \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$.
Since each of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4$ contains an induced K_4, from Corollary 2.2 it follows that $D(G) = 5$.

Case 2: $G[N[u_1]] = \mathcal{G}_5$, as shown in Figure 3.
In this case we observe that $|N^2(u_1)| > 1$, else G is isomorphic to $G_{3,2}$. Therefore, there exist at least two vertices, say $u_6, u_7 \in N^2(u_1)$. Suppose we have a pair of adjacent vertices, say $u_2, u_3 \in N^1(u_1)$ such that u_2 is adjacent to u_6 and u_3 is adjacent to u_7. Then, $\pi = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4, u_7\}, \{u_5, u_6\} \cup V'\}$, where $V' = V(G) \backslash \{u_1, u_2, \dots, u_7\}$, is an upper domatic partition of G. In the case of a non-adjacent pair of vertices, say $u_2, u_5 \in N^1(u_1)$ having adjacency between the pair of vertices (u_2, u_6) and (u_5, u_7), the vertex u_3 is adjacent either to u_6 or to another vertex $u_8 \in N^2(u_1)$. In the former case $\{\{u_1\}, \{u_3\}, \{u_5\}, \{u_4, u_6\}, \{u_2, u_7\} \cup V'\}$, where $V' = V(G) \backslash \{u_1, u_2, \dots, u_7\}$ forms an upper domatic partition of G, whereas in the latter case $\{\{u_1\}, \{u_3\}, \{u_5\}, \{u_4, u_8\}, \{u_2, u_7, u_6\} \cup V'\}$, where $V' = V(G) \backslash \{u_1, u_2, \dots, u_8\}$ constitute an upper domatic partition of G. Hence, $D(G) = 5$.

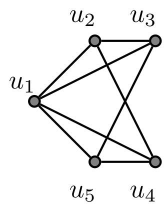

Figure 3: The graph $\mathcal{G}_5$.

Case 3: $G[N[u_1]] = \mathcal{G}_6$, as shown in Figure 4.
From the structure of $\mathcal{G}_6$, we note that there exist vertices $u_6, u_7 \in N^2(u_1)$ that are adjacent to u_3. Then, the sets $\{u_1\}, \{u_2\}, \{u_3\}, \{u_4, u_7\}$ along with $V' = V(G) \backslash \{u_1, u_2, u_3, u_4, u_7\}$ constitutes an upper domatic partition of cardinality five, proving that $D(G) = 5$.

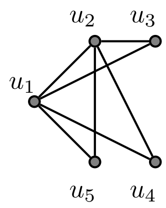

Figure 4: The graph $\mathcal{G}_6$.

Case 4: $G[N[u_1]] \in \{\mathcal{G}_7, \mathcal{G}_8\}$, as shown in Figure 5.
When $G[N[u_1]] = \mathcal{G}_7$ or $\mathcal{G}_8$, there is a vertex $u_6 \in N^2(u_1)$ such that u_3 is adjacent to u_6 but u_2 is not adjacent to u_6. Then $\pi = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_5, u_6\}, \{u_4\} \cup V'\}$, where $V' = V(G) \backslash \{u_1, u_2, \dots, u_6\}$, is an upper domatic partition of G proving that $D(G) = 5$.

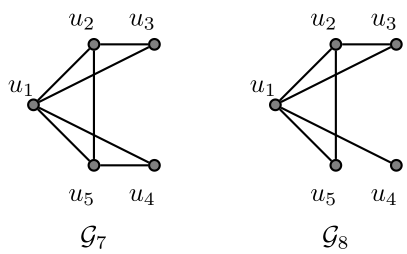

Figure 5: The graphs $\mathcal{G}_7$ and $\mathcal{G}_8$.

Case 5: $G[N[u_1]] \in \{\mathcal{G}_9, \mathcal{G}_{10}\}$, as shown in Figure 6.
In this case, suppose there is a vertex $u_6 \in N^2(u_1)$ such that u_2 and u_3 are adjacent to u_6. Then considering $V' = V(G) \backslash \{u_1, u_2, \dots, u_6\}$, we get an upper domatic partition $\pi = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4, u_6\}, \{u_5\} \cup V'\}$ of G proving that $D(G) = 5$. Else, there exist vertices $u_6, u_7 \in N^2(u_1)$ such that u_2 is adjacent to u_6 and u_3 is adjacent to u_7. Then $\pi = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4, u_6, u_7\}, \{u_5\} \cup V'\}$ is an upper domatic partition of G with $V' = V(G) \backslash \{u_1, u_2, \dots, u_7\}$ proving that $D(G) = 5$.

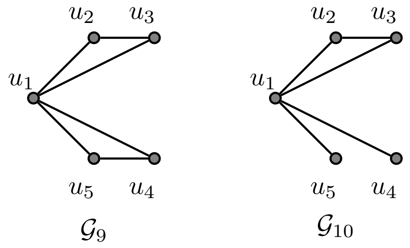

Figure 6: The graphs $\mathcal{G}_9$ and $\mathcal{G}_{10}$.

Case 6: $G[N[u_1]] = \mathcal{G}_{11}$, as shown in Figure 7.
Note that $\mathcal{G}_{11}$ is acyclic. If G contains a C_3, then G also contains one of the graphs $\mathcal{G}_i$, where $1 \leq i \leq 10$ as a subgraph. Hence it suffices to consider the case when G is C_3-free. We now examine the adjacency relation between the vertices of $N^1(u_1)$ and $N^2(u_1)$.

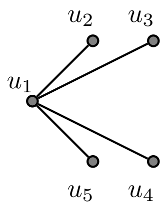

Figure 7: The graph $\mathcal{G}_{11}$.

Subcase 6.1: $|N^1(u_6) \cap N^1(u_1)| = 4$, for some $u_6 \in N^2(u_1)$.
In this case, if there is a vertex $u_7 \in N^2(u_1)$, such that u_7 is adjacent to at least two vertices in $N^1(u_1)$, say u_2 and u_3, then $\{\{u_1\}, \{u_2\}, \{u_3, u_6\}, \{u_4, u_7\}, \{u_5\} \cup V'\}$, where $V' = V(G)\backslash\{u_1, u_2, \ldots, u_7\}$, is an upper domatic partition of G proving that $D(G) = 5$. If every vertex in $N^2(u_1)$ except u_6 is adjacent to exactly one vertex in $N^1(u_1)$, then the partition $\{\{u_1\}, \{u_2\}, \{u_3, u_6\}, \{u_4, u_7, u_8\}, \{u_5\} \cup V'\}$, where $u_7, u_8 \in N^2(u_1)$ are adjacent to u_2 and u_3 respectively and $V' = V(G)\backslash\{u_1, u_2, \ldots, u_8\}$, is an upper domatic partition of G proving that $D(G) = 5$.

Subcase 6.2: $|N^1(u_6) \cap N^1(u_1)| = 3$, for some $u_6 \in N^2(u_1)$.
Let $u_4 \in N^1(u_1)$ be the vertex not adjacent to u_6. Consider another vertex $u_7 \in N^2(u_1)$ adjacent to the vertex u_2. Let $u_8 \in N^3(u_1)$ be a vertex adjacent to u_7. Then, the partition $\pi = \{\{u_1\}, \{u_2\}, \{u_3, u_7\}, \{u_4, u_6, u_8\}, \{u_5\} \cup V'\}$, where $V' = V(G)\backslash\{u_1, u_2, \ldots, u_8\}$, is an upper domatic partition of G proving that $D(G) = 5$.

Subcase 6.3: $|N^1(u_6) \cap N^1(u_1)| = 2$, for some $u_6 \in N^2(u_1)$.
Let u_3 and u_5 be the vertices in $N^1(u_1)$ not adjacent to u_6 implying that $u_2, u_4 \in N^1(u_1)$ are adjacent to u_6. If there exists a vertex $u_7 \in N^2(u_1)$ that is adjacent to both u_2 and u_3, then the partition $\pi = \{\{u_1\}, \{u_2\}, \{u_3, u_6\}, \{u_4, u_7\}, \{u_5\} \cup V'\}$, where $V' = V(G)\backslash\{u_1, u_2, \ldots, u_7\}$, is an upper domatic partition of G proving that $D(G) = 5$. Else if $u_7 \in N^2(u_1)$ is adjacent to u_2 but not u_3, then there exists $u_8 \in N^2(u_1)$ such that $u_3u_8 \in E(G)$, so that the partition $\{\{u_1\}, \{u_2\}, \{u_3, u_6\}, \{u_4, u_7, u_8\}, \{u_5\} \cup V'\}$, where $V' = V(G)\backslash\{u_1, u_2, \ldots, u_8\}$, is an upper domatic partition of G proving that $D(G) = 5$. In case u_4 is adjacent to both u_6 and u_7, the vertex u_8 is chosen such that $u_4u_8 \notin E(G)$.

Subcase 6.4: $|N^1(u) \cap N^1(u_1)| = 1$, for each $u \in N^2(u_1)$.
Consider the vertices $u_6, u_7 \in N^1(u_2)$, $u_8 \in N^1(u_3)$ and $u_9 \in N^1(u_6)$ along with $N^1(u_1) = \{u_2, u_3, u_4, u_5\}$. Then, either the partition $\{\{u_1\}, \{u_2\}, \{u_3, u_6\}, \{u_4, u_7, u_8\}, \{u_5, u_9\} \cup V'\}$ or the partition $\{\{u_1\}, \{u_2\}, \{u_3, u_6\}, \{u_4, u_7, u_8, u_9\}, \{u_5\} \cup V'\}$, where $V' = V(G)\backslash\{u_1, u_2, \ldots, u_9\}$, is an upper domatic partition of G of cardinality five, the former being the case when u_6 is adjacent to u_8.

Thus, in all the six cases we have demonstrated the existence of an upper domatic partition of cardinality five of the graph G, proving that $D(G) = 5$. This completes the proof. □

Corollary 2.3. *If G is a 4-regular graph such that $G \neq G_{3,2}$, then $D(G) = 5$.*

Remark 2.1. *There exist r-regular graphs of same order having different upper domatic numbers.*

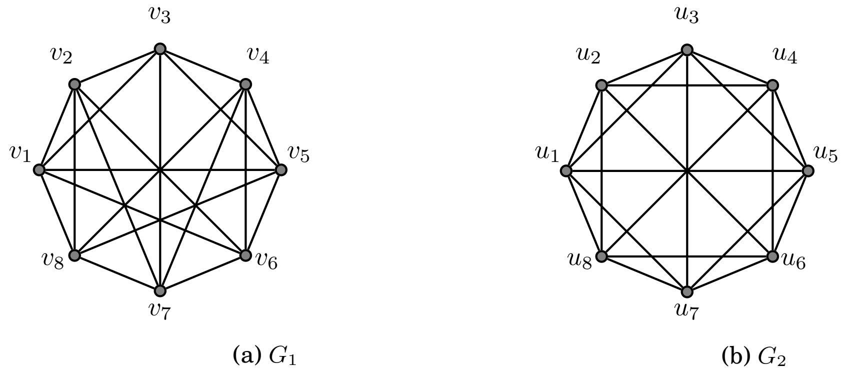

Figure 8: Regular graphs of same order with different upper domatic numbers.

The partition $\pi_1 = \{\{v_5, v_6, v_7\}, \{v_4, v_8\}, \{v_1\}, \{v_2\}, \{v_3\}\}$ is an upper domatic partition of the graph G_1. If $D(G_1) > 5$, then a D-partition of G_1 contains at least four singleton sets, but it can be verified that G_1 contains no K_4. Therefore, $D(G_1) = 5$. It can be easily seen that $D(G_2) = 6$ with the D-partition $\pi_2 = \{\{u_2, u_6\}, \{u_4, u_8\}, \{u_1\}, \{u_3\}, \{u_5\}, \{u_7\}\}$.

Remark 2.2. *$G_{3,2}$ is the graph with least order such that $D(G_{3,2}) < \delta(G_{3,2}) + 1$, which leads to the following conjecture.*

Conjecture. If a graph G is $G_{3,2}$-free, then $D(G) \geq \delta(G) + 1$.

Theorem 2.7. *If G is a 5-regular graph of girth at least five, then $D(G) = 6$.*

Proof. Let G be a 5-regular graph of girth at least five. We prove this result by employing a colouring protocol where the set of colour classes form an upper domatic partition of the graph G. Choose an induced cycle C_k, where $k \geq 5$. Colour four consecutive vertices of the cycle with colours $3, 1, 2$ and 3 respectively and the remaining vertices of the cycle with colour 6. Further, assign colours 4, 5 and 6 to vertices lying outside C_k and are adjacent to the vertices that have been coloured 1, 2 and 3 as shown in Figure 9.

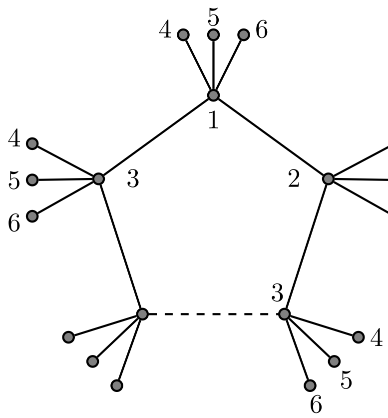

Figure 9: A subgraph of a 5-regular graph G of girth at least 5.

Next colour the neighbours of vertices coloured 4 in such a way that at least two of these vertices are assigned colours 5 and 6. Note that in case there is adjacency among the vertices that have been already coloured, this condition automatically follows. Finally, colour all the remaining vertices with colour 6. The resulting vertex partition of colour classes is an upper domatic partition of cardinality six, thus proving that $D(G) = 6$. $\square$

Next, we discuss the upper domatic number of the complement $\overline{C_n}$ of a cycle C_n. Note that the complement of a cycle of order n is an $(n-3)$-regular graph whose clique number is $\omega(\overline{C_n}) = \lfloor \frac{n}{2} \rfloor$.

Theorem 2.8. *For the cycle C_n, $D(\overline{C_n}) = \left\lfloor \frac{3n}{4} \right\rfloor$.*

Proof. Consider the cyclic labelling of the vertices of $\overline{C_n}$ where u_i is not adjacent to $u_{i-1 (\mathrm{mod}\ n)}$ and $u_{i+1 (\mathrm{mod}\ n)}$, see the labelling of $\overline{C_8}$ shown in Figure 10. Depending on the value of n, the vertex set is partitioned as follows.

Case 1: When $n (\mathrm{mod}\ 4) = 1$, partition the vertex set of $\overline{C_n}$ in the following manner, where $t = \left\lfloor \frac{n}{2} \right\rfloor$.

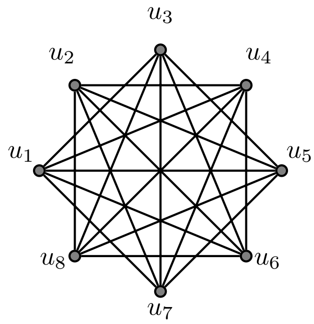

Figure 10: The graph $\overline{C_8}$.

$$V_i = \begin{cases} \{u_{2i-1}\}, & \text{if } 1 \le i \le t, \\ \{u_{2(i-t)}, u_{2(i-t)+t}\}, & \text{if } t+1 \le i \le \left\lfloor \frac{3n}{4} \right\rfloor - 1, \\ \{u_t, u_{n-1}, u_n\}, & \text{if } i = \left\lfloor \frac{3n}{4} \right\rfloor. \end{cases}$$

Case 2: When $n (\text{mod } 4) = 2$, the following vertex partition is considered, where $t = \left\lfloor \frac{n}{2} \right\rfloor$.

$$V_i = \begin{cases} \{u_{2i-1}\}, & \text{if } 1 \le i \le t, \\ \{u_{2(i-t)}, u_{2(i-t)+t-1}\}, & \text{if } t+1 \le i \le \left\lfloor \frac{3n}{4} \right\rfloor - 1, \\ \{u_{t-1}, u_{n-2}, u_n\}, & \text{if } i = \left\lfloor \frac{3n}{4} \right\rfloor. \end{cases}$$

Case 3: When $n (\text{mod } 4) = 3$, we consider the following partition where $t = \left\lfloor \frac{n}{2} \right\rfloor$.

$$V_i = \begin{cases} \{u_{2i-1}\}, & \text{if } 1 \le i \le t, \\ \{u_{2(i-t)}, u_{2(i-t)+t+1}\}, & \text{if } t+1 \le i \le \left\lfloor \frac{3n}{4} \right\rfloor - 1, \\ \{u_{t+1}, u_n\}, & \text{if } i = \left\lfloor \frac{3n}{4} \right\rfloor. \end{cases}$$

Case 4: When $n (\text{mod } 4) = 0$, partition $V(\overline{C_n})$ in the following manner, where $t = \left\lfloor \frac{n}{2} \right\rfloor$.

$$V_i = \begin{cases} \{u_{2i-1}\}, & \text{if } 1 \le i \le t, \\ \{u_{2(i-t)}, u_{2(i-t)+t}\}, & \text{if } t+1 \le i \le \left\lfloor \frac{3n}{4} \right\rfloor. \end{cases}$$

The partitions thus obtained in all these cases are upper domatic partitions. The sets $V_1, V_2, \ldots, V_t$ are singleton sets and contain the vertices $u_1, u_3, \ldots, u_{2t-1}$, respectively, which together induce a clique of $\overline{C_n}$. In $\overline{C_n}$, any set $\{u_i, u_j\}$, where $j \notin \{(i-2) \text{mod } n,\ (i+2) \text{mod } n\}$ is a dominating set, hence the remaining sets in the partition are dominating sets. Since $\omega(\overline{C_n}) = t$, from Theorem 2.3 it follows that

$$D(\overline{C_n}) = \left\lfloor \frac{3n}{4} \right\rfloor.$$

□

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References

[1] E. J. Cockayne, S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977) 247–261.
[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.
[3] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. McRae, N. Phillips, The upper domatic number of a graph, *AKCE Int. J. Graphs Comb.* **17** (2020) 139–148.
[4] S. T. Hedetniemi, P. Slater, T. W. Haynes, *Fundamentals of Domination in Graphs*, CRC Press, Boca Raton, 1998.
[5] L. C. Samuel, M. Joseph, New results on upper domatic number of graphs, *Commun. Comb. Optim.* **5** (2020) 125–137.
[6] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 1996.