A note on the identification numbers of caterpillars

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Abstract

A red-white coloring of a nontrivial connected graph $G$ of diameter $d$ is an assignment of red and white colors to the vertices of $G$ where at least one vertex is colored red. Associated with each vertex $v$ of $G$ is a $d$-vector, called the code of $v$, whose $i$th coordinate is the number of red vertices at distance $i$ from $v$. A red-white coloring of $G$ for which distinct vertices have distinct codes is called an identification coloring or ID-coloring of $G$. A graph $G$ possessing an ID-coloring is an ID-graph. The minimum number of red vertices among all ID-colorings of an ID-graph $G$ is the identification number or ID-number of $G$. A caterpillar is a tree of order 3 or more, the removal of whose leaves produces a path. A caterpillar possessing an ID-coloring is an ID-caterpillar. In this note, we characterize all ID-caterpillars, determine all possible values of the ID-numbers of ID-caterpillars, and show that each value is realizable.

Keywords: distance; ID-coloring; ID-number; ID-graph; caterpillar.

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1. Introduction

Let $G$ be a nontrivial connected graph. The distance $d(u, v)$ between vertices $u$ and $v$ in $G$ is the minimum number of edges in a $u - v$ path in $G$. The eccentricity $e(v)$ of a vertex $v$ of $G$ is the distance between $v$ and a vertex farthest from $v$ in $G$. The diameter $\text{diam}(G)$ of $G$ is the largest eccentricity among the vertices of $G$. Equivalently, the diameter of $G$ is the greatest distance between any two vertices of $G$.

Let $G$ be a connected graph of diameter $d \geq 2$ and let there be given a red-white vertex coloring $c$ of $G$ where at least one vertex is colored red. That is, the color $c(v)$ of a vertex $v$ in $G$ is either red or white and $c(v)$ is red for at least one vertex $v$ of $G$. With each vertex $v$ of $G$, there is associated a $d$-vector $\vec{d}(v) = (a_1, a_2, \ldots, a_d)$ called the code of $v$ corresponding to $c$, where the $i$th coordinate $a_i$ is the number of red vertices at distance $i$ from $v$ for $1 \leq i \leq d$. If distinct vertices of $G$ have distinct codes, then $c$ is called an identification coloring or ID-coloring of $G$. A graph possessing an identification coloring is an ID-graph. The minimum number of red vertices among all ID-colorings of an ID-graph $G$ is the identification number or ID-number $\text{ID}(G)$ of $G$. These concepts were introduced by Gary Chartrand and first studied in [1]. In this note, we study a well-known class of trees, namely caterpillars.

A caterpillar $T$ is a tree of order 3 or more, the removal of whose leaves produces a path called the spine of $T$. A star is therefore a caterpillar with a trivial spine and a double star (a tree of diameter 3) is a caterpillar whose spine is the path $P_2$ of order 2. A caterpillar possessing an ID-coloring is therefore an ID-caterpillar. Here, we determine all those caterpillars that are ID-caterpillars and their possible ID-numbers. For this purpose, it is useful to present some results obtained in [1, 2].

For an integer $t \geq 2$, the members of a set $S$ of $t$ vertices in a graph $G$ are called $t$-tuplets (twins if $t = 2$ and triplets if $t = 3$) if either (1) $S$ is an independent set in $G$ and every two vertices in $S$ have the same neighborhood or (2) $S$ is a clique, that is the subgraph $G[S]$ induced by $S$ is complete and every two vertices in $S$ have the same closed neighborhood.

**Proposition 1.1.** Let $G$ be a connected graph with twins $u$ and $v$. If $c$ is an ID-coloring of $G$, then $c(u) \neq c(v)$. Consequently, if $G$ is an ID-graph, then $G$ is triplet-free.

**Proposition 1.2.** Let $c$ be a red-white coloring of a connected graph $G$ where there is at least one vertex of each color. If $x$ is a red vertex and $y$ is a white vertex, then $\vec{d}(x) \neq \vec{d}(y)$.

**Theorem 1.1.** For a positive integer $r$, there exists a connected graph $G$ with $\text{ID}(G) = r$ if and only if $r \neq 2$.

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Theorem 1.2. A nontrivial connected graph \( G \) has \( \text{ID}(G) = 1 \) if and only if \( G \) is a path.

Theorem 1.3. A connected graph \( G \) of diameter 2 is an ID-graph if and only if \( G = P_3 \).

Since every caterpillar of maximum degree 5 or more contains triplets, the following observation is an immediate consequence of Proposition 1.1.

Observation 1.1. No caterpillar of maximum degree 5 or more is an ID-tree.

By Theorems 1.2 and 1.3 and Observation 1.1, we need only consider caterpillars of diameter 3 or more and maximum degree 3 or 4. In this note, we present the following results on ID-caterpillars and their ID-numbers.

1. A caterpillar is an ID-caterpillar if and only if it is triplet-free.
2. If \( T \) is a caterpillar of diameter 3 or more and maximum degree 3, then \( \text{ID}(T) = 3 \).
3. If \( T \) be a \( k \)-twin caterpillar (a caterpillar with exactly \( k \) pairs of twins) of diameter 4 or more and maximum degree 4, then \( \max\{3, k\} \leq \text{ID}(T) \leq k + 3 \). Furthermore, for each pair \((k, t)\) of integers where \( k \geq 0 \) and \( t \in \{0, 1, 2, 3\} \) such that \( k + t \geq 3 \), there is a \( k \)-twin caterpillar \( T_{k,t} \) for which \( \text{ID}(T_{k,t}) = k + t \).

2. Which caterpillars are ID-graphs?

We begin with caterpillars of diameter 3 or more and maximum degree 3.

Theorem 2.1. If \( T \) is a caterpillar of diameter 3 or more and maximum degree 3, then \( T \) is an ID-caterpillar and \( \text{ID}(T) = 3 \).

Proof. Let \( T \) be a caterpillar of diameter \( d \geq 3 \) and maximum degree 3. By Theorems 1.1 and 1.2, it suffices to show that \( T \) has an ID-coloring with exactly three red vertices. If \( d = 3 \), then \( T \) is a double star of order 5 or 6. Since these two double stars have ID-colorings with exactly three red vertices (as shown in Figure 1), we may assume that \( d \geq 4 \).

![Figure 1: ID-colorings of two double stars.](image1)

Let \( P = (v_0, v_1, \ldots, v_d) \) be a longest path in \( T \), where \( d = \text{diam}(T) \geq 4 \). For \( 1 \leq i \leq d - 1 \), if \( \text{deg} v_i = 3 \), then let \( u_i \) be the end-vertex of \( T \) that is not in \( P \) and adjacent to \( v_i \). Define a red-white coloring \( c \) of \( T \) by assigning the color red to \( v_i \) if \( i = 0, 1, d \) and the color white to the remaining vertices of \( T \). This red-white coloring is illustrated for the caterpillar of diameter 8 in Figure 2. We show that \( c \) is an ID-coloring of \( T \). Let \( x \) and \( y \) be two distinct vertices of \( T \). By Proposition 1.2, we may assume that \( x \) and \( y \) have the same color in \( T \). Since (i) the first and the last coordinates of \( \vec{d}(v_i) \) are both 1, (ii) the first coordinate of \( \vec{d}(v_1) \) is 1 and the last coordinate of \( \vec{d}(v_1) \) is 0, and (iii) the first coordinate of \( \vec{d}(v_d) \) is 0 and the last coordinate of \( \vec{d}(v_d) \) is 1, it follows that the three red vertices \( v_0, v_1 \) and \( v_d \) have distinct codes. Thus, we may assume that \( x \) and \( y \) are white vertices. Let \( \vec{d}(x) = (a_1, a_2, \ldots, a_d) \) and \( \vec{d}(y) = (b_1, b_2, \ldots, b_d) \).

![Figure 2: An ID-coloring of a caterpillar of diameter 8.](image2)

First, observe that for each vertex \( z \) of \( T \), the eccentricity of \( z \) is \( e(z) = \max\{d(z, v_0), d(z, v_d)\} \). Since \( v_0 \) and \( v_d \) are red vertices of \( T \), it follows that the \( e(z) \)th coordinate of \( \vec{d}(z) \) is the final coordinate of \( \vec{d}(z) \) that is not 0. Consequently, if \( e(x) \neq e(y) \), say \( e(x) < e(y) = s \), then \( a_s = 0 \) but \( b_s \neq 0 \), which implies that \( \vec{d}(x) \neq \vec{d}(y) \). Hence, we may assume that \( e(x) = e(y) = s \). We consider three cases, according to the location of \( x \) and \( y \).

Case 1. \( x = u_i \) and \( y = u_j \) where \( 1 \leq i < j \leq d - 1 \). Since \( e(u_i) = e(u_j) = s \) and \( 1 \leq i < j \leq d - 1 \), it follows that \( e(u_i) = d(u_i, v_d) \) and \( e(u_j) = d(u_j, v_0) \). Then \( b_{s-1} = 1 \) (since \( d(u_j, v_1) = s - 1 \)) and \( a_{s-1} = 0 \). Hence, \( \vec{d}(x) \neq \vec{d}(y) \).
Case 2. Let \( v_i = x \) and \( v_j = y \) where \( 2 \leq i < j \leq d - 1 \). Since \( e(v_i) = e(v_j) = s \) and \( 2 \leq i < j \leq d - 1 \), it follows that \( e(v_i) = d(v_i, v_d) \) and \( e(v_j) = d(v_j, v_0) \). Then \( b_{s-1} = 1 \) (since \( d(v_j, v_i) = s - 1 \)) and \( a_{s-1} = 0 \). Hence, \( d(x) \neq d(y) \).

Case 3. Let \( v_i = u \) where \( 1 \leq i \leq d - 1 \) and \( v_j = v \) where \( 2 \leq i < j \leq d - 1 \). Since \( e(u_i) = e(v_j) = s \), it follows that \( y \in \{v_{i-1}, v_{i+1}\} \).

If \( y = v_{i-1} \), then \( d(u_i, v_d) = d(v_{i-1}, v_d) = s = d - i + 1 \). Since \( d(v_i, v_1) = i - 2 \) and \( d(u_i, v_1) = i \), it follows that \( a_{s-2} = 0 \) and \( b_{s-1} = 1 \). Hence, \( d(x) \neq d(y) \). If \( y = v_{i+1} \), then \( d(u_i, v_0) = d(v_{i+1}, v_0) = s = i + 1 \) and \( d(u_i, v_1) = d(v_{i+1}, v_1) \). If \( a_s \neq b_s \) or \( a_{s-1} \neq b_{s-1} \), then \( d(x) \neq d(y) \). Thus, we may assume that \( a_s = b_s \) and \( a_{s-1} = b_{s-1} \). Since \( d(u_i, v_d) = d(v_{i+1}, v_d) + 2 \), it follows that \( a_s = b_s = 1 \) and \( a_{s-1} = b_{s-1} = 1 \). This implies that \( b_{d-i-1} = 1 \), where \( d - i - 1 = d(v_{i+1}, v_d) \), and \( a_{d-i-1} = 0 \), implying that \( d(x) \neq d(y) \).

We now consider triplet-free caterpillars of diameter at least 4 and maximum degree 4. A triplet-free caterpillar with exactly \( k \) pairs of twins is called a \( k \)-twin caterpillar. Figure 3 shows a 2-twin caterpillar of diameter 4 and a 4-twin caterpillar of diameter 5. Next we show that every triplet-free caterpillar of diameter at least 4 and maximum degree 4 is an ID-caterpillar and present bounds for the ID-number of a \( k \)-twin caterpillar, which shows that its ID-number is one of four numbers (in terms of \( k \)).

![Figure 3: A 2-twin caterpillar and a 4-twin caterpillar.](image)

**Theorem 2.2.** Let \( T \) be a caterpillar of diameter \( d \geq 4 \) and maximum degree 4. If \( T \) is a \( k \)-twin for some positive integer \( k \), then \( T \) is an ID-caterpillar and

\[
\text{max}\{3, k\} \leq \text{ID}(T) \leq k + 3.
\]

**Proof.** Since \( \text{ID}(T) \geq \text{max}\{3, k\} \) by Proposition 1.1 and Theorems 1.1–1.3, it remains to show that \( \text{ID}(T) \leq k + 3 \). Let \( P = (v_1, v_2, \ldots, v_{d-1}) \) be the spine of \( T \). For each integer \( i \) with \( 1 \leq i \leq d - 1 \), let \( L_i \) be the set of end-vertices of \( T \) that are adjacent to \( v_i \). Then \( 1 \leq |L_i| \leq 3 \) for \( i = 1, d - 1 \) and \( 0 \leq |L_i| \leq 2 \) for \( 2 \leq i \leq d - 2 \). We may assume that \( \text{deg} v_2 \geq \text{deg} v_{d-2} \). Define a red-white coloring \( c \) of \( T \) by

1. assigning the color red to \( v_1 \), exactly one vertex \( u_1 \) in \( L_1 \), exactly one vertex \( u_{d-1} \) in \( L_{d-1} \), and exactly one vertex \( u_i \) in \( L_i \) if \( |L_i| = 2 \) for \( 2 \leq i \leq d - 2 \) and
2. assigning the color white to the remaining vertices of \( T \).

Thus, for \( 1 \leq i \leq d - 1 \), the red end-vertex adjacent to \( v_i \) (should it exist) is denoted by \( u_i \), and the white end-vertex adjacent to \( v_i \) (should it exist) is denoted by \( w_i \). In particular, the red vertex \( u_1 \) is adjacent to the red vertex \( v_1 \) and the red vertex \( u_{d-1} \) is adjacent to the white vertex \( v_{d-1} \). Let \( r \) denote the number of red vertices in \( T \). Then

**Case 1.** \( x \) and \( y \) are both red. Observe that

- \( u_1 \) and \( v_1 \) are the only red vertices whose first coordinate is 1 and \( e(u_1) > e(v_1) \);
- for \( 2 \leq i \leq d - 1 \), if \( u_i \) exists, then the first coordinate of \( d(u_i) \) is 0 and \( e(u_{d-1}) > e(u_i) \) for \( 2 \leq i \leq d - 2 \).
Thus, we may assume that \( x = u_i \) and \( y = u_j \) where \( 2 \leq i < j \leq d - 2 \). Since \( e(u_i) = e(u_j) = s \) and \( 2 \leq i < j \leq d - 2 \), it follows that \( s = d(u_i, u_{d-1}) = d(u_j, u_1) \). Since (1) \( d(u_i, v_1) = d(u_i, v_{d-1}) = s - 1 \), (2) \( v_1 \) is red and \( v_{d-1} \) is white, and (3) \( \deg v_2 \geq \deg v_{d-2} \), it follows that \( b_{s-1} > a_{s-1} \). Hence, \( \vec{d}(x) \neq \vec{d}(y) \).

Case 2. \( x \) and \( y \) are both white. Observe that

- \( u_1 \) (should it exist) is the only white vertex whose code begins with 111
- \( u_{d-1} \) (should it exist) is only white peripheral vertex whose code begins with 0, and
- \( u_2 \) (should it exist) is the only white vertex whose code begins with 02.

Thus, we may assume that \( x, y \in \{ u_i : 3 \leq i \leq d - 2 \} \cup \{ v_i : 2 \leq i \leq d - 1 \} \). We consider three cases, according to the location of \( x \) and \( y \).

Subcase 2.1. \( x = u_i \) and \( y = u_j \) where \( 3 \leq i < j \leq d - 2 \). Since \( e(u_i) = e(u_j) = s \) and \( 3 \leq i < j \leq d - 2 \), it follows that \( s = d(u_i, u_{d-1}) = d(u_j, u_1) \). Since (1) \( \deg v_2 \geq \deg v_{d-2} \), (2) \( v_1 \) is red and \( v_{d-1} \) is white, and (3) \( d(u_i, v_1) = d(u_j, v_1) = d(u_j, u_2) = s - 1 \) (should \( u_2 \) exist), we have \( b_{s-1} > a_{s-1} \). Hence, \( \vec{d}(x) \neq \vec{d}(y) \).

Subcase 2.2. \( x = v_i \) and \( y = v_j \) where \( 2 \leq i < j \leq d - 1 \). Since \( e(v_i) = e(v_j) = s \) and \( 2 \leq i < j \leq d - 1 \), it follows that \( s = d(v_i, u_{d-1}) = d(v_j, u_1) \). Since (1) \( \deg v_2 \geq \deg v_{d-2} \), (2) \( v_1 \) is red and \( v_{d-1} \) is white, and (3) \( d(v_j, v_1) = d(v_j, u_2) = s - 1 \) (should \( u_2 \) exist), we have \( b_{s-1} > a_{s-1} \). Hence, \( \vec{d}(x) \neq \vec{d}(y) \).

Subcase 2.3. \( x = u_i \) where \( 3 \leq i \leq d - 2 \) and \( y = v_j \) where \( 2 \leq i < j \leq d - 1 \). Since \( e(u_i) = e(v_j) = s \), it follows that \( y \in \{ u_{i+1} \} \). Since \( v_i = u_{i+1} \) where \( i \geq 3 \), then \( d(u_i, u_{d-1}) = d(v_i, u_{d-1}) = s \). Since (1) \( \deg v_2 \geq \deg v_{d-2} \) and (2) \( d(u_i, v_1) = d(u_i, u_2) = i - 2 \) (should \( u_2 \) exist) and \( d(u_i, v_1) = i \), it follows that \( b_{i-2} > a_{i-2} = 1 \). Hence, \( \vec{d}(x) \neq \vec{d}(y) \).

By Theorem 2.2, if \( T \) is a \( k \)-twin caterpillar, then \( \max\{3, k\} \leq \ID(T) \leq k + 3 \). In fact, every integer between \( \max\{3, k\} \) and \( k + 3 \) is realizable as the ID-number of some \( k \)-twin caterpillar, as we show next.

Theorem 2.3. For each pair \((k, t)\) of integers where \( k \geq 0 \) and \( t \in \{0, 1, 2, 3\} \) such that \( k + t \geq 3 \), there is a \( k \)-twin caterpillar \( T \) for which \( \ID(T) = k + t \).

Proof. We verify the following four statements.

1. For each integer \( k \geq 0 \), there exists a \( k \)-twin caterpillar \( T \) with \( \ID(T) = k + 3 \).
2. For each integer \( k \geq 1 \), there exists a \( k \)-twin caterpillar \( T \) with \( \ID(T) = k + 2 \).
3. For each integer \( k \geq 2 \), there exists a \( k \)-twin caterpillar \( T \) with \( \ID(T) = k + 1 \).
4. For each integer \( k \geq 3 \), there exists a \( k \)-twin caterpillar \( T \) with \( \ID(T) = k \).

We provide a complete proof for Statements 1 and 2 and provide an outline of a proof for Statements 3 and 4.

First, we verify Statement 1. By Theorem 2.1, if \( T \) is a caterpillar of diameter 3 or more and maximum degree 3, then \( \ID(T) = 3 \). Thus, the statement is true for \( k = 0 \) and so we may assume that \( k \geq 1 \). Let \( T \) be a caterpillar of diameter \( d = k + 3 \geq 4 \) and let \( P \) be the spine of \( T \) such that each end-vertex of \( P \) is adjacent to exactly one end-vertex and each interior vertex of \( P \) is adjacent to exactly two end-vertices. Thus, \( T \) contains exactly \( k = d - 3 \) twins. We show that \( \ID(T) = d = k + 3 \). Let \( (v_0, v_1, \ldots, v_d) \) be the longest path in \( T \) and so \( P = (v_1, v_2, \ldots, v_{d-1}) \) is the spine of \( T \). For each integer \( i \) with \( 2 \leq i \leq d - 2 \), let \( u_i \) and \( v_i \) be the two end-vertices of \( T \) that are adjacent to \( v_i \). Since \( T \) contains exactly \( d - 3 \) twins, it follows by Theorem 2.2 that \( \ID(T) \leq d \). Thus, it remains to show that \( \ID(T) \geq d \). Assume, to the contrary, that \( \ID(T) \leq d - 1 \) and let \( c \) be an ID-coloring of \( T \) with exactly \( \ID(T) \) red vertices. Since \( u_i \) and \( v_i \) are twins in \( T \) for \( 2 \leq i \leq d - 2 \), we may assume that \( c(u_i) \) is red and \( c(v_i) \) is white. Since \( d(v_1, z) = d(v_2, z) \) for each \( z \in V(T) - \{v_0\} \), it follows that if \( v_0 \) and \( v_1 \) are both white, then \( \vec{d}(v_1) = \vec{d}(v_2) \), which is impossible. Thus, at least one of \( v_0 \) and \( v_1 \) is red. If \( v_0 \) is white and \( v_1 \) is red, then \( \vec{d}(v_1) = \vec{d}(u_2) \). Thus, \( v_0 \) must be red. Similarly, \( v_0 \) must be red. Hence, \( \ID(T) = d - 1 \) and \( v_0, v_d, \) and \( u_i \), where \( 2 \leq i \leq d - 2 \), are the \( d - 1 \) red vertices of \( T \). However then, \( \vec{d}(v_0) = \vec{d}(v_{d-1}) \), for example, which is impossible. Therefore, \( \ID(T) = d = k + 3 \) and so Statement 1 holds.
To verify Statement 2, let $T$ be a caterpillar of diameter $d = k + 4 \geq 5$ where $(v_0, v_1, \ldots, v_d)$ be the longest path in $T$ such that $\deg v_i = 2$ for $i = 1, d - 2, d - 1$ and $\deg v_i = 4$ for $2 \leq i \leq d - 3$. Thus, $T$ contains exactly $k = d - 4 \geq 1$ twins. We show that $\ID(T) = d - 2 = k + 2 \geq 3$. For each integer $i$ with $2 \leq i \leq d - 3$, let $u_i$ and $w_i$ be the two end-vertices of $T$ that are adjacent to $v_i$. First, we show that $\ID(T) \geq k + 2$. Let $c$ be an $\ID$-coloring of $T$. Since $u_i$ and $w_i$ are twins in $T$ for $2 \leq i \leq d - 3$, we may assume that $c(u_i)$ is red and $c(w_i)$ is white. Since $d(v_i, z) = d(u_i, z)$ for each $z \in V(T) - \{v_0\}$, it follows that if $v_0$ and $v_1$ are both white, then $d(v_1) = d(u_2)$, which is impossible. Thus, at least one of $v_0$ and $v_1$ must be red. Similarly, at least one vertex in $\{v_{d-2}, v_{d-1}, v_d\}$ must be red. Thus, $\ID(T) \geq k + 2$.

Next, we show that $T$ has an $\ID$-coloring with exactly $k + 2$ red vertices. Define a red-white coloring $c$ of $T$ by (1) assigning the color red to $v_0$, $v_d$, and $u_i$ for $2 \leq i \leq d - 3$ and (2) assigning the color white to the remaining vertices of $T$. Thus, $T$ has exactly $k + 2$ red vertices. We show that $c$ is an $\ID$-coloring of $T$. Let $x$ and $y$ be two distinct vertices of the same color in $T$, where $d(x) = (a_1, a_2, \ldots, a_d)$ and $d(y) = (b_1, b_2, \ldots, b_d)$. First, observe that for each vertex $z$ of $T$, the eccentricity of $z$ is $e(z) = \max\{d(z, u_1), d(z, u_{d-1})\}$. Since $u_1$ and $u_{d-1}$ are red vertices of $T$, it follows that the $e(z)$th coordinate of $d(z)$ is the final coordinate of $d(z)$ that is not 0. Consequently, if $e(x) \neq e(y)$, say $e(x) < e(y) = s$, then $a_s = 0$ but $b_s \neq 0$, which implies that $d(x) \neq d(y)$. Hence, we may assume that $e(x) = e(y) = s$. We consider two cases, according to whether $x$ and $y$ are both red or both white.

**Case 1.** $x$ and $y$ are both red. Since (1) $v_0$ and $v_d$ are the only red peripheral vertices and (2) $d(v_d)$ begins with 001, and $d(v_d)$ begins with 000, we may assume that $x = u_i$ and $y = u_j$ where $2 \leq i < j \leq d - 2$. Since $e(u_i) = e(u_j) = s$ and $2 \leq i < j \leq d - 2$, it follows that $s = d(u_i, v_d) = d(u_j, v_0)$. Then $b_{s-1} = 1$ and $a_{s-1} = 0$. Hence, $d(x) \neq d(y)$.

**Case 2.** $x$ and $y$ are both white. Observe that

- $v_{d-2}$ is the only white vertex whose code begins with 02,
- $v_{d-1}$ is the only white vertex whose code begins with 10,
- for $2 \leq i \leq d - 3$, the first coordinate of $d(v_i)$ is 0 and for $1 \leq j \leq d - 3$, the first coordinate of $d(v_j)$ is 1, which implies that $d(v_i) \neq d(v_j)$, and
- $e(v_i) > e(v_j)$ for $2 \leq i \leq d - 3$, which implies that $d(v_i) \neq d(v_j)$.

Thus, we may assume that $x = w_i$ and $y = w_j$ where $2 \leq i < j \leq d - 3$ or $x = v_i$ and $y = v_j$ where $2 \leq i < j \leq d - 3$. First, suppose that $x = w_i$ and $y = w_j$ where $2 \leq i < j \leq d - 3$. Since $e(w_i) = e(w_j) = s$ and $2 \leq i < j \leq d - 3$, it follows that $s = d(w_i, v_d) = d(u_j, v_0)$. Then $b_{s-1} > a_{s-1}$ (since $d(u_j, u_2) = s - 1$). Hence, $d(x) \neq d(y)$. Next, suppose that $x = v_i$ and $y = v_j$ where $2 \leq i < j \leq d - 3$. Since $e(v_i) = e(v_j) = s$ and $2 \leq i < j \leq d - 3$, it follows that $s = d(v_i, v_d) = d(v_j, v_0)$. Then $b_{s-1} > a_{s-1}$ (since $d(v_j, u_2) = s - 1$). Hence, $d(x) \neq d(y)$.

Hence, $c$ is an $\ID$-coloring of $T$ with exactly $k + 2$ red vertices and so $\ID(T) \leq k + 2$. Therefore, $\ID(T) = k + 2$ and Statement 2 holds.

We now provide an outline of a proof of Statement 3. By Theorem 2.1, every caterpillar of diameter 3 or more and maximum degree 3 has $\ID$-number 3. Thus, if $T$ is a caterpillar of diameter 3 or more and maximum degree 3 such that each of two end-vertices of its spine is adjacent to two end-vertices, then $T$ has $k = 2$ twins such that $\ID(T) = 3$. Consequently, we may assume that $k \geq 3$. Let $T$ be a caterpillar of diameter $d = k + 1 \geq 4$ and let $P$ be the spine of $T$ such that every vertex of $P$ is adjacent to exactly two end-vertices. Thus, $T$ contains exactly $k = d - 1$ twins. We show that $\ID(T) = d = k + 1$. Let $P=(v_1, v_2, \ldots, v_{d-1})$ be the spine of $T$. For each integer $i$ with $1 \leq i \leq d - 1$, let $u_i$ and $w_i$ be the two end-vertices of $T$ that are adjacent to $v_i$. Since $u_i$ and $w_i$ are twins in $T$ for $1 \leq i \leq d - 1$, it follows by Proposition 1.1 that $\ID(T) \geq d - 1$. Assume, to the contrary, that there is an $\ID$-coloring of $T$ with exactly $d - 1$ red vertices. We may assume that $u_i$ is red for $1 \leq i \leq d - 1$. However then, $v_1$ and $v_{d-1}$ have the same code, for example, which is impossible. Therefore, $\ID(T) \geq d$. To show $\ID(T) \leq d$, we define a red-white coloring $c$ of $T$ by (1) assigning the color red to $v_1$ and $w_i$ for $1 \leq i \leq d - 1$ and (2) assigning the color white to the remaining vertices of $T$. Thus, $T$ has exactly $d$ red vertices. It can be shown that $c$ is an $\ID$-coloring of $T$. Therefore, $\ID(T) = d = k + 1$.

To provide an outline of a proof Statement 4, let $T$ be a caterpillar of diameter $d = k + 2 \geq 5$ and let $P=(v_1, v_2, \ldots, v_{d-1})$ be the spine of $T$ such that each vertex of $P$ is adjacent to exactly two end-vertices except $v_{d-2}$ having degree 2. Thus, $T$ contains exactly $k = d - 2$ twins. We show that $\ID(T) = d - 2 = k$. For each integer $i$ with $1 \leq i \leq d - 1$ and $i \neq d - 2$, let $u_i$ and $w_i$ be the two end-vertices of $T$ that are adjacent to $v_i$. Since $T$ contains exactly $d - 2$ twins, it follows that $\ID(T) \geq d - 2$. To show that $\ID(T) \leq d - 2$, we define a red-white coloring $c$ of $T$ by (1) assigning the color red to $u_i$ for $1 \leq i \leq d - 3$ and $w_{d-1}$ and (2) assigning the color white to the remaining vertices of $T$. It can be shown that $c$ is an $\ID$-coloring of $T$ with exactly $d - 2$ red vertices. Therefore, $\ID(T) = d - 2 = k$. □
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