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Research Article

A note on the identification numbers of caterpillars

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Abstract

A red-white coloring of a nontrivial connected graph G of diameter d is an assignment of red and white colors to the vertices of G where at least one vertex is colored red. Associated with each vertex v of G is a d-vector, called the code of v, whose ith coordinate is the number of red vertices at distance i from v. A red-white coloring of G for which distinct vertices have distinct codes is called an identification coloring or ID-coloring of G. A graph G possessing an ID-coloring is an ID-graph. The minimum number of red vertices among all ID-colorings of an ID-graph G is the identification number or ID-number of G. A caterpillar is a tree of order 3 or more, the removal of whose leaves produces a path. A caterpillar possessing an ID-coloring is an ID-caterpillar. In this note, we characterize all ID-caterpillars, determine all possible values of the ID-numbers of ID-caterpillars, and show that each value is realizable.

Keywords: distance; ID-coloring; ID-number; ID-graph; caterpillar.

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1. Introduction

Let G be a nontrivial connected graph. The *distance* $d(u, v)$ between vertices u and v in G is the minimum number of edges in a $u - v$ path in G. The *eccentricity* $e(v)$ of a vertex v of G is the distance between v and a vertex farthest from v in G. The *diameter* $\operatorname{diam}(G)$ of G is the largest eccentricity among the vertices of G. Equivalently, the diameter of G is the greatest distance between any two vertices of G.

Let G be a connected graph of diameter $d \geq 2$ and let there be given a red-white vertex coloring c of G where at least one vertex is colored red. That is, the *color* $c(v)$ of a vertex v in G is either red or white and $c(v)$ is red for at least one vertex v of G. With each vertex v of G, there is associated a *d-vector* $\vec{d}(v) = (a_1, a_2, \ldots, a_d)$ called the *code* of v corresponding to c, where the ith coordinate a_i is the number of red vertices at distance i from v for $1 \leq i \leq d$. If distinct vertices of G have distinct codes, then c is called an *identification coloring* or *ID-coloring* of G. A graph possessing an identification coloring is an *ID-graph*. The minimum number of red vertices among all ID-colorings of an ID-graph G is the *identification number* or *ID-number* $\mathrm{ID}(G)$ of G. These concepts were introduced by Gary Chartrand and first studied in [1]. In this note, we study a well-known class of trees, namely caterpillars.

A *caterpillar* T is a tree of order 3 or more, the removal of whose leaves produces a path called the *spine* of T. A star is therefore a caterpillar with a trivial spine and a double star (a tree of diameter 3) is a caterpillar whose spine is the path P_2 of order 2. A caterpillar possessing an ID-coloring is therefore an *ID-caterpillar*. Here, we determine all those caterpillars that are ID-caterpillars and their possible ID-numbers. For this purpose, it is useful to present some results obtained in [1,2].

For an integer $t \geq 2$, the members of a set S of t vertices in a graph G are called *t-tuplets* (*twins* if $t = 2$ and *triplets* if $t = 3$) if either (1) S is an independent set in G and every two vertices in S have the same neighborhood or (2) S is a clique, that is the subgraph $G[S]$ induced by S is complete and every two vertices in S have the same closed neighborhood.

Proposition 1.1. *Let G be a connected graph with twins u and v. If c is an ID-coloring of G, then $c(u) \neq c(v)$. Consequently, If G is an ID-graph, then G is triplet-free.*

Proposition 1.2. *Let c be a red-white coloring of a connected graph G where there is at least one vertex of each color. If x is a red vertex and y is a white vertex, then $\vec{d}(x) \neq \vec{d}(y)$.*

Theorem 1.1. *For a positive integer r, there exists a connected graph G with $\mathrm{ID}(G) = r$ if and only if $r \neq 2$.*

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Theorem 1.2. *A nontrivial connected graph G has $\mathrm{ID}(G) = 1$ if and only if G is a path.*

Theorem 1.3. *A connected graph G of diameter 2 is an ID-graph if and only if $G = P_3$.*

Since every caterpillar of maximum degree 5 or more contains triplets, the following observation is an immediate consequence of Proposition 1.1.

Observation 1.1. *No caterpillar of maximum degree 5 or more is an ID-tree.*

By Theorems 1.2 and 1.3 and Observation 1.1, we need only consider caterpillars of diameter 3 or more and maximum degree 3 or 4. In this note, we present the following results on ID-caterpillars and their ID-numbers.

1. A caterpillar is an ID-caterpillar if and only if it is triplet-free.
2. If T is a caterpillar of diameter 3 or more and maximum degree 3, then $\mathrm{ID}(T) = 3$.
3. If T be a k-twin caterpillar (a caterpillar with exactly k pairs of twins) of diameter 4 or more and maximum degree 4, then $\max\{3, k\} \le \mathrm{ID}(T) \le k + 3$. Furthermore, for each pair (k, t) of integers where $k \ge 0$ and $t \in \{0, 1, 2, 3\}$ such that $k + t \ge 3$, there is a k-twin caterpillar $T_{k,t}$ for which $\mathrm{ID}(T_{k,t}) = k + t$.

2. Which caterpillars are ID-graphs?

We begin with caterpillars of diameter 3 or more and maximum degree 3.

Theorem 2.1. *If T is a caterpillar of diameter 3 or more and maximum degree 3, then T is an ID-caterpillar and $\mathrm{ID}(T) = 3$.*

Proof. Let T be a caterpillar of diameter $d \ge 3$ and maximum degree 3. By Theorems 1.1 and 1.2, it suffices to show that T has an ID-coloring with exactly three red vertices. If $d = 3$, then T is a double star of order 5 or 6. Since these two double stars have ID-colorings with exactly three red vertices (as shown in Figure 1), we may assume that $d \ge 4$.

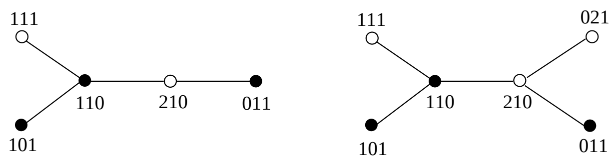

Figure 1: ID-colorings of two double stars.

Let $P = (v_0, v_1, \ldots, v_d)$ be a longest path in T, where $d = \mathrm{diam}(T) \ge 4$. For $1 \le i \le d-1$, if $\deg v_i = 3$, then let u_i be the end-vertex of T that is not in P and adjacent to v_i. Define a red-white coloring c of T by assigning the color red to v_i if $i = 0, 1, d$ and the color white to the remaining vertices of T. This red-white coloring is illustrated for the caterpillar of diameter 8 in Figure 2. We show that c is an ID-coloring of T. Let x and y be two distinct vertices of T. By Proposition 1.2, we may assume that x and y have the same color in T. Since (i) the first and the last coordinates of $\vec{d}(v_0)$ are both 1, (ii) the first coordinate of $\vec{d}(v_1)$ is 1 and the last coordinate of $\vec{d}(v_1)$ is 0, and (iii) the first coordinate of $\vec{d}(v_d)$ is 0 and the last coordinate of $\vec{d}(v_d)$ is 1, it follows that the three red vertices v_0, v_1 and v_d have distinct codes. Thus, we may assume that x and y are white vertices. Let $\vec{d}(x) = (a_1, a_2, \ldots, a_d)$ and $\vec{d}(y) = (b_1, b_2, \ldots, b_d)$.

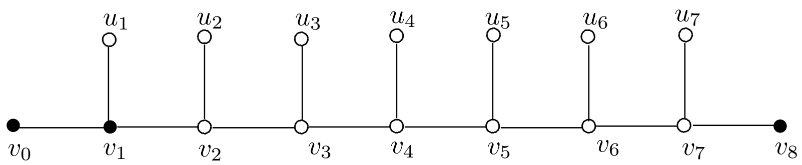

Figure 2: An ID-coloring of a caterpillar of diameter 8.

First, observe that for each vertex z of T, the eccentricity of z is $e(z) = \max\{d(z, v_0), d(z, v_d)\}$. Since v_0 and v_d are red vertices of T, it follows that the $e(z)$th coordinate of $\vec{d}(z)$ is the final coordinate of $\vec{d}(z)$ that is not 0. Consequently, if $e(x) \ne e(y)$, say $e(x) < e(y) = s$, then $a_s = 0$ but $b_s \ne 0$, which implies that $\vec{d}(x) \ne \vec{d}(y)$. Hence, we may assume that $e(x) = e(y) = s$. We consider three cases, according to the location of x and y.

Case 1. *$x = u_i$ and $y = u_j$ where $1 \le i < j \le d-1$.* Since $e(u_i) = e(u_j) = s$ and $1 \le i < j \le d-1$, it follows that $e(u_i) = d(u_i, v_d)$ and $e(u_j) = d(u_j, v_0)$. Then $b_{s-1} = 1$ (since $d(u_j, v_1) = s - 1$) and $a_{s-1} = 0$. Hence, $\vec{d}(x) \ne \vec{d}(y)$.

Case 2. *$x = v_i$ and $y = v_j$ where $2 \le i < j \le d-1$.* Since $e(v_i) = e(v_j) = s$ and $2 \le i < j \le d-1$, it follows that $e(v_i) = d(v_i, v_d)$ and $e(v_j) = d(v_j, v_0)$. Then $b_{s-1} = 1$ (since $d(v_j, v_1) = s-1$) and $a_{s-1} = 0$. Hence, $\vec{d}(x) \neq \vec{d}(y)$.

Case 3. *$x = u_i$ where $1 \le i \le d-1$ and $y = v_j$ where $2 \le i < j \le d-1$.* Since $e(u_i) = e(v_j) = s$, it follows that $y \in \{v_{i-1}, v_{i+1}\}$. If $y = v_{i-1}$, then $d(u_i, v_d) = d(v_{i-1}, v_d) = s = d - i + 1$. Since $d(v_{i-1}, v_1) = i - 2$ and $d(u_i, v_1) = i$, it follows that $a_{i-2} = 0$ and $b_{i-2} = 1$. Hence, $\vec{d}(x) \neq \vec{d}(y)$. If $y = v_{i+1}$, then $d(u_i, v_0) = d(v_{i+1}, v_0) = s = i + 1$ and $d(u_i, v_1) = d(v_{i+1}, v_1)$. If $a_s \neq b_s$ or $a_{s-1} \neq b_{s-1}$, then $\vec{d}(x) \neq \vec{d}(y)$. Thus, we may assume that $a_s = b_s$ and $a_{s-1} = b_{s-1}$. Since $d(u_i, v_d) = d(v_{i+1}, v_d) + 2$, it follows that $a_s = b_s = 1$ and $a_{s-1} = b_{s-1} = 1$. This implies that $b_{d-i-1} = 1$, where $d - i - 1 = d(v_{i+1}, v_d)$, and $a_{d-i-1} = 0$, implying that $\vec{d}(x) \neq \vec{d}(y)$. □

We now consider triplet-free caterpillars of diameter at least 4 and maximum degree 4. A triplet-free caterpillar with exactly k pairs of twins is called a *k-twin caterpillar*. Figure 3 shows a 2-twin caterpillar of diameter 4 and a 4-twin caterpillar of diameter 5. Next we show that every triplet-free caterpillar of diameter at least 4 and maximum degree 4 is an ID-caterpillar and present bounds for the ID-number of a k-twin caterpillar, which shows that its ID-number is one of four numbers (in terms of k).

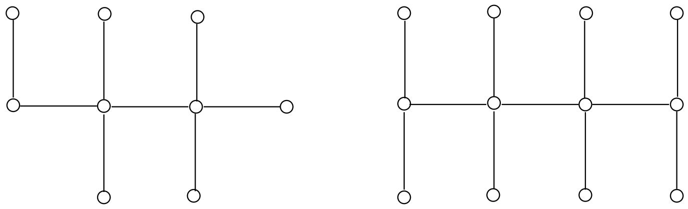

Figure 3: A 2-twin caterpillar and a 4-twin caterpillar.

Theorem 2.2. *Let T be a caterpillar of diameter $d \ge 4$ and maximum degree 4. If T is a k-twin for some positive integer k, then T is an ID-caterpillar and*

$$\max\{3, k\} \le \mathrm{ID}(T) \le k + 3.$$

Proof. Since $\mathrm{ID}(T) \ge \max\{3, k\}$ by Proposition 1.1 and Theorems 1.1–1.3, it remains to show that $\mathrm{ID}(T) \le k+3$. Let $P=(v_1, v_2, \ldots, v_{d-1})$ be the spine of T. For each integer i with $1 \le i \le d-1$, let L_i be the set of end-vertices of T that are adjacent to v_i. Then $1 \le |L_i| \le 3$ for $i = 1, d-1$ and $0 \le |L_i| \le 2$ for $2 \le i \le d-2$. We may assume that $\deg v_2 \ge \deg v_{d-2}$. Define a red-white coloring c of T by

(1) assigning the color red to v_1, exactly one vertex u_1 in L_1, exactly one vertex u_{d-1} in L_{d-1}, and exactly one vertex u_i in L_i if $|L_i| = 2$ for $2 \le i \le d-2$ and

(2) assigning the color white to the remaining vertices of T.

Thus, for $1 \le i \le d-1$, the red end-vertex adjacent to v_i (should it exist) is denoted by u_i and the white end-vertex adjacent to v_i (should it exist) is denoted by w_i. In particular, the red vertex u_1 is adjacent to the *red* vertex v_1 and the red vertex u_{d-1} is adjacent to the *white* vertex v_{d-1}. Let r denote the number of red vertices in T. Then

$$r = \begin{cases} k+3 & \text{if } |L_1| = |L_{d-1}| = 1 \\ k+2 & \text{if } \{|L_1|, |L_{d-1}|\} = \{1, 2\} \\ k+1 & \text{if } |L_1| = |L_{d-1}| = 2. \end{cases}$$

Thus, $r \ge 3$ if $k \in \{0, 1, 2\}$. It remains to show that c is an ID-coloring of T. Let x and y be two distinct vertices of T. By Proposition 1.2, we may assume that x and y have the same color in T. Let $\vec{d}(x) = (a_1, a_2, \ldots, a_d)$ and $\vec{d}(y) = (b_1, b_2, \ldots, b_d)$. First, observe that for each vertex z of T, the eccentricity of z is $e(z) = \max\{d(z, u_1), d(z, u_{d-1})\}$. Since u_1 and u_{d-1} are red vertices of T, it follows that the $e(z)$th coordinate of $\vec{d}(z)$ is the final coordinate of $\vec{d}(z)$ that is not 0. Consequently, if $e(x) \neq e(y)$, say $e(x) < e(y) = s$, then $a_s = 0$ but $b_s \neq 0$, which implies that $\vec{d}(x) \neq \vec{d}(y)$. Hence, we may assume that $e(x) = e(y) = s$. We consider two cases, according to whether x and y are both red or both white.

Case 1. *x and y are both red.* Observe that

- u_1 and v_1 are the only red vertices whose first coordinate is 1 and $e(u_1) > e(v_1)$;
- for $2 \le i \le d-1$, if u_i exists, then the first coordinate of $\vec{d}(u_i)$ is 0 and $e(u_{d-1}) > e(u_i)$ for $2 \le i \le d-2$.

Thus, we may assume that $x = u_i$ and $y = u_j$ where $2 \le i < j \le d-2$. Since $e(u_i) = e(u_j) = s$ and $2 \le i < j \le d-2$, it follows that $s = d(u_i, u_{d-1}) = d(u_j, u_1)$. Since (1) $d(u_j, v_1) = d(u_i, v_{d-1}) = s-1$, (2) v_1 is red and v_{d-1} is white, and (3) $\deg v_2 \ge \deg v_{d-2}$, it follows that $b_{s-1} > a_{s-1}$. Hence, $\vec{d}(x) \ne \vec{d}(y)$.

Case 2. *x and y are both white.* Observe that

- w_1 (should it exist) is the only white vertex whose whose code begins with 111
- w_{d-1} (should it exist) is only white peripheral vertex whose code begins with 0, and
- w_2 (should it exist) is the only white vertex whose whose code begins with 02.

Thus, we may assume that $x, y \in \{w_i : 3 \le i \le d-2\} \cup \{v_i : 2 \le i \le d-1\}$. We consider three cases, according to the location of x and y.

Subcase 2.1. *$x = w_i$ and $y = w_j$ where* $3 \le i < j \le d-2$. Since $e(w_i) = e(w_j) = s$ and $3 \le i < j \le d-2$, it follows that $s = d(w_i, u_{d-1}) = d(w_j, u_1)$. Since (1) $\deg v_2 \ge \deg v_{d-2}$, (2) v_1 is red and v_{d-1} is white, and (3) $d(w_i, v_{d-1}) = d(w_j, v_1) = d(w_j, u_2) = s-1$ (should u_2 exist), we have $b_{s-1} > a_{s-1}$. Hence, $\vec{d}(x) \ne \vec{d}(y)$.

Subcase 2.2. *$x = v_i$ and $y = v_j$ where* $2 \le i < j \le d-1$. Since $e(v_i) = e(v_j) = s$ and $2 \le i < j \le d-1$, it follows that $s = d(v_i, u_{d-1}) = d(v_j, u_1)$. Since (1) $\deg v_2 \ge \deg v_{d-2}$, (2) v_1 is red and v_{d-1} is white, and (3) $d(v_j, v_1) = d(v_j, u_2) = s-1$ (should u_2 exist), we have $b_{s-1} > a_{s-1}$. Hence, $\vec{d}(x) \ne \vec{d}(y)$.

Subcase 2.3. *$x = w_i$ where $3 \le i \le d-2$ and $y = v_j$ where* $2 \le i < j \le d-1$. Since $e(w_i) = e(v_j) = s$, it follows that $y \in \{v_{i-1}, v_{i+1}\}$. If $y = v_{i-1}$ where $i \ge 3$, then $d(w_i, u_{d-1}) = d(v_{i-1}, u_{d-1}) = s$. Since (1) $\deg v_2 \ge \deg v_{d-2}$ and (2) $d(v_{i-1}, v_1) = d(v_{i-1}, u_2) = i-2$ (should u_2 exist) and $d(w_i, v_1) = i$, it follows that $b_{i-2} > a_{i-2} = 1$. Hence, $\vec{d}(x) \ne \vec{d}(y)$. If $y = v_{i+1}$, then $d(w_i, u_1) = d(v_{i+1}, u_1) = s$. Observe that if $z \in \{v_p, u_p : 1 \le p \le i\} \cup \{w_p : 1 \le p \le i-1\}$, then $d(w_i, z) = d(v_{i+1}, z)$. Let t be the smallest integer with $i+1 \le t \le d-1$ such that u_t exists (where it is possible that $t = d-1$) and so u_t is red. Since $d(w_i, u_t) = d(v_{i+1}, u_t) + 2$ and $t - i = d(v_{i+1}, u_t)$, it follows that $b_{t-i} > a_{t-i}$. Hence, $\vec{d}(x) \ne \vec{d}(y)$. □

By Theorem 2.2, if T is a k-twin caterpillar, then $\max\{3, k\} \le \mathrm{ID}(T) \le k+3$. In fact, every integer between $\max\{3, k\}$ and $\le k+3$ is realizable as the ID-number of some k-twin caterpillar, as we show next.

Theorem 2.3. *For each pair (k, t) of integers where $k \ge 0$ and $t \in \{0, 1, 2, 3\}$ such that $k+t \ge 3$, there is a k-twin caterpillar T for which* $\mathrm{ID}(T) = k+t$.

Proof. We verify the following four statements.

1. For each integer $k \ge 0$, there exists a k-twin caterpillar T with $\mathrm{ID}(T) = k+3$.
2. For each integer $k \ge 1$, there exists a k-twin caterpillar T with $\mathrm{ID}(T) = k+2$.
3. For each integer $k \ge 2$, there exists a k-twin caterpillar T with $\mathrm{ID}(T) = k+1$.
4. For each integer $k \ge 3$, there exists a k-twin caterpillar T with $\mathrm{ID}(T) = k$.

We provide a complete proof for Statements 1 and 2 and provide an outline of a proof for Statements 3 and 4.

First, we verify Statement 1. By Theorem 2.1, if T is a caterpillar of diameter 3 or more and maximum degree 3, then $\mathrm{ID}(T) = 3$. Thus, the statement is true for $k = 0$ and so we may assume that $k \ge 1$. Let T be a caterpillar of diameter $d = k+3 \ge 4$ and let P be the spine of T such that each end-vertex of P is adjacent to exactly one end-vertex and each interior vertex of P is adjacent to exactly two end-vertices. Thus, T contains exactly $k = d-3$ twins. We show that $\mathrm{ID}(T) = d = k+3$. Let $(v_0, v_1, \ldots, v_d)$ be the longest path in T and so $P = (v_1, v_2, \ldots, v_{d-1})$ is the spine of T. For each integer i with $2 \le i \le d-2$, let u_i and w_i be the two end-vertices of T that are adjacent to v_i. Since T contains exactly $d-3$ twins, it follows by Theorem 2.2 that $\mathrm{ID}(T) \le d$. Thus, it remains to show that $\mathrm{ID}(T) \ge d$. Assume, to the contrary, that $\mathrm{ID}(T) \le d-1$ and let c be an ID-coloring of T with exactly $\mathrm{ID}(T)$ red vertices. Since u_i and w_i are twins in T for $2 \le i \le d-2$, we may assume that $c(u_i)$ is red and $c(w_i)$ is white. Since $d(v_1, z) = d(w_2, z)$ for each $z \in V(T) - \{v_0\}$, it follows that if v_0 and v_1 are both white, then $\vec{d}(v_1) = \vec{d}(w_2)$, which is impossible. Thus, at least one of v_0 and v_1 is red. If v_0 is white and v_1 is red, then $\vec{d}(v_1) = \vec{d}(u_2)$. Thus, v_0 must be red. Similarly, v_d must be red. Hence, $\mathrm{ID}(T) = d-1$ and v_0, v_d, and u_i, where $2 \le i \le d-2$, are the $d-1$ red vertices of T. However then, $\vec{d}(v_0) = \vec{d}(v_{d-1})$, for example, which is impossible. Therefore, $\mathrm{ID}(T) = d = k+3$ and so Statement 1 holds.

To verify Statement 2, let T be a caterpillar of diameter $d = k + 4 \geq 5$ where $(v_0, v_1, \ldots, v_d)$ be the longest path in T such that $\deg v_i = 2$ for $i = 1, d-2, d-1$ and $\deg v_i = 4$ for $2 \leq i \leq d-3$. Thus, T contains exactly $k = d - 4 \geq 1$ twins. We show that $\mathrm{ID}(T) = d - 2 = k + 2 \geq 3$. For each integer i with $2 \leq i \leq d-3$, let u_i and w_i be the two end-vertices of T that are adjacent to v_i. First, we show that $\mathrm{ID}(T) \geq k + 2$. Let c be an ID-coloring of T. Since u_i and w_i are twins in T for $2 \leq i \leq d-3$, we may assume that $c(u_i)$ is red and $c(w_i)$ is white. Since $d(v_1, z) = d(w_2, z)$ for each $z \in V(T) - \{v_0\}$, it follows that if v_0 and v_1 are both white, then $\vec{d}(v_1) = \vec{d}(w_2)$, which is impossible. Thus, at least one of v_0 and v_1 must be red. Similarly, at least one vertex in $\{v_{d-2}, v_{d-1}, v_d\}$ must be red. Thus, $\mathrm{ID}(T) \geq k + 2$.

Next, we show that T has an ID-coloring with exactly $k + 2$ red vertices. Define a red-white coloring c of T by (1) assigning the color red to v_0, v_d, and u_i for $2 \leq i \leq d-3$ and (2) assigning the color white to the remaining vertices of T. Thus, T has exactly $k+2$ red vertices. We show that c is an ID-coloring of T. Let x and y be two distinct vertices of the same color in T, where $\vec{d}(x) = (a_1, a_2, \ldots, a_d)$ and $\vec{d}(y) = (b_1, b_2, \ldots, b_d)$. First, observe that for each vertex z of T, the eccentricity of z is $e(z) = \max\{d(z, u_1), d(z, u_{d-1})\}$. Since u_1 and u_{d-1} are red vertices of T, it follows that the $e(z)$th coordinate of $\vec{d}(z)$ is the final coordinate of $\vec{d}(z)$ that is not 0. Consequently, if $e(x) \neq e(y)$, say $e(x) < e(y) = s$, then $a_s = 0$ but $b_s \neq 0$, which implies that $\vec{d}(x) \neq \vec{d}(y)$. Hence, we may assume that $e(x) = e(y) = s$. We consider two cases, according to whether x and y are both red or both white.

Case 1. *x and y are both red.* Since (1) v_0 and v_d are the only red peripheral vertices and (2) $\vec{d}(v_0)$ begins with 001, and $\vec{d}(v_d)$ begins with 000, we may assume that $x = u_i$ and $y = u_j$ where $2 \leq i < j \leq d-2$. Since $e(u_i) = e(u_j) = s$ and $2 \leq i < j \leq d-2$, it follows that $s = d(u_i, v_d) = d(u_j, v_0)$. Then $b_{s-1} = 1$ and $a_{s-1} = 0$. Hence, $\vec{d}(x) \neq \vec{d}(y)$.

Case 2. *x and y are both white.* Observe that

- v_{d-2} is the only white vertex whose whose code begins with 02,
- v_{d-1} is the only white vertex whose whose code begins with 10,
- for $2 \leq i \leq d-3$, the first coordinate of $\vec{d}(w_i)$ is 0 and for $1 \leq j \leq d-3$, the first coordinate of $\vec{d}(v_j)$ is 1, which implies that $\vec{d}(w_i) \neq \vec{d}(v_j)$, and
- $e(v_1) > e(v_i)$ for $2 \leq i \leq d-3$, which implies that $\vec{d}(v_1) \neq \vec{d}(v_i)$.

Thus, we may assume that $x = w_i$ and $y = w_j$ where $2 \leq i < j \leq d-3$ or $x = v_i$ and $y = v_j$ where $2 \leq i < j \leq d-3$. First, suppose that $x = w_i$ and $y = w_j$ where $2 \leq i < j \leq d-3$. Since $e(w_i) = e(w_j) = s$ and $2 \leq i < j \leq d-3$, it follows that $s = d(w_i, v_d) = d(u_j, v_0)$. Then $b_{s-1} > a_{s-1}$ (since $d(u_j, u_2) = s - 1$). Hence, $\vec{d}(x) \neq \vec{d}(y)$. Next, suppose that $x = v_i$ and $y = v_j$ where $2 \leq i < j \leq d-3$. Since $e(v_i) = e(v_j) = s$ and $2 \leq i < j \leq d-3$, it follows that $s = d(v_i, v_d) = d(v_j, v_0)$. Then $b_{s-1} > a_{s-1}$ (since $d(v_j, u_2) = s - 1$). Hence, $\vec{d}(x) \neq \vec{d}(y)$.

Hence, c is an ID-coloring of T with exactly $k + 2$ red vertices and so $\mathrm{ID}(T) \leq k + 2$. Therefore, $\mathrm{ID}(T) = k + 2$ and Statement 2 holds.

We now provide an outline of a proof of Statement 3. By Theorem 2.1, every caterpillar of diameter 3 or more and maximum degree 3 has ID-number 3. Thus, if T is a caterpillar of diameter 3 or more and maximum degree 3 such that each of two end-vertices of its spine is adjacent to two end-vertices, then T has $k = 2$ twins such that $\mathrm{ID}(T) = 3$. Consequently, we may assume that $k \geq 3$. Let T be a caterpillar of diameter $d = k + 1 \geq 4$ and let P be the spine of T such that every vertex of P is adjacent to exactly two end-vertices. Thus, T contains exactly $k = d - 1$ twins. We show that $\mathrm{ID}(T) = d = k + 1$. Let P=$(v_1, v_2, \ldots, v_{d-1})$ be the spine of T. For each integer i with $1 \leq i \leq d-1$, let u_i and w_i be the two end-vertices of T that are adjacent to v_i. Since u_i and w_i are twins in T for $1 \leq i \leq d-1$, it follows by Proposition 1.1 that $\mathrm{ID}(T) \geq d - 1$. Assume, to the contrary, that there is an ID-coloring of T with exactly $d - 1$ red vertices. We may assume that u_i is red for $1 \leq i \leq d-1$. However then, v_1 and v_{d-1} have the same code, for example, which is impossible. Therefore, $\mathrm{ID}(T) \geq d$. To show $ID(T) \leq d$, we define a red-white coloring c of T by (1) assigning the color red to v_1 and u_i for $1 \leq i \leq d-1$ and (2) assigning the color white to the remaining vertices of T. Thus, T has exactly d red vertices. It can be shown that c is an ID-coloring of T. Therefore, $\mathrm{ID}(T) = d = k + 1$.

To provide an outline of a proof Statement 4, let T be a caterpillar of diameter $d = k+2 \geq 5$ and let $P = (v_1, v_2, \ldots, v_{d-1})$ be the spine of T such that each vertex of P is adjacent to exactly two end-vertices except v_{d-2} having degree 2. Thus, T contains exactly $k = d - 2$ twins. We show that $\mathrm{ID}(T) = d - 2 = k$. For each integer i with $1 \leq i \leq d-1$ and $i \neq d-2$, let u_i and w_i be the two end-vertices of T that are adjacent to v_i. Since T contains exactly $d-2$ twins, it follows that $\mathrm{ID}(T) \geq d-2$. To show that $\mathrm{ID}(T) \leq d - 2$, we define a red-white coloring c of T by (1) assigning the color red to u_i for $1 \leq i \leq d-3$ and u_{d-1} and (2) assigning the color white to the remaining vertices of T. It can be shown that c is an ID-coloring of T with exactly $d - 2$ red vertices. Therefore, $\mathrm{ID}(T) = d - 2 = k$. $\square$

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