Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **8** (2022) 1–5
DOI: 10.47443/dml.2021.0059

Research Article

Two explicit formulas for degenerate Peters numbers and polynomials*

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(Received: 8 June 2021. Received in revised form: 12 August 2021. Accepted: 20 August 2021. Published online: 1 September 2021.)

Abstract

In the present paper, by virtue of the Faà di Bruno formula and some identities of the Bell polynomials of the second kind, the authors derive two explicit formulas for degenerate Peters numbers and polynomials.

Keywords: Peters number; Peters polynomial; Boole number; Boole polynomial; explicit formula; generating function.

2020 Mathematics Subject Classification: 05A19, 05A10, 11B37, 11B83, 11Y55.

1. Introduction

The Boole polynomials $\mathrm{Bl}_n(x;\alpha)$ are defined in [8, p. 127, Section 4.5] by

$$\frac{(1+t)^x}{1+(1+t)^\alpha}=\sum_{n=0}^{\infty}\mathrm{Bl}_n(x;\alpha)\frac{t^n}{n!}.$$

For $x=0$, we call $\mathrm{Bl}_n(0;\alpha)=\mathrm{Bl}_n(\alpha)$ the Boole numbers. As a generalization of the Boole polynomials $\mathrm{Bl}_n(x;\alpha)$, the Peters polynomials (or say, higher-order Boole polynomials) $s_n(x;\alpha,\mu)$ are given in [1] and [8, p. 128, Section 4.6] by

$$\frac{(1+t)^x}{[1+(1+t)^\alpha]^\mu}=\sum_{n=0}^{\infty}s_n(x;\alpha,\mu)\frac{t^n}{n!}.$$

These two polynomials are the members of the family of the Sheffer polynomials which play important roles in umbral calculus. It is easy to see that $s_n(x;\alpha,1)=\mathrm{Bl}_n(x;\alpha)$. In the last decade, these polynomials and related ones have been extensively considered in many works. In [4], the Witt-type formulas and several new identities for the Boole polynomials $\mathrm{Bl}_n(x;\alpha)$ were investigated. In [5], via so-called nonlinear differential equations satisfied by the generating function of the Boole numbers $\mathrm{Bl}_n(\alpha)$, some significant formulas involving the Boole numbers $\mathrm{Bl}_n(\alpha)$ and their higher-order type were presented. Hereafter, by applying related generating functions and infinite series, Simsek and his coauthor So established in [9–12] a plenty of new identities, inequalities, recursive relations, and observations for polynomials of the Peters type and for some combinatorial numbers and polynomials. For more information, we refer the reader to the papers [4,5] and a number of related references cited therein.

Let

$$\frac{\mathrm{e}^{x[(1+t)^\lambda-1]/\lambda}}{\left(1+\mathrm{e}^{\alpha[(1+t)^\lambda-1]/\lambda}\right)^\mu}=\sum_{n=0}^{\infty}s_n(x;\alpha,\mu;\lambda)\frac{t^n}{n!}.$$

Since

$$\lim_{\lambda\to0}\frac{(1+t)^\lambda-1}{\lambda}=\ln(1+t),$$

we have

$$\lim_{\lambda\to0}s_n(x;\alpha,\mu;\lambda)=s_n(x;\alpha,\mu).$$

Therefore, we call the sequence $s_n(x;\alpha,\mu;\lambda)$ degenerate Peters polynomials, call the sequence $s_n(0;\alpha,\mu;\lambda)=s_n(\alpha,\mu;\lambda)$ degenerate Peters numbers which are generated by

$$\frac{1}{\left(1+\mathrm{e}^{\alpha[(1+t)^\lambda-1]/\lambda}\right)^\mu}=\sum_{n=0}^{\infty}s_n(\alpha,\mu;\lambda)\frac{t^n}{n!},$$

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and call $s_n(x;\alpha,1;\lambda)=\mathrm{Bl}_n(x;\alpha,\lambda)$ degenerate Boole polynomials.

In this paper, via the Faà di Bruno formula (1) and with the help of several identities of the second kind Bell polynomials stated in the next section, we present an explicit formula for degenerate Peters numbers $s_n(\alpha,\mu;\lambda)$ and degenerate Peters polynomials $s_n(x;\alpha,\mu;\lambda)$, respectively.

2. Identities of the Bell polynomials of the second kind

In this section, in order to present our main results, we recall several identities of the Bell polynomials of the second kind.

The Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ were defined in [2, p. 134] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i},\quad n\ge k\ge 0.$$

For $n\in\mathbb{N}$, the Faà di Bruno formula is described in [2, p. 139] in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=1}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big). \tag{1}$$

In [2, p. 135], there is an identity

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{2}$$

where $n\ge k\ge 0$ and a,b,λ,α are any complex numbers. In [2, p. 135], there is the relation

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k), \tag{3}$$

where the Stirling numbers of the second kind $S(n,k)$ can be generated by

$$\frac{(\mathrm{e}^t-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!}.$$

In [2, p. 136, Eq. [3n]], it was given that the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy

$$\mathrm{B}_{n,k}(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1})=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}(x_1,x_2,\dots,x_{\ell-r+1})\,\mathrm{B}_{m,s}(y_1,y_2,\dots,y_{m-s+1}). \tag{4}$$

In [6, Remark 1], there exists the formula

$$\mathrm{B}_{n,k}\left(1,1-\lambda,(1-\lambda)(1-2\lambda),\dots,\prod_{\ell=0}^{n-k}(1-\ell\lambda)\right)=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda). \tag{5}$$

The explicit formula (5) is equivalent to

$$\mathrm{B}_{n,k}(\langle\lambda\rangle_1,\langle\lambda\rangle_2,\dots,\langle\lambda\rangle_{n-k+1})=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\lambda\ell\rangle_n \tag{6}$$

which was presented in [7, Theorems 2.1 and 4.1], where the falling factorial $\langle x\rangle_n$ is defined for $x\in\mathbb{C}$ by

$$\langle x\rangle_n=\prod_{k=0}^{n-1}(x-k)=\begin{cases}x(x-1)\cdots(x-n+1), & n\ge 1;\\ 1, & n=0;\end{cases}$$

When $n\in\mathbb{N}$, the explicit formulas (5) and (6) were rearranged in [3, Remark 7.5] as

$$\mathrm{B}_{n,k}\left(1,1-\lambda,(1-\lambda)(1-2\lambda),\dots,\prod_{\ell=0}^{n-k}(1-\ell\lambda)\right)=(-1)^k\frac{\lambda^{n-1}(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\ell/\lambda-1}{n-1}$$

for $\lambda\ne 0$ and

$$\mathrm{B}_{n,k}(\langle\lambda\rangle_1,\langle\lambda\rangle_2,\dots,\langle\lambda\rangle_{n-k+1})=(-1)^k\lambda\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}, \tag{7}$$

where the generalized binomial coefficient $\binom{z}{w}$ is defined by

$$\binom{z}{w}=\begin{cases}\dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z,w,z-w\in\mathbb{C}\setminus\{-1,-2,\dots\};\\ 0, \quad z\in\mathbb{C}\setminus\{-1,-2,\dots\}, & w,z-w\in\{-1,-2,\dots\}.\end{cases}$$

3. An explicit formula of degenerate Peters numbers

In this section, we state and prove an explicit formula of degenerate Peters numbers $s_n(\alpha,\mu;\lambda)$.

Theorem 3.1. *For $n\in\mathbb{N}$, degenerate Peters numbers $s_n(\alpha,\mu;\lambda)$ can be explicitly computed by*

$$s_n(\alpha,\mu;\lambda)=(n-1)!\sum_{k=1}^{n}\frac{(-1)^k}{k!}\frac{\alpha^k}{\lambda^{k-1}}\left[\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}S(k,\ell)\right]\left[\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]. \tag{8}$$

Proof. For $n\in\mathbb{N}$, applying $f(u)=\frac{1}{(1+e^{\alpha u})^\mu}$ and $u=h_\lambda(t)=\frac{(1+t)^\lambda-1}{\lambda}\to0$ as $t\to0$ to the Faà di Bruno formula (1) and straightforwardly computing give

$$\frac{d^n}{d\,t^n}\left[\frac{1}{\left(1+e^{\alpha[(1+t)^\lambda-1]/\lambda}\right)^\mu}\right]=\sum_{k=1}^{n}\frac{d^k}{d\,u^k}\left[\frac{1}{(1+e^{\alpha u})^\mu}\right]B_{n,k}\Big(h_\lambda'(t),h_\lambda''(t),\dots,h_\lambda^{(n-k+1)}(t)\Big),$$

$$\begin{aligned}\lim_{u\to0}\frac{d^k}{d\,u^k}\left[\frac{1}{(1+e^{\alpha u})^\mu}\right]&=\lim_{u\to0}\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{(1+e^{\alpha u})^{\mu+\ell}}B_{k,\ell}\big(\alpha e^{\alpha u},\alpha^2e^{\alpha u},\dots,\alpha^{k-\ell+1}e^{\alpha u}\big)\\&=\alpha^k\lim_{u\to0}\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{(1+e^{\alpha u})^{\mu+\ell}}e^{\alpha u\ell}B_{k,\ell}(1,1,\dots,1)\\&=\alpha^k\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}S(k,\ell),\end{aligned}$$

and

$$\begin{aligned}B_{n,k}\Big(h_\lambda'(t),h_\lambda''(t),\dots,h_\lambda^{(n-k+1)}(t)\Big)&=\frac{(1+t)^{k\lambda-n}}{\lambda^k}B_{n,k}(\langle\lambda\rangle_1,\langle\lambda\rangle_2,\dots,\langle\lambda\rangle_{n-k+1})\\&=\frac{(1+t)^{k\lambda-n}}{\lambda^k}(-1)^k\lambda\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\\&\to\frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1},\quad t\to0,\end{aligned}$$

where we used the identities (2), (3), and (7). Consequently, for $n\in\mathbb{N}$, we obtain

$$\begin{aligned}s_n(\alpha,\mu;\lambda)&=\lim_{t\to0}\frac{d^n}{d\,t^n}\left[\frac{1}{\left(1+e^{\alpha[(1+t)^\lambda-1]/\lambda}\right)^\mu}\right]\\&=\sum_{k=1}^{n}\alpha^k\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}S(k,\ell)\frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\\&=(n-1)!\sum_{k=1}^{n}\frac{(-\alpha)^k}{\lambda^{k-1}k!}\left[\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}S(k,\ell)\right]\left[\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right].\end{aligned}$$

The proof of the explicit formula (8) in Theorem 3.1 is thus complete. □

Remark 3.1. *It is clear that $s_0(\alpha,\mu;\lambda)=\frac{1}{2^\mu}$. From the explicit formula (8), we can derive the first few values of $s_n(\alpha,\mu;\lambda)$ for $1\le n\le6$ as follows:*

$$\begin{aligned}s_1(\alpha,\mu;\lambda)&=-\frac{\alpha\mu}{2^{\mu+1}},\\s_2(\alpha,\mu;\lambda)&=\frac{\alpha\mu}{2^{\mu+2}}[\alpha(\mu-1)-2\lambda+2],\\s_3(\alpha,\mu;\lambda)&=-\frac{\alpha\mu}{2^{\mu+3}}\big[\alpha^2(\mu-3)\mu+6\alpha(\lambda+\mu-\lambda\mu-1)+4(\lambda^2-3\lambda+2)\big],\\s_4(\alpha,\mu;\lambda)&=\frac{\alpha\mu}{2^{\mu+4}}\big[\alpha^3(\mu^3-6\mu^2+3\mu+2)-12\alpha^2(\lambda-1)(\mu-3)\mu\\&\quad+4\alpha(7\lambda^2-18\lambda+11)(\mu-1)-8(\lambda^3-6\lambda^2+11\lambda-6)\big],\\s_5(\alpha,\mu;\lambda)&=-\frac{\alpha\mu}{2^{\mu+5}}\big[\alpha^4\mu(\mu^3-10\mu^2+15\mu+10)-20\alpha^3(\lambda-1)(\mu^3-6\mu^2+3\mu+2)+20\alpha^2(5\lambda^2-12\lambda+7)(\mu-3)\mu\\&\quad-40\alpha(3\lambda^3-14\lambda^2+21\lambda-10)(\mu-1)+16(\lambda^4-10\lambda^3+35\lambda^2-50\lambda+24)\big],\end{aligned}$$

$$
\begin{aligned}
s_6(\alpha,\mu;\lambda) = \frac{\alpha\mu}{2^{\mu+6}}&(\alpha^5(\mu^5-15\mu^4+45\mu^3+15\mu^2-30\mu-16)-30\alpha^4(\lambda-1)\mu(\mu^3-10\mu^2+15\mu+10)\\
&+20\alpha^3(13\lambda^2-30\lambda+17)(\mu^3-6\mu^2+3\mu+2)-120\alpha^2(6\lambda^3-25\lambda^2+34\lambda-15)(\mu-3)\mu\\
&+16\alpha(31\lambda^4-225\lambda^3+595\lambda^2-675\lambda+274)(\mu-1)-32(\lambda^5-15\lambda^4+85\lambda^3-225\lambda^2+274\lambda-120)).
\end{aligned}
$$

4. An explicit formula of degenerate Peters polynomials

Now, we establish an explicit formula of degenerate Peters polynomials $s_n(x;\alpha,\mu;\lambda)$ for $n\in\mathbb{N}$.

Theorem 4.1. *For $n\in\mathbb{N}$, degenerate Peters polynomials $s_n(x;\alpha,\mu;\lambda)$ can be computed by*

$$
\begin{aligned}
s_n(x;\alpha,\mu;\lambda) = (n-1)!\sum_{k=1}^{n}&\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\\
&\times\left[\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}\sum_{r+s=\ell}\sum_{i+j=k}\binom{k}{i}\left(-\frac{x}{\mu}\right)^i\left(\alpha-\frac{x}{\mu}\right)^j S(i,r)S(j,s)\right].
\end{aligned}
\tag{9}
$$

Proof. It is clear that

$$
\frac{e^{x[(1+t)^\lambda-1]/\lambda}}{\left(1+e^{\alpha[(1+t)^\lambda-1]/\lambda}\right)^\mu} = \frac{1}{\left(e^{-(x/\mu)[(1+t)^\lambda-1]/\lambda}+e^{(\alpha-x/\mu)[(1+t)^\lambda-1]/\lambda}\right)^\mu} = \frac{1}{\left[e^{Ah_\lambda(t)}+e^{Bh_\lambda(t)}\right]^\mu},
$$

where $A=-\frac{x}{\mu}$, $B=\alpha+A$, and $u=h_\lambda(t)=\frac{(1+t)^\lambda-1}{\lambda}\to0$ as $t\to0$. Then, by virtue of the Faà di Bruno formula (1) and the identity (2), we have

$$
\begin{aligned}
\lim_{t\to0}\frac{d^n}{d\,t^n}&\left[\frac{e^{x[(1+t)^\lambda-1]/\lambda}}{\left(1+e^{\alpha[(1+t)^\lambda-1]/\lambda}\right)^\mu}\right] = \lim_{t\to0}\frac{d^n}{d\,t^n}\left(\frac{1}{\left[e^{Ah_\lambda(t)}+e^{Bh_\lambda(t)}\right]^\mu}\right)\\
&=\lim_{t\to0}\sum_{k=1}^{n}\frac{d^k}{d\,u^k}\left[\frac{1}{(e^{Au}+e^{Bu})^\mu}\right]B_{n,k}\left(h_\lambda'(t),h_\lambda''(t),\dots,h_\lambda^{(n-k+1)}(t)\right)\\
&=\sum_{k=1}^{n}\lim_{t\to0}B_{n,k}\left(h_\lambda'(t),h_\lambda''(t),\dots,h_\lambda^{(n-k+1)}(t)\right)\lim_{u\to0}\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{(e^{Au}+e^{Bu})^{\mu+\ell}}\\
&\quad\times B_{k,\ell}\big(Ae^{Au}+Be^{Bu},A^2e^{Au}+B^2e^{Bu},\dots,A^{k-\ell+1}e^{Au}+B^{k-\ell+1}e^{Bu}\big)\\
&=(n-1)!\sum_{k=1}^{n}\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}B_{k,\ell}\big(A+B,A^2+B^2,\dots,A^{k-\ell+1}+B^{k-\ell+1}\big)\\
&=(n-1)!\sum_{k=1}^{n}\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}\\
&\quad\times\sum_{r+s=\ell}\sum_{i+j=k}\binom{k}{i}B_{i,r}\big(A,A^2,\dots,A^{i-r+1}\big)B_{j,s}\big(B,B^2,\dots,B^{j-s+1}\big)\\
&=(n-1)!\sum_{k=1}^{n}\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}\sum_{r+s=\ell}\sum_{i+j=k}\binom{k}{i}A^iB^jB_{i,r}(1,1,\dots,1)B_{j,s}(1,1,\dots,1)\\
&=(n-1)!\sum_{k=1}^{n}\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}\sum_{r+s=\ell}\sum_{i+j=k}\binom{k}{i}A^iB^jS(i,r)S(j,s)\\
&=(n-1)!\sum_{k=1}^{n}\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\left[\sum_{\ell=1}^{k}\frac{\langle-\mu\rangle_\ell}{2^{\mu+\ell}}\sum_{r+s=\ell}\sum_{i+j=k}\binom{k}{i}\left(-\frac{x}{\mu}\right)^i\left(\alpha-\frac{x}{\mu}\right)^jS(i,r)S(j,s)\right],
\end{aligned}
$$

where we used the identities (3) and (4). The explicit formula (9) is thus proved. The proof of Theorem 4.1 is complete. □

Remark 4.1. *If we set $x=0$ in (9), we obtain (8).*

Acknowledgements

The first author (Yue-Wu Li) is partially supported by the Doctors Foundation of Hulunbuir University, China (through Grant No. 2018BS12). The authors thank anonymous referees for corrections, valuable comments, and helpful suggestions on the original version of this paper.

References

[1] R. P. Boas, R. C. Buck, *Polynomial Expansions of Analytic Functions*, Second Edition, Springer-Verlag, Berlin, 1964.

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel, Dordrecht, 1974.

[3] B. N. Guo, D. Lim, F. Qi, Maclaurin series expansions for powers of inverse (hyperbolic) sine, for powers of inverse (hyperbolic) tangent, and for incomplete gamma functions, with applications, *arXiv*: 2101.10686v6 [math.CO], (2021).

[4] D. S. Kim, T. Kim, A note on Boole polynomials, *Integral Transforms Spec. Funct.* **25** (2014) 627–633.

[5] T. Kim, D. S. Kim, Nonlinear differential equations arising from Boole numbers and their applications, *Filomat* **31** (2017), 2441–2448.

[6] F. Qi, B. N. Guo, Viewing some ordinary differential equations from the angle of derivative polynomials, *Iran. J. Math. Sci. Inform.* **16** (2021) 77–95.

[7] F. Qi, D. W. Niu, D. Lim, B. N. Guo, Closed formulas and identities for the Bell polynomials and falling factorials, *Contrib. Discrete Math.* **15** (2020) 163–174.

[8] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.

[9] Y. Simsek, Peters type polynomials and numbers and their generating functions: approach with p-adic integral method, *Math. Methods Appl. Sci.* **42** (2019) 7030–7046.

[10] Y. Simsek, A new family of combinatorial numbers and polynomials associated with Peters numbers and polynomials, *Appl. Anal. Discrete Math.* **14** (2020) 627–640.

[11] Y. Simsek, J. S. So, Identities, inequalities for Boole type polynomials: approach to generating functions and infinite series, *J. Inequal. Appl.* **2019** (2019) #62.

[12] Y. Simsek, J. S. So, On generating functions for Boole type polynomials and numbers of higher order and their applications, *Symmetry* **11** (2019) #352.