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Research Article

Perfect Roman domination in middle graphs

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Abstract

The middle graph $M(G)$ of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all these newly introduced vertices of adjacent edges of G. A perfect Roman dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ is adjacent to exactly one vertex u for which $f(u) = 2$. The weight of a perfect Roman dominating function f is the sum of weights of vertices. The perfect Roman domination number is the minimum weight of a perfect Roman dominating function on G. In this paper, a characterization of middle graphs with equal Roman domination and perfect Roman domination numbers is given.

Keywords: perfect domination; Roman domination; perfect Roman domination; middle graph.

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1. Introduction

Let $G = (V, E)$ be an undirected graph with the vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is defined as the cardinality of V. The *open neighborhood* of $v \in V(G)$ is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$. In [5], Hamada and Yoshimura defined the middle graph of a graph. The *middle graph* $M(G)$ of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all these newly introduced vertices of adjacent edges of G. The precise definition of $M(G)$ is as follows. The vertex set $V(M(G))$ is $V(G) \cup E(G)$. Two vertices $v, w \in V(M(G))$ are adjacent in $M(G)$ if (i) $v, w \in E(G)$ and v, w are adjacent in G or (ii) $v \in V(G)$, $w \in E(G)$ and v, w are incident in G.

The study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274–337 AD. The concept of Roman domination was introduced in [3, 10, 11]. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* on G if every vertex $v \in V(G)$ for which $f(v) = 0$ is adjacent to at least one vertex $u \in V(G)$ for which $f(u) = 2$. The weight of a Roman dominating function is the value $\omega(f) := \sum_{v \in V(G)} f(v)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of a Roman dominating function on G. As a variant of Roman domination, a function $f : V(G) \to \{0, 1, 2\}$ is a *perfect Roman dominating function* on G if every vertex $v \in V(G)$ for which $f(v) = 0$ is adjacent to exactly one vertex $u \in V(G)$ for which $f(u) = 2$. The *perfect Roman domination number* $\gamma_{pR}(G)$ is the minimum weight of a perfect Roman dominating function on G. In [6], Henning et al. introduced the notion of perfect Roman domination and showed that if T is a tree on $n \geq 3$ vertices, then $\gamma_{pR}(T) \leq \frac{4}{5}n$. In [4], Darkooti et al. proved that it is NP-complete to decide whether a graph has a perfect Roman dominating function, even if the graph is bipartite. This suggests determining the exact value of perfect Roman domination numbers for special classes of graphs. Recently, the following result was proved in [1, 9].

Theorem 1.1. *Let G be a graph of order n. Then $\gamma_R(M(G)) = n$.*

Based on this result, we characterize all graphs G such that $\gamma_R(M(G)) = \gamma_{pR}(M(G))$. We try to determine the exact value of perfect Roman domination numbers in middle graphs of special classes of graphs. Note that recent results of domination number in middle graphs were given in [7, 8].

2. Main results

In this section, we introduce the concept of a middle Roman dominating function to study the Roman domination number of the middle graph $M(G)$ for a given graph G. A *middle Roman dominating function* (MRDF) on a graph G is a function $f : V \cup E \to \{0, 1, 2\}$ satisfying the following conditions: (i) every element $x \in V$ for which $f(x) = 0$ is incident to at least one

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element $y \in E$ for which $f(y) = 2$, (ii) every element $x \in E$ for which $f(x) = 0$ is adjacent or incident to at least one element $y \in V \cup E$ for which $f(y) = 2$. A MRDF f gives an ordered partition $(V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ (or $(V_0^f \cup E_0^f, V_1^f \cup E_1^f, V_2^f \cup E_2^f)$ to refer to f) of $V \cup E$, where $V_i := \{x \in V \mid f(x) = i\}$ and $E_i := \{x \in E \mid f(x) = i\}$. The weight of a middle Roman dominating function f is $\sum_{x \in V \cup E} f(x)$. The *middle Roman domination number* $\gamma_R^\star(G)$ of G is the minimum weight of a middle Roman dominating function of G. A $\gamma_R^\star(G)$-*function* is a MRDF on G with weight $\gamma_R^\star(G)$. Similarly, we can define a perfect middle Roman dominating function (PMRDF) and related definitions. By the definition of middle graphs, we state the following remark.

Remark 2.1. *For any graph G, $\gamma_R^\star(G) = \gamma_R(M(G))$ and $\gamma_{pR}^\star(G) = \gamma_{pR}(M(G))$.*

For a subset S of G, the subgraph obtained from G by deleting all vertices in S and all edges incident with S is denoted by $G - S$. For terminology and notation on graph theory not given here, the reader is referred to [2]. We make use of the following result.

Proposition 2.1 ([12]). *Let G be a graph with components $G_1, \ldots, G_t$. Then $\gamma_{pR}(G) = \sum_{i=1}^{t} \gamma_{pR}(G_i)$.*

The following is our main theorem.

Theorem 2.1. *Let G be a graph. Then $\gamma_R^\star(G) = \gamma_{pR}^\star(G)$ if and only if there exists a $\gamma_R^\star(G)$-function such that (i) vertices incident to edges in E_2 are adjacent to vertices in V_1 and (ii) $G - \{u, v \in V(G) \mid uv \in E_2\}$ is an empty graph.*

Proof. If G is an empty graph, then the statement holds. By Proposition 2.1, from now on we assume that G is connected and not empty.

($\Rightarrow$): Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be a $\gamma_{pR}^\star(G)$-function such that $\gamma_R^\star(G) = \gamma_{pR}^\star(G)$. We proceed by proving six claims.

Claim 1. $V_2 = \emptyset$.

Suppose that there exists $v \in V_2$. Consider $G - \{v\}$. Define $g : V(G - \{v\}) \cup E(G - \{v\}) \to \{0, 1, 2\}$ by $g(x) = f(x)$. Then g is a MRDF on $G - \{v\}$ with $\omega(g) = n - 2$. Since $G - \{v\}$ has order $n - 1$, by Theorem 1.1 $\omega(g) \geq \gamma_R^\star(G - \{v\}) = n - 1$, a contradiction.

Claim 2. $E_1 = \emptyset$.

Suppose that there exists $e \in E_1$. Let u and v be vertices incident to e. We divides the following three cases depending on the values of u and v assigned under f.

Case 1. $u, v \in V_0$. There exist $e_1, e_2 \in E_2$ such that u, v are incident to e_1, e_2, respectively. Also, there exist $u', v' \in V(G) \setminus \{u, v\}$ such that u', v' are incident to e_1, e_2, respectively. Consider $G - \{u, u', v, v'\}$. Define $g : V(G - \{u, u', v, v'\}) \cup E(G - \{u, u', v, v'\}) \to \{0, 1, 2\}$ by $g(x) = f(x)$. Since there is no edge in E_2 adjacent to e_1 or e_2, g is a MRDF on $G - \{u, u', v, v'\}$. Then g is a MRDF with $\omega(g) \leq n - 5$. Since $G - \{u, u', v, v'\}$ has order $n - 4$, by Theorem 1.1 $\omega(g) \geq \gamma_R^\star(G - \{u, u', v, v'\}) = n - 4$, a contradiction.

Case 2. $u \in V_0, v \in V_1$ or $v \in V_0, u \in V_1$. By symmetry, assume that $u \in V_0$ and $v \in V_1$. There exists $e_1 \in E_2$ incident to u. Also, there exist $u' \in V(G)$ incident to e_1. Consider $G - \{u, u', v\}$. Define $g : V(G - \{u, u', v\}) \cup E(G - \{u, u', v\}) \to \{0, 1, 2\}$ by $g(x) = \min\{f(x) + f(xv), 2\}$ for each $x \in N(v) \setminus \{u\}$ and $g(x) = f(x)$ otherwise. Then g is a MRDF with $\omega(g) \leq n - 4$. Since $G - \{u, u', v\}$ has order $n - 3$, by Theorem 1.1 $\omega(g) \geq \gamma_R^\star(G - \{u, u', v\}) = n - 3$, a contradiction.

Case 3. $u, v \in V_1$. Consider $G - \{u, v\}$. Define $g : V(G - \{u, v\}) \cup E(G - \{u, v\}) \to \{0, 1, 2\}$ by $g(x) = \min\{f(x) + f(xu), 2\}$ for each $x \in N(u) \setminus \{v\}$, $g(x) = \min\{f(x) + f(xv), 2\}$ for each $x \in N(v) \setminus \{u\}$ and $g(x) = f(x)$ otherwise. Then g is a MRDF with $\omega(g) \leq n - 3$. Since $G - \{u, v\}$ has order $n - 2$, by Theorem 1.1 $\omega(g) \geq \gamma_R^\star(G - \{u, v\}) = n - 2$, a contradiction.

Claim 3. Every edge in E_2 is incident to vertices in V_0.

Let $e \in E_2$ and $e = uv$. Suppose that $u \notin V_0$ or $v \notin V_0$. Consider $G - \{u, v\}$. Define $g : V(G - \{u, v\}) \cup E(G - \{u, v\}) \to \{0, 1, 2\}$ by $g(x) = \min\{f(x) + f(xu), 2\}$ for each $x \in N(u) \setminus \{v\}$, $g(x) = \min\{f(x) + f(xv), 2\}$ for each $x \in N(v) \setminus \{u\}$ and $g(x) = f(x)$ otherwise. Then g is a MRDF with $\omega(g) \leq n - 3$. Since $G - \{u, v\}$ has order $n - 2$, by Theorem 1.1 $\omega(g) \geq \gamma_R^\star(G - \{u, v\}) = n - 2$, a contradiction.

Claim 4. Every edge in E_2 is adjacent to edges in E_0.

By Claims 2, 3 and the hypothesis that f is a PMRDF, Claim 4 follows.

Claim 5. Vertices incident to edges in E_2 are adjacent to vertices in V_1.

Let $uv = e \in E_2$. Suppose that u is adjacent to $w \in V_0$. Then w must be incident to some $e' \in E_2$. Since $uw \in E_0$ is adjacent to e and e', this is a contradiction.

Claim 6. $G - \{u, v \in V(G) \mid uv \in E_2\}$ is an empty graph.

If $H := G - \{u, v \in V(G) \mid uv \in E_2\}$ is not empty, then every edge of H must be assigned 1 under f. The weight of f is not equal to the order G, a contradiction.

($\Leftarrow$): The conditions (i) and (ii) imply that $\gamma^\star_R(G) = \gamma^\star_{pR}(G)$. □

By Remark 2.1 and Theorem 2.1, we can characterize all graphs G such that $\gamma_R(M(G)) = \gamma_{pR}(M(G))$. Based on Theorem 2.1, we determine the exact value of perfect Roman domination numbers for middle graphs of paths, cycles and Kneser graphs.

Proposition 2.2. *For a path P_n of order n, $\gamma^\star_{pR}(P_n) = n$*

Proof. Let $P_n = v_1v_2 \ldots v_n$. Clearly $\gamma^\star_{pR}(P_2) = 2$. For $n \geq 3$, we divides the following three cases.

Case 1. $n \equiv 0 \pmod 3$. Define $f : V(P_n) \cup E(P_n) \to \{0, 1, 2\}$ by $f(v_{3i+1}) = 1$, $f(v_{3i+2}v_{3i+3}) = 2$ for $0 \leq i \leq \frac{n-3}{3}$ and $f(x) = 0$ otherwise.

Case 2. $n \equiv 1 \pmod 3$. Define $f : V(P_n) \cup E(P_n) \to \{0, 1, 2\}$ by $f(v_{3i+1}) = 1$, $f(v_{3i+2}v_{3i+3}) = 2$ for $0 \leq i \leq \frac{n-4}{3}$, $f(v_n) = 1$ and $f(x) = 0$ otherwise.

Case 3. $n \equiv 2 \pmod 3$. Define $f : V(P_n) \cup E(P_n) \to \{0, 1, 2\}$ by $f(v_{3i+1}v_{3i+2}) = 2$, $f(v_{3i+3}) = 1$ for $0 \leq i \leq \frac{n-5}{3}$, $f(v_{n-1}v_n) = 2$ and $f(x) = 0$ otherwise.

In any case, it is easy to see that f is a PMRDF with the weight n. This completes the proof. □

Proposition 2.3. *For a cycle C_n of order n, $\gamma^\star_{pR}(C_n) = n$ if $n \equiv 0 \pmod 3$, $n + 1$ otherwise.*

Proof. Let $C_n = v_1v_2 \ldots v_nv_1$, and let f be a $\gamma^\star_{pR}(C_n)$-function. Suppose that $\omega(f) < |V(C_n) \cup E(C_n)|$. Then there exists an element $x \in V(C_n) \cup E(C_n)$ such that $f(x) = 2$. If there exists an element $x \in V(C_n)$ such that $f(x) = 2$, then it follows from Theorem 2.1 that $\gamma^\star_{pR}(C_n) > \gamma^\star_R(C_n)$.

Without loss of generality, now we assume that $f(v_{n-1}v_n) = 2$. Consider $C_n - \{v_{n-1}, v_n\} \cong P_{n-2}$. If $\gamma^\star_R(C_n) = \gamma^\star_{pR}(C_n)$, then it follows from Theorem 2.1 that $f(v_1) = f(v_{n-2}) = 1$.

If $n - 2 \equiv 1 \pmod 3$, then by the Case 2 of Proposition 2.2 P_{n-2} has a PMRDF g such that $g(v_1) = g(v_{n-2}) = 1$ and $\gamma^\star_{pR}(P_{n-2}) = n - 2$. This implies that $\gamma^\star_{pR}(C_n) = n$.

If $n - 2 \not\equiv 1 \pmod 3$, then Theorem 2.1 implies that $\gamma^\star_{pR}(C_n) > \gamma^\star_R(C_n)$. Now we can define a PMRDF f with the weight $n + 1$ by giving $f(v_1v_n) = 1$ in the Cases 2 and 3 of Proposition 2.2. Thus, $\gamma^\star_{pR}(C_n) = n + 1$. □

Proposition 2.4. *For a Kneser graph $K(2m + 1, m)$, $\gamma^\star_{pR}(K(2m + 1, m)) = \gamma^\star_R(K(2m + 1, m))$.*

Proof. For $m = 1$, $K(3, 1) \cong K_3$ implies that $\gamma^\star_{pR}(K(3, 1)) = \gamma^\star_R(K(3, 1))$.

Now we assume that $m \geq 2$. Let A be the family of m-element subsets of $[2m] := \{1, 2, \ldots, 2m\}$ and B the family of $(m - 1)$-element subsets of $[2m]$. Let $C = \{\{2m + 1\} \cup b \mid b \in B\}$. Then $V(K(2m + 1, m)) = A \cup C$ and

$$E(K(2m + 1, m)) = \{aa' \mid a \cap a' = \emptyset \text{ for } a, a' \in A\} \cup \{ac \mid a \cap c = \emptyset \text{ for } a \in A, c \in C\}.$$

Consider the subgraph of $K(2m + 1, m)$ induced by A. Then it has a unique perfect matching M. Define

$$f : V(K(2m + 1, m)) \cup E(K(2m + 1, m)) \to \{0, 1, 2\}$$

by $f(e) = 2$ for $e \in M$, $f(c) = 1$ for $c \in C$ and $f(x) = 0$ otherwise. Then f is a MRDF with weight $|V(K(2m + 1, m))|$. It follows from Theorem 2.1 that f is a PMRDF such that $\gamma^\star_{pR}(K(2m + 1, m)) = \gamma^\star_R(K(2m + 1, m))$. □

Finally, we conclude our paper by suggesting the following problems.

Problem 2.1. *For a complete bipartite graph $K_{m,n}$, what is the exact value of $\gamma^\star_{pR}(K_{m,n})$?*

Problem 2.2. *For a complete graph K_n, what is the exact value of $\gamma^\star_{pR}(K_n)$?*

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