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Research Article

Restricted coloring of 1-by-n chessboards

Ronit Mansour[1,*], Amal Sharif-Rasslan[2]

[1]*AL-Qasemi Academy, (R.A)-Academic College of Education, Baqa-El-Gharbia, Israel*
[2]*Department of Mathematics, Academic Arab College, Haifa, Israel*

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Abstract

In this note, we answer the following question: in how many ways one can color a $1 \times n$ chessboard using colors from the set $\{1, 2, \dots, m\}$ such that the color i occurs a multiple of t_i number of times for all $i = 1, 2, \dots, m$?

Keywords: coloring chessboard; generating function.

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1. Introduction

The results on counting combinatorial objects in a discrete structure are often used for solving problems at the high school level and undergraduate level. Recently, Abboud, Saleh, and Sharif-Rasslan (one of the present authors) [3] studied the following problem.

Problem 1.1. *In how many ways one can color a $1 \times n$ chessboard using m colors if one of the colors occurs a multiple of t number of times?*

It needs to be noted here that Problem 1.1 is motivated by a problem posed in the book of Brualdi [2] (this problem is also posed in [1]). Figure 1 presents all possible ways to color a 1×4 chessboard in black and white.

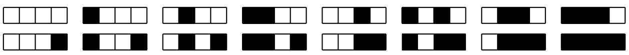

Figure 1: Coloring a 1×4 chessboard by black and white.

In the present note, we answer the following generalized version of Problem 1.1.

Problem 1.2. *In how many ways one can color a $1 \times n$ chessboard using colors from the set $[m] = \{1, 2, \dots, m\}$ such that the color i occurs a multiple of t_i number of times for all $i = 1, 2, \dots, m$?*

Clearly, Problem 1.2 reduced to Problem 1.1 when $t_2 = t_3 = \cdots = t_m = 1$.

2. A combinatorial solution to Problem 1.2

Let $a(n) = a(n; t_1, t_2, \dots, t_m)$ be the number of the different ways to color a $1 \times n$ chessboard using colors from the set $[m]$ such that the color i occurs a multiple of t_i number of times for all $i = 1, 2, \dots, m$.

Assume that the $1 \times n$ chessboard is produced from n unit squares. So the number of ways $a(n)$ is equal to choosing $t_i j_i$ squares coloring them by color i, for all $i \in [m]$, such that $t_1 j_1 + \cdots + t_m j_m = n$. Thus, $a(n)$ is given by the following multiple sum formula:

$$a(n, t_1, \dots, t_m) = \sum_{\substack{j_1, \dots, j_m \geq 0, \\ t_1 j_1 + \cdots + t_m j_m = n}} \binom{n}{j_1 t_1} \binom{n - j_1 t_1}{j_2 t_2} \cdots \binom{n - j_1 t_1 - \cdots - j_{m-1} t_{m-1}}{j_m t_m},$$

where $\binom{a}{b}$ is defined to be zero whenever $a < b$.

*Corresponding author (ronitmansour77@gmail.com).

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Note that when we take $t_2 = \cdots = t_m = 1$ and $t_1 = t$, we obtain

$$a(n; t, 1, \ldots, 1) = \sum_{\substack{j_1, \ldots, j_m \geq 0, \\ t_1 j_1 + j_2 + \cdots + j_m = n}} \binom{n}{j_1 t}\binom{n - j_1 t}{j_2} \cdots \binom{n - j_1 t - j_2 - \cdots - j_{m-1}}{j_m}.$$

Since the multisum is conditioned by $t_1 j_1 + j_2 + \cdots + j_m = n$, we see that

$$\binom{n - j_1 t - j_2 - \cdots - j_{m-1}}{j_m}$$

contributes 1. Thus,

$$a(n; t, 1, \ldots, 1) = \sum_{\substack{j_1, \ldots, j_{m-1} \geq 0, \\ t_1 j_1 + j_2 + \cdots + j_{m-1} \leq n}} \binom{n}{j_1 t}\binom{n - j_1 t}{j_2} \cdots \binom{n - j_1 t - j_2 - \cdots - j_{m-2}}{j_{m-1}}.$$

By separating the sum over j_{m-1} from the multiple sum, we obtain

$$\begin{aligned} a(n; t, 1, \ldots, 1) &= \sum_{\substack{j_1, \ldots, j_{m-1} \geq 0, \\ t_1 j_1 + j_2 + \cdots + j_{m-1} \leq n}} \left(\binom{n}{j_1 t} \cdots \binom{n - j_1 t - j_2 - \cdots - j_{m-3}}{j_{m-2}} \sum_{j_m = 0}^{n - j_1 t - j_2 - \cdots - j_{m-2}} \binom{n - j_1 t - j_2 - \cdots - j_{m-2}}{j_{m-1}} \right) \\ &= \sum_{\substack{j_1, \ldots, j_{m-2} \geq 0, \\ t_1 j_1 + j_2 + \cdots + j_{m-2} \leq n}} \left(\binom{n}{j_1 t} \cdots \binom{n - j_1 t - j_2 - \cdots - j_{m-3}}{j_{m-2}} 2^{n - j_1 t - j_2 - \cdots - j_{m-2}} \right). \end{aligned}$$

Again, by separating the sum over j_{m-2} from the multiple sum, we have

$$a(n; t, 1, \ldots, 1) = \sum_{\substack{j_1, \ldots, j_{m-3} \geq 0, \\ t_1 j_1 + j_2 + \cdots + j_{m-3} \leq n}} \left(\binom{n}{j_1 t} \cdots \binom{n - j_1 t - j_2 - \cdots - j_{m-4}}{j_{m-3}} 3^{n - j_1 t - j_2 - \cdots - j_{m-3}} \right).$$

Thus, by induction on m, we have

$$\begin{aligned} a(n; t, 1, \ldots, 1) &= \sum_{\substack{j_1, j_2 \geq 0, \\ t_1 j_1 + j_2 \leq n}} \left(\binom{n}{j_1 t}\binom{n - j_1 t}{j_2} (m-2)^{n - j_1 t - j_2} \right) \\ &= \sum_{j_1 = 0}^{\lfloor n/2 \rfloor} \binom{n}{j_1 t} (m-1)^{n - j_1 t}, \end{aligned}$$

as have been shown in [3].

For another example, we see that $a(n; t, r, 1, 1, \ldots, 1)$ is the number of ways to color a $1 \times n$ chessboard using colors from the set $[m]$ such that the colors 1 and 2 occurs a multiple of t and r number of times, respectively. From the above general formula, we have

$$a(n; t, r, 1, 1, \ldots, 1) = \sum_{j=0}^{\lfloor n/t \rfloor} \sum_{i=0}^{\lfloor (n-jt)/r \rfloor} \binom{n}{jt}\binom{n - jt}{ir} (m-2)^{n - jt - ir}. \tag{1}$$

3. A solution to Problem 1.2 by using exponential generating functions

Let $A(x) = A(x; t_1, \ldots, t_m)$ be the exponential generating function for the sequence $a(n)$, that is

$$A(x) = \sum_{n \geq 0} a(n) \frac{x^n}{n!}.$$

A multiset is a collection of elements, some of which may be repeated a finite or an infinite number of times. Recall that $a(n) = a(n; t_1, \ldots, t_m)$ equals the number of ways to color a $1 \times n$ chessboard using colors from the set $[m] = \{1, 2, \ldots, m\}$ such that the color i occurs a multiple of t_i number of times for all $i = 1, 2, \ldots, m$. Then, $a(n)$ equals the number of arrangements of the elements of the multiset $\{1^{j_1t_1}, 2^{j_2t_2}, \dots, m^{j_mt_m} \mid j_1t_1 + j_2t_2 + \cdots + j_mt_m = n\}$. Thus, the exponential generating function is given by

$$A(x;t_1,\dots,t_m) = \sum_{j_1t_1+\cdots+t_mj_m=n} \frac{x^n}{\prod_{i=1}^m (j_it_i)!}.$$

Since $j_1t_1 + \cdots + t_mj_m = n$, we have

$$A(x;t_1,\dots,t_m) = \sum_{j_1,\dots,j_m\geq 0} \prod_{i=1}^m \frac{x^{j_it_i}}{(j_it_i)!} = \prod_{i=1}^m \left(\sum_{j_i\geq 0} \frac{x^{j_it_i}}{(j_it_i)!}\right).$$

Again, for $t_1 = t$ and $t_2 = \cdots = t_m = 1$, we obtain the same formula as in [3]. Here is another example

$$A(x;t,t,\dots,t) = \left(\sum_{j\geq 0} \frac{x^{jt}}{(jt)!}\right)^m.$$

In particular, if $t = 2$, then we obtain

$$A(x;2,2,\dots,2) = \left(\sum_{j\geq 0} \frac{x^{2j}}{(2j)!}\right)^m = \left(\frac{e^x + e^{-x}}{2}\right)^m,$$

which leads to

$$A(x;2,2,\dots,2) = \frac{1}{2^m}\sum_{j=0}^m \binom{m}{j} e^{(m-2j)x} = \frac{1}{2^m}\sum_{j=0}^m\sum_{i\geq 0}\binom{m}{j}\frac{(m-2j)^i x^i}{i!}.$$

Hence,

$$A(n;2,2,\dots,2) = \frac{1}{2^m}\sum_{j=0}^m \binom{m}{j}(m-2j)^n.$$

For instance, the words with 4 letters over the alphabet $\{1,2,3\}$ such that each letter occurs even number of times are 1111, 1122, 1133, 1212, 1221, 1313, 1331, 2112, 2121, 2211, 2222, 2233, 2323, 2332, 3113, 3131, 3311, 3223, 3232, 3322 and 3333. From the formula, we see that the number of such words is equal to

$$\frac{1}{2^3}\sum_{j=0}^3 \binom{3}{j}(3-2j)^4 = \frac{3^4+3+3+3^4}{8} = 21.$$

Note that by using the fact that

$$\sum_{j\geq 0}\frac{x^{jt}}{(jt)!} = \frac{1}{t}(e^x + e^{\omega x} + \cdots + e^{\omega^{t-1}x})$$

where ω is a primitive t-root of the unity (that is, $\omega^t = 1$), we have

$$A(x;t_1,\dots,t_m) = \prod_{i=1}^m \left(\frac{e^x + e^{\omega_i x} + \cdots + e^{\omega_i^{t_i-1}x}}{t_i}\right),$$

where ω_i is any primitive t_i-root of the unity. For instance, if $m = 3$, $t_1 = t_2 = 2$ an $t_3 = 1$, then $\omega_1 = \omega_2 = -1$ and $\omega_3 = 1$, so

$$A(x;2,2,1) = \frac{e^x+e^{-x}}{2}\cdot\frac{e^x+e^{-x}}{2}\cdot e^x = \frac{e^{3x}+2e^x+e^{-x}}{4}.$$

Thus, the number of ways to color $1\times n$ chessboard by 3 colors such that the first color occurs an even number of times and the second color occurs an even number of times is $\frac{1}{4}(3^n + 2 + (-1)^n)$.

Next, we consider another example, which answers a question that we found on a web page[†]. The question asks in how many ways one can color a $1\times n$ chessboard by four colors such that the first color occurs an even number of times and the second color occurs an even number of times. The answer to this question is of course $a(n;2,2,1,1)$, which has the exponential generating function

$$\frac{1}{4}(e^x+e^{-x})^2e^{2x} = \frac{1}{4}(e^{4x}+2e^{2x}+1).$$

Hence, $a(n;2,2,1,1) = 4^{n-1} + 2^{n-1}$ for all $n \geq 1$. More generally, if $t_i = 2$ for all $i = 1,2,\dots,m'$ and $t_i = 1$ for all $i = m'+1, m'+2, \dots, m$, then

$$A(x;t_1,\dots,t_m) = \frac{1}{2^{m'}}(e^x+e^{-x})^{m'}e^{(m-m')x},$$

[†] https://math.stackexchange.com/questions/35063/coloring-of-an-1-times-n-board-using-4-colors

which leads to

$$a(n;2,2,\ldots,2,1,1,\ldots,1)=\frac{1}{2^{m'}}\sum_{j=0}^{m'}\binom{m'}{j}(m-2j)^n,$$

for all $j\geq 1$. Note that if $m'=2$ then by using (1), we get the following result.

Corollary 3.1. *For all $n\geq 0$, it holds that*

$$\sum_{j=0}^{\lfloor n/2\rfloor}\sum_{i=0}^{\lfloor n/2-j\rfloor}\binom{n}{2j}\binom{n-2j}{2i}(m-2)^{n-2j-2i}=\frac{1}{4}(m^n+2(m-2)^n+(m-4)^n).$$

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