DML Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **7** (2021) 86–89
DOI: 10.47443/dml.2021.0075

Research Article

The average domination polynomial of graphs is unimodal

Mohammad Reza Oboudi*

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran

(Received: 23 July 2021. Received in revised form: 25 August 2021. Accepted: 25 August 2021. Published online: 27 August 2021.)

Abstract

Let G be a simple graph on n vertices. A dominating set of G is a subset of the vertex set $V(G)$ of G, say S, such that every vertex in $V(G)\setminus S$ is adjacent to at least one vertex of S. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=1}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. For every $n\geq 1$, let $\Phi_n(x)$ be the average of the domination polynomials of all labeled graphs on n vertices. In this paper, the polynomial $\Phi_n(x)$ is studied and it is shown that $\Phi_n(x)$ is log-concave and unimodal.

Keywords: dominating set; domination polynomial; unimodal polynomial; log-concave polynomial.

2020 Mathematics Subject Classification: 05C31, 05C69, 05C70.

1. Introduction

Throughout this paper, we consider only simple graphs (the graphs with no loops and multiple edges). Let $G=(V(G),E(G))$ be a simple graph. The *order* of G is the number of vertices of G. For a vertex $v\in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$ (for simplicity we write $deg(v)$ instead of $deg_G(v)$). For a vertex $v\in V(G)$, the *open neighborhood* of v is the set $N(v)=\{u\in V: uv\in E(G)\}$ and the *closed neighborhood* is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V(G)$, the open neighborhood of S is $N(S)=\bigcup_{v\in S}N(v)$ and the closed neighborhood of S is $N[S]=N(S)\cup S$. A set $S\subseteq V(G)$ is a *dominating set* of G if $N[S]=V(G)$, or equivalently, every vertex in $V(G)\setminus S$ is adjacent to at least one vertex of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum of the cardinality of the dominating sets of G. We denote the complete graph of order n, the cycle of order n and the path of order n, by K_n, C_n, and P_n, respectively.

There are numerous polynomials associated with graphs. For example *chromatic polynomial*, *clique polynomial*, *independence polynomial*, *matching polynomial*, *edge cover polynomial*, *edge elimination polynomial*, *domination polynomial* and *Tutte polynomial*. For more details, see [1]– [22] and references therein. By studying these polynomials one can obtain some properties of a graph. For instance the roots of these polynomials reflect some important information about the structure of graphs. Let G be a graph of order n. The *domination polynomial* of G that is denoted by $D(G,x)$ is the one variable polynomial such that the coefficient of x^k is d_k, where d_k is the number of dominating sets of G with size k. More precisely

$$D(G,x)=\sum_{k=1}^{n} d_k x^k.$$

For example, the domination polynomial of the complete graph K_n is $D(K_n,x)=(x+1)^n-1$. The domination polynomial was first introduced in [5, 8].

One of the most interesting properties of graph polynomials is unimodality. A polynomial $f(x)=\sum_{i=0}^{n} a_i x^i$ with real coefficients (or a sequence $(a_0,\ldots,a_n)$) is called *unimodal* if there is $k\in\{0,\ldots,n\}$, k is called the *mode* of $f(x)$, such that

$$a_0\leq\cdots\leq a_{k-1}\leq a_k\geq a_{k+1}\geq\cdots\geq a_n.$$

Also, $f(x)$ (or a sequence $(a_0,\ldots,a_n)$) is called *logarithmically concave* (or simply, *log-concave*), if for every $1\leq i\leq n-1$, $a_i^2\geq a_{i-1}a_{i+1}$. The polynomial $f(x)$ (or a sequence $(a_0,\ldots,a_n)$) is called *symmetric* (or *palindromic*) if $a_i=a_{n-i}$ for $i=0,1,\ldots,n$. It is known that any log-concave polynomial with positive coefficients (or a sequence of positive numbers) is also unimodal. See [21] for more details on these definitions.

*E-mail addresses: mr_oboudi@yahoo.com, mr_oboudi@shirazu.ac.ir

The unimodality problems of graph polynomials have always been of great interest to researchers in graph polynomials. In [20], it has been conjectured that the chromatic polynomial of a graph is unimodal. Recently, in [14], this conjectured has been proved. It is conjectured that the domination polynomial of every graph is unimodal [5]. Also, there is a famous conjecture due to Alavi et al. [1] on the unimodality of the independence polynomial of trees. The unimodality of graph polynomials, in particular the unimodality of independence polynomial, have been extensively studied, see [1,7,10,11,14, 16,22]. In [11], the authors show that the average of the independence polynomial of graphs is unimodal. Motivated by these papers, in this paper we study the unimodality of domination polynomial. For every $n \geq 1$, let $\Phi_n(x)$ be the average of the domination polynomials of all labeled graphs on n vertices. In this paper first we indicate $\Phi_n(x)$ and then show that this polynomial is log-concave. Since the coefficients of $\Phi_n(x)$ are positive, we conclude that $\Phi_n(x)$ is unimodal.

2. Results

For every integer $n \geq 1$, let $\mathcal{G}_n$ be the set of all simple graphs on the vertices $v_1, \ldots, v_n$. In other words $\mathcal{G}_n$ is the set of all labeled graphs on the vertices $v_1, \ldots, v_n$. Hence the cardinality of $\mathcal{G}_n$ is $2^{\binom{n}{2}}$. Let $\Phi_n(x)$ be the average of all domination polynomial of graphs of $\mathcal{G}_n$. In other words,

$$\Phi_n(x) = 2^{-\binom{n}{2}} \sum_{G \in \mathcal{G}_n} D(G, x).$$

For example, $\Phi_1(x) = x$, $\Phi_2(x) = x + x^2$, $\Phi_3(x) = \frac{3x+9x^2+4x^3}{4}$ and $\Phi_4(x) = \frac{4x+27x^2+28x^3+8x^4}{8}$. In this section, we determine the coefficients of polynomial $\Phi_n(x)$ and show that $\Phi_n(x)$ is unimodal (in fact we prove that this polynomial is log-concave).

Theorem 2.1. *For every integer $n \geq 1$ we have*

$$\Phi_n(x) = \sum_{k=1}^{n} \binom{n}{k} (1 - 2^{-k})^{n-k} x^k.$$

Proof. Let $n \geq 1$ be an integer and

$$S_n(x) = 2^{\binom{n}{2}} \Phi_n(x).$$

For a graph G, let $T(G)$ be the set of all dominating sets of G. Therefore $D(G, x) = \sum_{I \in T(G)} x^{|I|}$. Note that the summation is taken over all dominating sets of G. Hence

$$S_n(x) = \sum_{G \in \mathcal{G}_n} D(G, x) = \sum_{G \in \mathcal{G}_n} \sum_{I \in T(G)} x^{|I|}.$$

Therefore

$$S_n(x) = \sum_{\varnothing \neq I \subseteq \{v_1, \ldots, v_n\}} \lambda_I x^{|I|}, \tag{1}$$

where λ_I is the number of graphs G with the vertex set $\{v_1, \ldots, v_n\}$ such that I is a dominating set of G.

Hence to complete the proof it suffices to find λ_I. For every $1 \leq k \leq n$, let $I_k = \{v_1, \ldots, v_k\}$. Since the number of graphs G on $\{v_1, \ldots, v_n\}$ is $2^{\binom{n}{2}}$ and $V(G)$ is a dominating set of G, $\lambda_{I_n} = 2^{\binom{n}{2}}$. Thus assume that $k \leq n - 1$. First we find the number of bipartite graphs H with the parts $\{v_1, \ldots, v_k\}$ and $\{v_{k+1}, \ldots, v_n\}$ such that the degree of all vertices $v_{k+1}, \ldots, v_n$ in H is non-zero. In other words $N(v_i) \cap \{v_1, \ldots, v_k\} \neq \emptyset$ for $i = k+1, \ldots, n$. For $j = 1, \ldots, n-k$ define the property c_j as "the degree of v_{j+k} is zero". Let $M(c_{j_1} c_{j_2} \cdots c_{j_m})$ be the number of bipartite graphs T with the parts $\{v_1, \ldots, v_k\}$ and $\{v_{k+1}, \ldots, v_n\}$ such that the degree of all vertices $v_{k+j_1}, \ldots, v_{k+j_m}$ in T is zero. Therefore $M(c_{j_1} c_{j_2} \cdots c_{j_m}) = 2^{k(n-k-m)}$ (note that the number of bipartite graphs L with parts X and Y is $2^{|X||Y|}$). Let β be the number of bipartite graphs H with the parts $\{v_1, \ldots, v_k\}$ and $\{v_{k+1}, \ldots, v_n\}$ such that the degree of all vertices $v_{k+1}, \ldots, v_n$ in H is non-zero. Using the inclusion – exclusion principle we conclude that

$$\beta = 2^{k(n-k)} - \sum_{1 \leq j \leq n-k} M(c_j) + \sum_{1 \leq j < j' \leq n-k} M(c_j c_{j'}) - \sum_{1 \leq j < j' < j'' \leq n-k} M(c_j c_{j'} c_{j''}) + \cdots + (-1)^{n-k} M(c_1 \cdots c_{n-k}).$$

Therefore

$$\beta = 2^{k(n-k)} - \binom{n-k}{1} 2^{k(n-k-1)} + \binom{n-k}{2} 2^{k(n-k-2)} + \cdots + (-1)^{n-k} \binom{n-k}{n-k} 2^{k(n-k-(n-k))}.$$

Hence by the binomial Theorem we obtain that

$$\beta = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} 2^{k(n-k-j)} = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (2^k)^{n-k-j} = (2^k - 1)^{n-k}. \tag{2}$$

Since the number of graphs on $\{v_1, \ldots, v_k\}$ is $2^{\binom{k}{2}}$ and the number of graphs on $\{v_{k+1}, \ldots, v_n\}$ is $2^{\binom{n-k}{2}}$ we find that

$$\lambda_{I_k} = 2^{\binom{k}{2}} 2^{\binom{n-k}{2}} \beta. \tag{3}$$

On the other hand

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + \binom{k}{1}\binom{n-k}{1}. \tag{4}$$

So by the Equations (2), (3) and (4) we obtain that

$$\lambda_{I_k} = 2^{\binom{n}{2} - k(n-k)} \ (2^k - 1)^{n-k} = 2^{\binom{n}{2}} \frac{(2^k-1)^{n-k}}{(2^k)^{n-k}} = 2^{\binom{n}{2}} \ (1 - 2^{-k})^{n-k}. \tag{5}$$

This equality shows that for every I where $\emptyset \neq I \subseteq \{v_1, \ldots, v_n\}$,

$$\lambda_I = 2^{\binom{n}{2}} \ (1 - 2^{-|I|})^{n-|I|}. \tag{6}$$

On the other hand the number of subset of $\{v_1, \ldots, v_n\}$ with cardinality k is $\binom{n}{k}$. So, by Equations (1) and (6) we conclude that the coefficient of x^k in $S_n(x)$ is

$$\binom{n}{k} \ 2^{\binom{n}{2}} \ (1 - 2^{-k})^{n-k}$$

for $k = 0, \ldots, n$. In other words, the coefficient of x^k in $\Phi_n(x)$ is

$$\binom{n}{k} \ (1 - 2^{-k})^{n-k}.$$

The proof is complete. □

Now we show that the average of the domination polynomials of graphs is unimodal.

Theorem 2.2. *For every integer $n \geq 1$ the polynomial $\Phi_n(x)$ is log-concave and so is unimodal.*

Proof. Let $n \geq 1$ be an integer. The result easily follows for $n \leq 2$. Hence let $n \geq 3$. For every $k \in \{0, 1, \ldots, n\}$ let $A_{n,k} = (1 - 2^{-k})^{n-k}$. Note that $A_{n,0} = 0$ and $A_{n,k} > 0$ for $k \geq 1$. We claim the sequence $A_{n,0}, \ldots, A_{n,n}$ is log-concave. In other words for every $k \in \{1, \ldots, n-1\}$, $A_{n,k}^2 \geq A_{n,k-1} A_{n,k+1}$. For $k = 1$ there is nothing to prove. So let $k \geq 2$. Since $(2^k - 1)^2 > (2^{k-1} - 1)(2^{k+1} - 1)$, we have $(2^k - 1)^{2n-2k} > (2^{k-1} - 1)^{n-k} (2^{k+1} - 1)^{n-k}$. On the other hand $2^{k+1} - 1 > 4(2^{k-1} - 1)$. Hence

$$(2^k - 1)^{2n-2k} > \frac{2^{k+1} - 1}{2^{k-1} - 1} \ (2^{k-1} - 1)^{n-k+1} \ (2^{k+1} - 1)^{n-k-1} > 4(2^{k-1} - 1)^{n-k+1} \ (2^{k+1} - 1)^{n-k-1}.$$

This implies that

$$(2^k - 1)^{2n-2k} > 4(2^{k-1} - 1)^{n-k+1} \ (2^{k+1} - 1)^{n-k-1}. \tag{7}$$

On the other hand

$$2k(n-k) = (k-1)(n-k+1) + (k+1)(n-k-1) + 2. \tag{8}$$

By dividing the sides of Equation (7) by $2^{2k(n-k)}$ and using Equation (8), we obtain

$$\left(\left(\frac{2^k - 1}{2^k} \right)^{n-k} \right)^2 > \left(\frac{2^{k-1} - 1}{2^{k-1}} \right)^{n-k+1} \left(\frac{2^{k+1} - 1}{2^{k+1}} \right)^{n-k-1}. \tag{9}$$

This shows that

$$((1 - 2^{-k})^{n-k})^2 > \left(1 - 2^{-(k-1)}\right)^{n-k+1} \left(1 - 2^{-(k+1)}\right)^{n-k-1}.$$

Hence the claim is proved. Note that the positive sequence

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

is log-concave. Thus (since the non-negative sequence $A_{n,0}, \ldots, A_{n,n}$ is log-concave) we conclude that the sequence

$$\binom{n}{0} A_{n,0}, \binom{n}{1} A_{n,1}, \ldots, \binom{n}{n} A_{n,n}$$

is also log-concave. In other words $\Phi_n(x)$ is log-concave. Since the coefficients of $\Phi_n(x)$ are positive, $\Phi_n(x)$ is unimodal. The proof is complete. □

References

[1] Y. Alavi, P. J. Malde, A. J. Schwenk, P. Erdős, The vertex independence sequence of a graph is not constrained, *Congr. Numer.* **58** (1987) 15–23.
[2] S. Akbari, S. Alikhani, Y. H. Peng, Characterization of graphs using domination polynomial, *European J. Combin.* **31** (2010) 1714–1724.
[3] S. Akbari, M. R. Oboudi, Cycles are determined by their domination polynomials, *Ars Combin.* **116** (2014) 353–358.
[4] S. Akbari, M. R. Oboudi, On the edge cover polynomial of a graph, *European J. Combin.* **34** (2013) 297–321.
[5] S. Alikhani, *Dominating Sets and Domination Polynomials of Graphs*, Ph.D. Dissertation, University Putra Malaysia, Seri Kembangan, 2008.
[6] S. Alikhani, J. I. Brown, S. Jahari, On the domination polynomials of friendship graphs, *Filomat* **30** (2016) 169–178.
[7] S. Alikhani, F. Jafari, On the unimodality of independence polynomial of certain classes of graphs, *Trans. Comb.* **2** (2013) 33–41.
[8] S. Alikhani, Y. H. Peng, Introduction to Domination Polynomial of a Graph, *Ars Combin.* **114** (2014) 257–266.
[9] I. Averbouch, B. Godlin, J. A. Makowsky, An extension of the bivariate chromatic polynomial, *European J. Combin.* **31** (2010) 1–17.
[10] P. Bahls, N. Salazar, Symmetry and unimodality of independence polynomials of path-like graphs, *Australas. J. Combin.* **47** (2010) 165–175.
[11] J. I. Brown, R. J. Nowakowski, Average independence polynomials, *J. Combin. Theory Series B* **93** (2005) 313–318.
[12] P. Csikvari, M. R. Oboudi, On the roots of edge cover polynomials of graphs, *European J. Combin.* **32** (2011) 1407–1416.
[13] K. Dohmen, P. Tittmann, Domination reliability, *Electron. J. Combin.* **19** (2012) Art# P15.
[14] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, *J. Amer. Math. Soc.* **25** (2012) 907–927.
[15] T. Kotek, J. Preen, P. Tittmann, Subset-sum representations of domination polynomials, *Graphs Combin.* **30** (2014) 647–660.
[16] M. R. Oboudi, On the number of perfect matchings of generalized theta graphs and the edge cover polynomials of friendship graphs, *Filomat* **33** (2019) 111–120.
[17] M. R. Oboudi, Some results on the independence polynomial of unicyclic graphs, *Discuss. Math. Graph Theory* **38** (2018) 515–524.
[18] M. R. Oboudi, On the largest real root of independence polynomials of trees, *Ars Combin.* **137** (2018) 149–164.
[19] M. R. Oboudi, On the roots of domination polynomial of graphs, *Discrete Appl. Math.* **205** (2016) 126–131.
[20] R. C. Read, An introduction to chromatic polynomials, *J. Combin. Theory* **4** (1968) 52–71.
[21] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics and geometry, Ann. New York Acad. Sci. **576** (1989) 500–534.
[22] Y. Wang, B. X. Zhu, On the unimodality of independence polynomials of some graphs, *European J. Combin.* **32** (2011) 10–20.