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Research Article

A note on hyper-plane arrangements in $\mathbb{R}^d$

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Abstract

This note addresses hyper-plane arrangements in $\mathbb{R}^d$ and functions that are constant in the interior of each of the d-dimensional faces of the arrangement. We show that such a function g can be expressed in a simple form using basis functions that are products of d or less indicator functions of the open half-spaces bounded by the hyper-planes in the arrangement. Moreover, we present a simple and efficient algorithm that can be used to express g as a linear combination of these basis functions.

Keywords: hyperplanes; hyperplane-arrangements; linear basis.

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1 Introduction

For hyper-planes $H_1, \ldots, H_n$ in $\mathbb{R}^d$ we denote by $\mathscr{A}$ the arrangement of the n hyper-planes $H_1, \ldots, H_n$. Hyper-plane arrangements is a broad and well studied topic in mathematics with relations and applications to many other fields in mathematics and computer science [5, 8, 9]. The objects of $\mathscr{A}$ are all the possible intersections of hyper-planes and open half-spaces defined by $H_1, \ldots, H_n$. Of particular interest to us will be the d-dimensional faces of the arrangement $\mathscr{A}$. These are in fact the connected components of $\mathbb{R}^d \setminus \cup_{i=1}^n H_i$.

We recall the function $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$ that is equal to 1 for every positive number and is equal to -1 for negative numbers. We artificially define $\mathrm{sign}\,(0)$ to be -1. As will be elaborated below, the value of $\mathrm{sign}\,(0)$ will not be important for us at all in the sequel.

In this note we address functions $g : \mathbb{R}^d \to \mathbb{R}$ that can be expressed in terms of the n sign functions $\mathrm{sign}\,(\langle x, v_i\rangle - c_i)$, $1 \le i \le n$, where $v_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ are some given vectors and constants, respectively. Such functions were encountered in the study of estimation and control of linear systems forced by Cauchy [3, 4, 6, 7] and Laplacian [1, 2] noises. When derived, these functions g have a very complex form that leads to numerical difficulties in applications. The goal of this note is to propose an alternative and simpler representation of those functions. In addition, it will show a simple and fast algorithm to construct this representation. In this paper we will not care about the values of the function g at points x for which $\langle x, v_i\rangle - c_i = 0$ for some $1 \le i \le n$. This is the reason why the value of $\mathrm{sign}\,(0)$ will not be of any importance in this paper. For convenience we set $\mathrm{sign}\,(0) = -1$.

It will be more convenient to work with the functions $\sigma_i(x) = \frac{1}{2}\left[\mathrm{sign}\,(\langle x, v_i\rangle - c_i) + 1\right]$. Notice that $\sigma_i(x)$ is the indicator function of the open half-space $H_i^+ = \{x \in \mathbb{R}^d \mid \langle x, v_i\rangle > c_i\}$. That is, $\sigma_i(x) = 1$ for every x in H_i^+ and is equal to 0 otherwise. Observe that a function g can be expressed in terms of the functions $\mathrm{sign}\,(\langle x, v_i\rangle - c_i)$ if and only if it can be expressed in terms of $\sigma_1, \ldots, \sigma_n$.

Because each of the functions $\sigma_1, \ldots, \sigma_n$ is constant in every d-dimensional face of $\mathscr{A}$, then so are the functions g that we study in this note.

It is easy to see that also the other way around is true. That is, given $H_1, \ldots, H_n$, any function $g : \mathbb{R}^d \setminus \cup_{i=1}^n H_i \to \mathbb{R}$ that is constant in every d-dimensional face of $\mathscr{A}$ can be expressed in terms of $\sigma_1, \ldots, \sigma_n$. To prove this observe that it is enough to address functions g that are equal to 1 for every x in some d-dimensional face F of $\mathscr{A}$ and are equal to 0 otherwise. Let $I_n = \{1, \ldots, n\}$ and $I \subset I_n$ be the set of all indices i such that H_i supports a $(d-1)$-dimensional facet of F. Then F is equal to the intersection of all half-spaces containing F and bounded by H_i for some $i \in I$. Let I_a be the set of all indices $i \in I$ such that $\sigma_i(x) = 1$ for every $x \in F$. Let $I_b = I \setminus I_a$. Then $g = \prod_{i \in I_a} \sigma_i \cdot \prod_{i \in I_b}(1 - \sigma_i)$. Consequently, any function

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$g : \mathbb{R}^d \setminus \cup_{i=1}^n H_i \to \mathbb{R}$ that is constant in every d-dimensional face of $\mathscr{A}$ can be written as a linear combination of products of n or less of $\sigma_1, \ldots, \sigma_n$.

The main result in this note is the following improvement that is also tight.

Theorem 1.1. *Let $\mathscr{A}$ be a hyper-plane arrangement of n affine hyper-planes $H_1, \ldots, H_n$ in $\mathbb{R}^d$ defined by $H_i = \{x \mid \langle x, v_i \rangle = c_i\}$, where $x \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$ is normal to H_i, and $c_i \in \mathbb{R}$. For every $1 \leq i \leq n$ let σ_i denote the indicator function of the open half-space $\{x \mid \langle x, v_i \rangle > c_i\}$ bounded by H_i. Let g be any function that is constant in the interior of every d-dimensional face in $\mathscr{A}$. Then there is a linear combination of products of d or less of the functions σ_i that is equal to g at any point in $\mathbb{R}^d \setminus \cup_{i=1}^n H_i$.*

Given a hyper-plane arrangement $\mathscr{A}$ of n affine hyper-planes $H_1, \ldots, H_n$ in $\mathbb{R}^d$ and a function $g : \mathbb{R}^d \setminus \cup_{i=1}^n H_i \to \mathbb{R}$ that is constant on every d-dimensional face of $\mathscr{A}$, Theorem 1.1 tells us that we can write g as a linear combination of products of d or less of the functions $\sigma_1, \ldots, \sigma_n$. Specifically, let $I \subset I_n$ be a subset of I_n with cardinality $|I| \leq d$ and denote by σ_I the product $\prod_{i \in I} \sigma_i$. When I is the empty set, we define $\sigma_\emptyset = 1$. Theorem 1.1 implies that the function g can be expressed as

$$g = \sum_{|I| \leq d} a_I \sigma_I \tag{1}$$

for all $I \subset I_n$ with $|I| \leq d$ with some coefficients a_I.

The number of such possible products σ_I and thus terms in the sum of (1) is equal to $N = \sum_{i=0}^{d} \binom{n}{i}$. This raises the question of an efficient computation of the N coefficients a_I in (1). In Section 3 we provide an algorithm that given the function g computes these coefficients with running time of $O(2^d \binom{n}{d})$. That is, number of operations in the algorithm is $O(2^d \binom{n}{d})$, where calling to the function g is considered as one operation.

2 Proof of Theorem 1.1

We start with a preliminary result that will be used to prove the main theorem presented next. When stated separately, not within the problem addressed in this note, its statement and proof can be greatly simplified, without hampering its generalization.

Lemma 2.1. *Let Δ_d be a d-simplex in $\mathbb{R}^d$. Let $H_1, \ldots, H_{d+1}$ be the $d+1$ affine hyper-planes supporting the facets of Δ_d. For $i = 1, \ldots, d+1$, let σ_i be the indicator function of the closed half-space bounded by H_i and containing Δ_d. Then*

$$\prod_{i=1}^{d+1} \left(1 - \sigma_i\right) = 0. \tag{2}$$

Proof. Assume, without loss of generality, that $0 \in \Delta_d$. For $i = 1, \ldots, d+1$ we write H_i as $H_i = \{x \mid \langle x, v_i \rangle = c_i\}$, where $v_i \in \mathbb{R}^d$ (orthogonal to H_i) is chosen such that $c_i > 0$. Then $\Delta_d = \{x \mid \forall\ 1 \leq i \leq d+1, \langle x, v_i \rangle \leq c_i\ \}$. Observe that the statement of the Lemma 2.1 is equivalent to saying that there is no vector $u \in \mathbb{R}^d$ such that $\langle u, v_i \rangle > c_i$ for every $1 \leq i \leq d+1$. Assume to the contrary that there is such a vector u. Then for every $\alpha > 0$ and every $1 \leq i \leq d+1$ we have $\langle -\alpha u, v_i \rangle = -\alpha \langle u, v_i \rangle < -\alpha c_i < 0 < c_i$. In other words, $-\alpha u \in \Delta_d$ for every $\alpha > 0$. This is impossible as Δ_d is bounded. □

We can now proceed to the proof of Theorem 1.1. Observe that in order to prove Theorem 1.1 it is enough to consider functions g that are indicator functions of d-dimensional faces in $\mathscr{A}$.

Let F be a d-dimensional face in $\mathscr{A}$ and let g be the indicator function of F. Let $I \subseteq I_n$ denote the set of indices i such that H_i supports F at a facet of dimension $d-1$. Let I_a and I_b be a partition of I into two parts such that if $i \in I_a$, then $F \subset \{x \mid \langle x, v_i \rangle > c_i\}$ and if $i \in I_b$, then $F \subset \{x \mid \langle x, v_i \rangle < c_i\}$. Observe that F is equal to the intersection of all open half-spaces containing F that are bounded by some hyper-plane H_i where $i \in I$. Therefore, the function

$$\tilde{g} = \prod_{i \in I_a} \sigma_i \cdot \prod_{i \in I_b} (1 - \sigma_i)$$

is equal to g at any point not in $\cup_{i=1}^n H_i$.

If the cardinality $|I|$ of I is smaller than or equal to d, we are done because $\tilde{g}$ can clearly be written as a linear combination of products of $|I|$ or less of the indicator functions $\sigma_1, \ldots, \sigma_n$. If the cardinality of I is larger than d, then $\tilde{g}$ can still be written as a linear combination of products of the indicator functions $\sigma_1, \ldots, \sigma_n$, however the number of terms in each product may exceed d. Therefore, Theorem 1.1 will follow if we can show that the product of every $d+1$ of the indicator functions $\sigma_1, \ldots, \sigma_n$ is equal, on $\mathbb{R}^d \setminus \cup_{i=1}^n H_i$, to a linear combination of products of d or less of the indicator functions $\sigma_1, \ldots, \sigma_n$.

We prove this by induction on d. The basis of the induction is the case $d = 1$. In this case we have two indicator functions, say σ_1 and σ_2. We would like to consider the function $\sigma_1\sigma_2$ and express it as a linear combination of products of zero or one of the functions σ_1 and σ_2.

This could easily be left to the reader, but for completeness we bring the simple analysis here. For $i = 1, 2$ there exists x_i such that the function σ_i is either the indicator function of $\{x \mid x < x_i\}$ or of $\{x \mid x > x_i\}$. Without loss of generality assume that $x_1 \leq x_2$. We consider four possible cases.

Case 1. σ_1 is the indicator function of $\{x \mid x < x_1\}$ and σ_2 is the indicator function of $\{x \mid x < x_2\}$. In this case $\sigma_1\sigma_2 = \sigma_1$.

Case 2. σ_1 is the indicator function of $\{x \mid x < x_1\}$ and σ_2 is the indicator function of $\{x \mid x > x_2\}$. In this case $\sigma_1\sigma_2 = 0$.

Case 3. σ_1 is the indicator function of $\{x \mid x > x_1\}$ and σ_2 is the indicator function of $\{x \mid x < x_2\}$. In this case $\sigma_1\sigma_2$ is equal to the function $\sigma_1 + \sigma_2 - 1$.

Case 4. σ_1 is the indicator function of $\{x \mid x > x_1\}$ and σ_2 is the indicator function of $\{x \mid x > x_2\}$. In this case $\sigma_1\sigma_2$ is equal to the function σ_2.

This concludes the case $d = 1$ being the basis of induction.

For $d > 1$ we consider two possible cases:

Case 1. $k+1$ of the vectors $v_1, \ldots, v_{d+1}$ are linearly dependent for some $1 \leq k < d$. Without loss of generality, assume that $v_1, \ldots, v_{k+1}$ are linearly dependent. By a possible rotation of $\mathbb{R}^d$, we can assume that $\text{span}\{v_1, \ldots, v_{k+1}\} \subseteq \text{span}\{e_1, \ldots, e_k\}$, where $e_1, \ldots, e_k$ are the first k elements of the standard basis of $\mathbb{R}^d$. Let $P : \mathbb{R}^d \to \mathbb{R}^k$ be the projection on the first k coordinates of $\mathbb{R}^d$. In $\mathbb{R}^k$, for every $1 \leq i \leq k+1$ we define $H'_i = \{x \in \mathbb{R}^k \mid \langle P(v_i), x\rangle = c_i\}$ and let $\sigma'_i : \mathbb{R}^k \to \mathbb{R}$ be the indicator function of $\{x \in \mathbb{R}^k \mid \langle P(v_i), x\rangle > c_i\}$. Observe that for every $x \in \mathbb{R}^d$ and $1 \leq i \leq k+1$,

$$\sigma_i(x) = \sigma'_i(P(x)). \tag{3}$$

Because $k < d$, we can apply the induction hypothesis for dimension k and conclude that $\prod_{i=1}^{k+1} \sigma'_i$ is equal to a linear combination of products of k or less of $\sigma'_1, \ldots, \sigma'_{k+1}$. Because of (3) it follows that $\prod_{i=1}^{k+1} \sigma_i$ is equal to a linear combination of products of k or less of $\sigma_1, \ldots, \sigma_{k+1}$. Consequently,

$$\prod_{i=1}^{d+1} \sigma_i = \prod_{i=1}^{k+1} \sigma_i \cdot \prod_{i=k+2}^{d+1} \sigma_i$$

is equal to a linear combination of products of d or less of $\sigma_1, \ldots, \sigma_{d+1}$.

Case 2. every set of d vectors from $v_1, \ldots, v_{d+1}$ is linearly independent. We split into two possible subcases.

Case 2a. $\cap_{i=1}^{d+1} H_i = \emptyset$. In this case $H_1, \ldots, H_{d+1}$ are the $d+1$ affine hyper-planes supporting the facets of the d-simplex Δ_d whose vertices are $v_j = \cap_{i=1, i\neq j}^{d+1} H_i$ for $1 \leq j \leq d+1$. We may assume, without loss of generality, that $0 \in \Delta_d$. Let $I = \{1, \ldots, d+1\}$. Let I_a and I_b be a partition of I into two parts such that if $i \in I_a$, then $c_i < 0$, and if $i \in I_b$, then $c_i > 0$. Then

$$\prod_{i \in I_a} \sigma_i \cdot \prod_{i \in I_b} (1 - \sigma_i)$$

is the indicator function of the interior of Δ_d. Applying (2) of Lemma 2.1 yields

$$\prod_{i \in I_a} (1 - \sigma_i) \cdot \prod_{i \in I_b} \sigma_i = 0,$$

which proves the theorem for this case.

Case 2b. $\cap_{i=1}^{d+1} H_i \neq \emptyset$. In this case $\cap_{i=1}^{d+1} H_i$ is a single point, because $v_1, \ldots, v_d$ are linearly independent. Without loss of generality we assume that this single point is 0. Consequently, $c_i = 0$ for $1 \leq i \leq d+1$. Let $\alpha_1, \ldots, \alpha_{d+1}$ be real numbers, not all zero, such that $\sum_{i=1}^{d+1} \alpha_i v_i = 0$. Notice that in this case $\alpha_i \neq 0$ for all $1 \leq i \leq d+1$, because every set of d vectors from $v_1, \ldots, v_{d+1}$ is linearly independent.

Define $I_a = \{1 \leq i \leq d+1 \mid \alpha_i > 0\}$ and $I_b = \{1 \leq i \leq d+1 \mid \alpha_i < 0\}$. Notice that I_a and I_b form a partition of $\{1, \ldots, d+1\}$. We claim that

$$\prod_{i \in I_a} \sigma_i \cdot \prod_{i \in I_b} (1 - \sigma_i) = 0. \tag{4}$$

Observe that once (4) is established we are done, as (4) implies that $\prod_{i=1}^{d+1} \sigma_i$ is a linear combination of products of d or less of $\sigma_1, \ldots, \sigma_{d+1}$.

To prove (4), notice that the contrary assumption is that there exists a vector u such that for every $i \in I_a$ we have $\langle u, v_i\rangle > 0$ and for every $i \in I_b$ we have $\langle u, v_i\rangle < 0$. It follows now from the definition of I_a and I_b that for every $1 \leq i \leq d+1$

we have $\alpha_i \langle u, v_i \rangle > 0$. This is a contradiction as

$$\sum_{i=1}^{d+1} \alpha_i \langle u, v_i \rangle = \left\langle u, \sum_{i=1}^{d+1} \alpha_i v_i \right\rangle = \langle u, 0 \rangle = 0. \tag{5}$$

3 Computational aspects

Assume we are given a hyper-plane arrangement $\mathscr{A}$ of n affine hyper-planes $H_1, \ldots, H_n$ in $\mathbb{R}^d$ and a function g that is constant on each of the d-dimensional faces of $\mathscr{A}$. For every $i = 1, \ldots, n$ we let σ_i be the indicator function of a open half-space bounded by H_i (we may choose any of the two). In this section we provide an algorithm of running time $O(2^d \binom{n}{d})$ that produces the representation of g as a linear combination of products of d or less of $\sigma_1, \ldots, \sigma_n$.

We start with a preliminary observation that we will need.

Observation 3.1. *Assume $H_1, \ldots, H_d$ are d hyper-planes in $\mathbb{R}^d$, passing through the origin, with normal vectors that are linearly independent. Then $H_1, \ldots, H_d$ partition $\mathbb{R}^d$ into 2^d regions and the horizontal hyper-plane $\{x_d = 0\}$ must avoid at least two of them. Moreover, if $\{x_d = 0\}$ is not parallel to any intersection of $d-1$ or less of $H_1, \ldots, H_d$, then the horizontal hyper-plane must avoid exactly two of the 2^d regions.*

Proof. Perhaps the easiest way to see this is to perform a linear transformation on $\mathbb{R}^d$ such that $H_1, \ldots, H_d$ coincide with the d axis-parallel hyper-planes through the origin. Then $H_1, \ldots, H_d$ partition $\mathbb{R}^d$ into 2^d region $F_1, \ldots, F_{2^d}$ that correspond to the 2^d different sign patterns of the d coordinates.

The linear transformation we performed takes the horizontal hyper-plane $\{x_d = 0\}$ to some hyper-plane H with a normal vector v. Observe that the condition that H is not parallel to any intersection of $d-1$ or less of $H_1, \ldots, H_d$ is equivalent to that no coordinate of v is equal to 0.

There are at least two of the regions $F_1, \ldots, F_{2^d}$ (and exactly two if the coordinates of v are all different from 0) whose coordinates sign patterns either always agree with the sign pattern of the coordinates of v, or always disagree with the sign pattern of the coordinates of v. These are exactly the regions avoided by H. □

We will first assume that the arrangement $\mathscr{A}$ is in general position in the sense that no $d+1$ hyper-planes from $H_1, \ldots, H_n$ share a common point. For reasons that will become clear shortly, we assume without loss of generality that no face in $\mathscr{A}$ is horizontal, i.e., is parallel to the hyper-plane $\{x_d = 0\}$. This can be achieved by a generic rotation of $\mathscr{A}$.

We start by computing all the $\binom{n}{d}$ vertices of $\mathscr{A}$. This can easily be done in time $O(d^3 \binom{n}{d})$ (we do not try to optimize here as long as we are at least as fast as $O(2^d \binom{n}{d})$), simply by considering every subset of d of hyper-planes from $H_1, \ldots, H_n$ and then solving a system of d linear equations in d variables.

It follows from Observation 3.1 that every vertex in $\mathscr{A}$ is the unique lowest point of a unique d-dimensional face in $\mathscr{A}$. Indeed, at every vertex X of $\mathscr{A}$ apply Observation 3.1, where $H_1, \ldots, H_d$ are taken to be the d hyper-planes meeting at X and let H in Observation 3.1 be the horizontal hyper-plane through X, that is, H is the hyper-plane through X parallel to $\{x_d = 0\}$.

Theorem 1.1 tells us that any function g that is constant on every d-dimensional face of $\mathscr{A}$ can be written as a linear combination of products of d or less of $\sigma_1, \ldots, \sigma_n$, given in (1). Notice that because $H_1, \ldots, H_n$ are in general position this implies that the $N = \sum_{i=0}^{d} \binom{n}{i}$ products of d or less of $\sigma_1, \ldots, \sigma_n$ form in fact a basis for the space of all functions g that are constant on every d-dimensional face of $\mathscr{A}$. This is because the number of d-dimensional faces in any arrangement of n hyper-planes in general position is precisely N (see [5, 8, 9]). Although we will not use the fact that the products of d or less of $\sigma_1, \ldots, \sigma_n$ are linearly independent, it is useful to observe this fact in order to understand better the proof.

We will use, however, the following simple observation. We claim that even if we replace any σ_i by $(1 - \sigma_i)$ in any of the products of d or less of $\sigma_1, \ldots, \sigma_n$ we will still remain with a basis for the space of all functions g that are constant on every d-dimensional face of $\mathscr{A}$. To see this, let I be a set of k indices from I_n, where $1 \leq k \leq d$ (the case $k = 0$ is trivial). Let $I' \subset I$ be any subset of I. We would like to show that we can replace $\prod_{i \in I} \sigma_i$ by $\prod_{i \in I'} (1 - \sigma_i) \cdot \prod_{i \in I \setminus I'} \sigma_i$ and remain with a basis for the space of all functions g that are constant on every d-dimensional face of $\mathscr{A}$. This is indeed true because $\prod_{i \in I'} (1 - \sigma_i) \cdot \prod_{i \in I \setminus I'} \sigma_i$ is equal to $\pm \prod_{i \in I} \sigma_i$ plus a linear combination of products of $k-1$ or less of $\sigma_1, \ldots, \sigma_n$. This fact can now be used to prove by induction on k that the replacement of any σ_i by $(1 - \sigma_i)$ does not change the linear span of these products.

We are given a function g that is constant on every d-dimensional face of $\mathscr{A}$ and we would like to find the representation of it, that surely exists because of Theorem 1.1, as in (1). In particular, we want to find the N coefficients a_I given the values of g in the N d-dimensional faces of $\mathscr{A}$.

For every $I \subset I_n$ such that $|I| = d$ we consider the point X_I that is the intersection of all the d hyper-planes H_i where $i \in I$. As we observed, X_I is the unique lowest point of a unique d-dimensional face that we denote by F_I. Let $I' \subset I$ be the set of all indices $i \in I$ such that σ_i is equal to 0 on F_I. Define $\sigma'_I = \prod_{i\in I'}(1-\sigma_i)\cdot\prod_{i\in I\setminus I'}\sigma_i$. Observe that σ'_I is equal to 0 at every point below X_I. Moreover, σ'_I is equal to 0 in all the d-dimensional faces that have X_I as a vertex, except for F_I where σ'_I is equal to 1.

As we already noticed, g can be written also as a linear combination of σ'_I for $|I| = d$ and σ_I for $|I| < d$. Let a'_I for $|I| \le d$ be the coefficients such that $g = \sum_{|I|=d} a'_I\sigma'_I + \sum_{|I|<d} a'_I\sigma_I$. Let H be a horizontal hyper-plane that is lower than the lowest vertex of $\mathscr{A}$. Notice that H intersects precisely all the d-dimensional faces that do not have a lowest point. Notice also that $\sum_{|I|=d} a'_I\sigma'_I$ is equal to 0 at every point on H simply because H is lower than all the vertices in $\mathscr{A}$. Therefore, on H we have $g = \sum_{|I|<d} a'_I\sigma_I$. We therefore consider the $(d-1)$-dimensional arrangement of $H\cap H_1,\ldots,H\cap H_n$ in the $d-1$ dimensional space H. We use induction and find the representation of $g = \sum_{|I|<d} a'_I\sigma_I$ on H with running time $T(d-1)$ that we will analyze later (we will show that $T(d) = O(2^d\binom{n}{d})$). Having found the coefficients a'_I for $|I| < d$ we proceed to finding the coefficients a'_I for $|I| = d$.

Fix $I \subset I_n$ such that $|I| = d$. We show how to find a'_I. The point X_I is the intersection of the d hyper-planes H_i, where $i \in I$. By Observation 3.1, these hyper-planes partition $\mathbb{R}^d$ into 2^d d-dimensional regions and therefore, X_I is a vertex of precisely 2^d d-dimensional faces of $\mathscr{A}$. Denote these d-dimensional faces by $F_1,\ldots,F_{2^d}$ and recall that F_I is one of these faces.

Let S_I be the function $\prod_{i\in I'}(2(1-\sigma_i)-1)\cdot\prod_{i\in I\setminus I'}(2\sigma_i-1)$. To understand the simple meaning of S_I observe that $2\sigma_i - 1$ is equal to 1 in the half-space where $\sigma_i = 1$ and is equal to -1 in the half-space where $\sigma_i = 0$ (the same is true for $(2(1-\sigma_i)-1)$ with the change of role of the two half-spaces bounded by H_i). Therefore, S_I is equal to 1 in F_I (where $(1-\sigma_i) = 1$ for all $i \in I'$ and $\sigma_i = 1$ for all $i \in I\setminus I'$). Moreover S_I changes sign every time we cross one of the hyper-planes H_i, where $i \in I$.

We claim that

$$a'_I = \sum_{i=1}^{2^d} g(F_i)S_I(F_i). \tag{6}$$

Notice that evaluating a'_I in this way takes time 2^d (that is, 2^d calls to the function g).

To prove (6) recall that $g = \sum_{|J|=d} a'_J\sigma'_J + \sum_{|J|<d} a'_J\sigma_J$. Hence, (6) is restated a

$$a'_I = \sum_{|J|=d}\sum_{i=1}^{2^d} a'_J\sigma'_J(F_i)S_I(F_i) + \sum_{|J|<d}\sum_{i=1}^{2^d} a'_J\sigma_J(F_i)S_I(F_i). \tag{7}$$

Observe that we have $\sum_{i=1}^{2^d} a'_I\sigma'_I(F_i)S_I(F_i) = a'_I\sigma'_I(F_I)S_I(F_I) = a'_I$. This is because σ'_I is equal to 0 on all the faces $F_1,\ldots,F_{2^d}$ except for F_I on which σ'_I is equal to 1. Therefore, in order to complete the proof of (6) it is enough to examine the sums in (7) with $J \ne I$ and show that $\sum_{i=1}^{2^d} a'_J\sigma'_J(F_i)S_I(F_i) = 0$ for $|J| = d$ and $\sum_{i=1}^{2^d} a'_J\sigma_J(F_i)S_I(F_i) = 0$ for $|J| < d$.

Let $J \subset I_n$, $|J| = d$, and $J \ne I$. Because $|I| = d$, it must be that $I \setminus J$ is not empty. Choose arbitrarily some $q \in I\setminus J$. For every face F_i the value $\sigma'_J(F_i)$ is either 1 or 0. If $\sigma'_J(F_i) = 0$ for all $1 \le i \le 2^d$, we are done. For every face F_i with $\sigma'_J(F_i) = 1$ consider the unique face $F_{i'}$ such that F_i and $F_{i'}$ share a common facet on the hyper-plane H_q. Observe that for $F_{i'}$ it is also true that $\sigma'_J(F_{i'}) = 1$ because $q \notin J$. However $S_I(F_i) = -S_I(F_{i'})$. Therefore, in $\sum_{i=1}^{2^d} a'_J\sigma'_J(F_i)S_I(F_i)$ the terms i and i' for which $\sigma'_J(F_i) = \sigma'_J(F_{i'}) = 1$ cancel the contributions of each other, while the other terms are identically zero because their respective $\sigma'_J = 0$. Consequently, $\sum_{i=1}^{2^d} a'_J\sigma'_J(F_i)S_I(F_i) = 0$. A similar argument as above holds when $|J| < d$ to yield that $\sum_{i=1}^{2^d} a'_J\sigma_J(F_i)S_I(F_i) = 0$. This concludes the proof of equation (6).

We can therefore find all the coefficients a'_I for $|I| = d$ in time $O(2^d\binom{n}{d})$, as there are $\binom{n}{d}$ subsets I of I_n with $|I| = d$. To summarize, we found all the coefficients a'_I in the representation of g as $g = \sum_{|I|=d} a'_I\sigma'_I + \sum_{|I|<d} a'_I\sigma_I$ in time $O(2^d\binom{n}{d}) + T(d-1)$.

We are not done yet because we need to find the representation of g as $g = \sum_{|I|\le d} a_I\sigma_I$. In order to do this we recall that every σ'_I for $|I| = d$ is a product of d terms each of which is equal either to σ_i or to $(1-\sigma_i)$ for some $i \in I$. We can therefore write it as linear sum of at most 2^d products of d or less of σ_i where $i \in I$. This requires an additional work of time $O(2^d\binom{n}{d})$. Then we get a representation of g as a linear sum of at most

$$2^d\binom{n}{d} + \binom{n}{d-1} + \binom{n}{d-2} + \ldots + \binom{n}{0}$$

products of d or less of $\sigma_1,\ldots,\sigma_n$. With additional work of $O(2^d\binom{n}{d})$ we can gather similar terms and get the required representation of g as in (1).

In order to analyze the running time $T(d)$ we observe that $T(d) = T(d-1) + O(2^d\binom{n}{d})$. This easily implies $T(d) = O(2^d\binom{n}{d})$.

We have thus proved that the desired representation of g can be computed in time $O(2^d\binom{n}{d})$ when the arrangement $\mathscr{A}$ is in general position. We next show how to conclude from here an algorithm with the same running time also when $\mathscr{A}$ may not be in general position. What we do is perturb by just a little bit the hyper-planes $H_1, \ldots, H_n$ and define g to be 0 (although it could take arbitrary values just as well) in the newly created d-dimensional faces. More precisely, we first take sample points from every face in $\mathscr{A}$ before the perturbation. Then we perturb $\mathscr{A}$. We find which new d-dimensional faces correspond to the old ones. This can be done very quickly, as we just need to find the new d-dimensional faces to which each of the old sample points belong. Now that we have the perturbed hyper-plane arrangement that is in general position and the new function g, we apply our algorithm and the representation that we get for the new g will work also for the original one.

A disadvantage of the above method for arrangements $\mathscr{A}$ that are not in a general position is that the representation we obtain for g has more terms in (1) than the number of d-dimensional faces in $\mathscr{A}$. Effectively it means that the basis we use to represent g is larger than needed and one can find a smaller one, i.e., g could be fully represented as in (1) with the number of terms that equals to the number of d-dimensional faces in $\mathscr{A}$. In the next section we present an algorithm that finds such a minimum-dimensional basis and shows how to find the coefficients of a representation of g with this basis.

4 The non-general-position case

Theorem 4.1. *Let $\mathscr{A}$ be a hyper-plane arrangement of n affine hyper-planes $H_1, \ldots, H_n$ in $\mathbb{R}^d$. For every $1 \leq i \leq n$ let σ_i denote the indicator function of one of the two open half-spaces bounded by H_i. Let $\mathscr{G}$ be the linear vector space consisting of all functions g that are constant in the interior of every d-dimensional face in $\mathscr{A}$. Then there is a basis for $\mathscr{G}$ that consists of products of d or less of the functions σ_i.*

Remark. The size of the basis guaranteed in Theorem 4.1 is clearly equal to the dimension of $\mathscr{G}$, that is, the number of d-dimensional faces in $\mathscr{A}$.

Proof. We prove the theorem by induction on d. For $d = 1$ there is nothing to prove because in this case the theorem is equivalent to Theorem 1.1.

Assume the theorem is true for dimension $d-1$ and we prove it for dimension d. Given d is fixed, we prove the theorem by induction on n. The theorem is clearly true for $n = 1$ because in this case $\mathscr{A}$ consists of just two d-dimensional faces, that is, the two open half-spaces bounded by H_1. Then a basis for $\mathscr{G}$ is $\{1, \sigma_1\}$ and this is regardless of the choice of the indicator function σ_1.

Assume the theorem is true for $n-1$ and we prove it for n. We would like to find a basis for $\mathscr{G}$ in which every element is a product of d or less of $\sigma_1, \ldots, \sigma_n$. Consider the hyper-plane arrangement $\tilde{\mathscr{A}}$ of the $n-1$ hyper-planes $H_1, \ldots, H_{n-1}$. By induction hypothesis, there is a basis whose elements are products of d or less of $\sigma_1, \ldots, \sigma_{n-1}$ for the linear space $\tilde{\mathscr{G}}$ of all the functions that are constant on the interior of every d-dimensional face of $\tilde{\mathscr{A}}$. Denote this basis by $a_1, \ldots, a_k$, where k is the dimension of $\tilde{\mathscr{G}}$.

Consider now the hyper-plane H_n. For $i = 1, \ldots, n-1$ denote by H'_i the intersection $H_i \cap H_n$. Therefore, H'_i is a hyper-plane of dimension $d-1$ in H_n. Let $\mathscr{A}'$ denote the hyper-plane arrangement of $H'_1, \ldots, H'_{n-1}$ inside H_n. Let $\mathscr{G}'$ be the linear space of all functions that are constant on every $(d-1)$-dimensional face of $\mathscr{A}'$. For $i = 1, \ldots, n-1$ let σ'_i denote the restriction of σ_i to H_n. Notice that σ'_i is the indicator function of one of the open half-spaces of H_n bounded by H'_i. By the induction hypothesis, there is a basis for $\mathscr{G}'$ whose elements are products of $d-1$ or less of $\sigma'_1, \ldots, \sigma'_{n-1}$. Denote the elements of this basis by $b'_1, \ldots, b'_\ell$, where ℓ is the dimension of $\mathscr{G}'$. For $j = 1, \ldots, \ell$ let I_j be the set of indices from $\{1, \ldots, n-1\}$ such that $b'_j = \prod_{i \in I_j} \sigma'_i$.

We observe that the number of d-dimensional faces in $\mathscr{A}$ is equal to the number of d-dimensional faces in $\tilde{\mathscr{A}}$ plus the number of $(d-1)$-dimensional faces in $\mathscr{A}'$. In other words, the dimension of $\mathscr{G}$ is equal to $k + \ell$, which is the sum of the dimension of $\tilde{\mathscr{G}}$ and the dimension of $\mathscr{G}'$.

We are now ready to define the desired basis for $\mathscr{G}$. For $j = 1, \ldots, \ell$, considering the subsets I_j used to define b'_j above, define $b_j = \prod_{i \in I_j} \sigma_i$. We claim that $B = \{a_1, \ldots, a_k\} \cup \{\sigma_n b_1, \ldots, \sigma_n b_\ell\}$ is a basis for $\mathscr{G}$. We remark that once we prove B is a basis for $\mathscr{G}$ we are done, as every member of B is a product of d or less of $\sigma_1, \ldots, \sigma_n$. Moreover, to conclude that B is indeed a basis for $\mathscr{G}$ it is enough to show that it spans $\mathscr{G}$, because $|B|$ is equal to $k + \ell$, the dimension of $\mathscr{G}$.

In order to show that B spans $\mathscr{G}$ it is enough to show that the indicator function of every d-dimensional face in $\mathscr{A}$ can be written as a linear combination of members of B. Let C be a d-dimensional face of $\mathscr{A}$. If none of the facets of C is supported by H_n, then C is also a d-dimensional face in $\tilde{\mathscr{A}}$ and therefore the indicator function of C can be written as a linear combination of $a_1, \ldots, a_k$.

Assume therefore that H_n supports a facet of C. There is a unique cell in $\tilde{\mathscr{A}}$ that contains C. We denote this cell by $\tilde{C}$. In fact C is equal to the intersection of $\tilde{C}$ with one of the two open half-spaces bounded by H_n.

Let $C' = H_n \cap \tilde{C}$. C' is a $(d-1)$-dimensional face in $\mathscr{A}'$. Therefore, the indicator function of C' can be written as $\sum_{j=1}^{\ell} \beta_j b'_j$ for some coefficients $\beta_1, \ldots, \beta_\ell$. Now consider the function $g = \sum_{j=1}^{\ell} \beta_j b_j$. Notice that g is constant on every d-dimensional face of $\mathscr{A}$. Moreover, because the restriction of g to H_n is equal to $\sum_{j=1}^{\ell} \beta_j b'_j$, that is, to the indicator function of C', then g must be equal to 1 on $\tilde{C}$ and must be equal to 0 on every other d-dimensional face of $\tilde{\mathscr{A}}$ whose interior is intersected by H_n.

Consider now the function $\sigma_n g$. This function is constant on every d-dimensional face of $\tilde{\mathscr{A}}$ except for $\tilde{C}$. Indeed, $\sigma_n g$ is constant on every d-dimensional face of $\tilde{\mathscr{A}}$ whose interior is not intersected by H_n. It is equal to 0 on every d-dimensional face of $\tilde{\mathscr{A}}$ whose interior is intersected by H_n except for $\tilde{C}$. The face $\tilde{C}$ is a union of two d-dimensional faces in $\mathscr{A}$, namely, C and $\tilde{C} \setminus C$. The function $\sigma_n g$ is equal to 1 on one of C and $\tilde{C} \setminus C$ and is equal to 0 on the other.

Let f be the function that is equal to $\sigma_n g$ except that f is equal to 0 on $\tilde{C}$. Because f is in $\tilde{\mathscr{G}}$, we can write $f = \sum_{j=1}^{k} \alpha_j a_j$ for some coefficients $\alpha_1, \ldots, \alpha_k$. Now, the function $\sigma_n g - f$ is the indicator function of either C or $\tilde{C} \setminus C$, the one for which $\sigma_n g$ is equal to 1. If $\sigma_n g - f$ is the indicator function of C, then we are done because

$$\sigma_n g - f = \sum_{j=1}^{\ell} \beta_j \sigma_n b_j - \sum_{j=1}^{k} \alpha_j a_j.$$

If $\sigma_n g - f$ is the indicator function of $\tilde{C} \setminus C$, then write the indicator function of $\tilde{C}$ as $\sum_{j=1}^{k} \gamma_j a_j$. Then the indicator function of C is equal to

$$\sum_{j=1}^{k} \gamma_j a_j - (\sigma_n g - f) = \sum_{j=1}^{k} (\gamma_j + \alpha_j) a_j - \sum_{j=1}^{\ell} \beta_j \sigma_n b_j,$$

and we are done again. □

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