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*Research Article*

# An asymptotic relation between the wirelength of an embedding and the Wiener index

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**Abstract**

Wirelength is an important criterion to validate the quality of an embedding of a graph into a host graph and is used in particular in VLSI (Very-Large-Scale Integration) layout designs. Wiener index plays a significant role in mathematical chemistry, cheminformatics, and elsewhere. In this note these two concepts are related by proving that the Wiener index of a host graph is an upper bound for the wirelength of a given embedding. The wirelength of embedding complete $2^p$-partite graphs into Cartesian products of paths and/or cycles as the function of the Wiener index is also determined. The latter result is an asymptotic approximation of the general upper bound.

**Keywords:** Wiener index; embedding; wirelength; complete $2^p$-partite graph; Cartesian product of graphs; integer labeling.

**2020 Mathematics Subject Classification:** 05C12, 05C60.

## 1. Introduction

The embedding of interconnection networks is aimed to analyze the interdependence between two graphs because it aims to determine whether a particular guest graph is included in the host graph or they are correlated with each other. Wirelength is among the most important criteria to validate the quality of an embedding. The wirelength of a graph embedding arises from VLSI (Very-Large-Scale Integration) design, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering, and so on [3, 12, 17, 18, 23, 24, 26, 29].

The Wiener index of a graph, defined as the sum of distances between all unordered pairs of vertices, besides its crucial role in the calculation of average distance, is the most famous topological index in mathematical chemistry [28]. In chemical graph theory, the Wiener index is used to study the relationship between molecular structure and physical and chemical properties of compounds. In computer science, the average distance is used as a fundamental parameter to measure the communication cost of networks. For a very selected further information on the average distance we refer to [4, 20], and on the Wiener index to [9–11, 14, 19, 30].

A thorough literature survey reveals that the wirelength of an embedding and the Wiener index were so far investigated separately, cf. [1, 2, 5, 7, 8, 16, 22]. In this note, we make a connection between these two important concepts. A trivial connection is that if the guest graph is a complete graph, then the wirelength of an embedding from a complete graph into a graph $H$ is the Wiener index of $H$, cf. [25]. In this paper, we obtain the wirelength of embedding a complete $2^p$-partite graph into Cartesian products of paths and/or cycles (as a host graph) using the Wiener index of a host graph by multiplying with a dynamic factor. As already indicated, this appears to be the first time that a non-trivial connection between these two concepts is established.

Given graphs $G$ (guest) and $H$ (host), an *embedding* of $G$ into $H$ is an injective mapping $f : V(G) \to V(H)$ together with an assignment that, to every edge $e = xy \in E(G)$, assigns a path $P_f(e)$ in $H$ between $f(x)$ and $f(y)$. The *wirelength* [24] *of embedding $G$ into $H$* is defined as

$$WL(G, H) = \min_{f:V(G)\to V(H)} \sum_{e=xy\in E(G)} m(P_f(e)),$$

where $m(P_f(e))$ is the number of edges of the path $P_f(e)$. The paths $P_f(e)$ in an embedding $f$ of $G$ into $H$ in general need not be shortest paths because one can be interested in other properties of the embedding but the wirelength. On the

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other hand, $WL(G,H)$ will be realized on an embedding in which all paths $P_f(e)$ are shortest, hence the definition of the wirelength can be equivalently written as follows:

$$WL(G,H) = \min_{f:V(G)\to V(H)} \sum_{e=xy\in E(G)} d_H(f(x),f(y))\,,$$

where $d_G(u,v)$ denotes the length of a shortest path (that is, the number of its edges) between the vertices $u$ and $v$ of $G$.

In the next section we prove that the Wiener index of a host graph $H$ in an upper bound for the wirelength of embedding $G$ into $H$. In the subsequent section we derive formulas for the Wiener index of Cartesian products of a finite number of paths and/or cycles. In Section 4 we combine these expressions with earlier results to determine the wirelength of embedding complete $2^p$-partite graphs into a Cartesian product as a function of the Wiener index. This result is an asymptotic approximation of the bound from Section 2. We conclude with examples demonstrating that embedding complete $2^p$-partite graphs into some other host graphs does not have this property.

### 1.1. Further definitions and more on Cartesian product networks

Graphs in this note are connected, unless stated otherwise. The order of a graph $G$ is denoted by $n(G)$ and the size of $G$ by $m(G)$. The complete $p$-partite graph $G = K_{n_1,\ldots,n_p}$ is a graph that contains $p$ independent sets with respective cardinalities $n_i$, $i \in [p] = \{1,\ldots,p\}$, and all possible edges between vertices from different parts. For a graph $G$, the Wiener index $W(G)$ is defined as

$$W(G) = \frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d_G(u,v)\,.$$

The *Cartesian product* $G\,\Box\,H$ of (not necessarily connected) graphs $G$ and $H$ is the graph with the vertex set $V(G)\times V(H)$, vertices $(u,v)$ and $(u',v')$ being adjacent if either $u=u'$ and $vv'\in E(H)$, or $v=v'$ and $uu'\in E(G)$. The graphs $G$ and $H$ are called *factors* of $G\,\Box\,H$. Networks generated by Cartesian product of networks are very powerful in creating a large network from given small graphs and it is an important technique for planning large-scale interconnection networks. For more information on Cartesian product graphs see the book [13].

Hypercubes, grids, cylinders, and torii are powerful interconnection networks used to execute parallel algorithms [6,29]. The Cosmic Cube from Caltech, which is a landmark in computer design, the iPSC/2 from Intel, and the Connection Machines are based on hypercubes and have been implemented commercially [6]. Now, the $r$-dimensional hypercube $Q^r$, $r\geq 1$, is the Cartesian product of $r$ copies of $P_2$. Similarly, an $r$-dimensional grid is the Cartesian product of $r$ paths, an $r$-dimensional torus is the Cartesian product of $r$ cycles, and an $r$-dimensional cylinder is the Cartesian product of some paths and some cycles, $r$ of them together. Moreover, different modifications and/or extensions of these networks were proposed, especially of hypercubes, for instance, very recently $CCC(r,n)$ networks proposed in [31].

## 2. The connection

If a graph $G$ is sparser than a graph $H$, then $WL(G,H)$ is expected to be relatively small. For instance, if $G$ is a spanning subgraph of $H$, then the identity mapping $V(G)\to V(H)$ obtained by considering $G$ to be spanned in $H$, yields $WL(G,H)\leq m(G)$. Since $m(G)$ is clearly a general lower bound for $WL(G,H)$, this means that $W(G,H)=m(G)$ provided that $G$ is a spanning graph of $H$. On the other hand, if $G$ is relatively dense, then we roughly expect that $WL(G,H)$ is relatively large. The large is explained in the next result which connects the wirelength with the Wiener index.

**Theorem 2.1.** *If $G$ and $H$ are graphs with $n(G)=n(H)$, then $WL(G,H)\leq W(H)$. The equality holds if and only if $G$ is a complete graph.*

*Proof.* Let $f$ be a mapping from $V(G)$ to $V(H)$ for which $WL(G,H)$ is realized, that is,

$$WL(G,H) = \sum_{xy\in E(G)} d_H(f(x),f(y))\,.$$

Since for every edge $xy\in E(G)$ we have $f(x)\neq f(y)$, and for every two different edges $xy, x'y'\in E(G)$ we also have $\{f(x),f(y)\}\neq\{f(x'),f(y')\}$, we can estimate as follow:

$$\begin{aligned} WL(G,H) &= \sum_{xy\in E(G)} d_H(f(x),f(y)) \\ &= \sum_{\substack{xy\in E(G)\\ \{f(x),f(y)\}\in\binom{V(H)}{2}}} d_H(f(x),f(y)) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\{u,v\}\in\binom{V(H)}{2}} d_H(u,v) \\
&= W(H)\,.
\end{aligned}$$

The equality in the inequality above holds if and only if the sets $\{f(x), f(y)\}$ run over all 2-subsets of $V(H)$. This holds if and only if $G$ has $|\binom{V(H)}{2}| = |\binom{V(G)}{2}|$ edges, that is, if and only if $G$ is a complete graph. $\square$

## 3. Wiener index of Cartesian products

The Wiener index of Cartesian products graphs has been independently obtained several times, the seminal papers being [11, 30]. The result says that if $G$ and $H$ are graphs, then

$$W(G\,\square\,H) = n(G)^2 \cdot W(H) + n(H)^2 \cdot W(G)\,. \tag{1}$$

The simplest way to obtain (1) is to apply the so-called Distance Lemma which asserts that if $G$ and $H$ are graphs, and $(g,h),(g',h')$ are vertices of $V(G\,\square\,H)$, then $d_{G\square H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$. From here the formula (1) follows by a straightforward computation.

Since the Cartesian product operation is associative, Distance Lemma naturally extends to more than two factors. More precisely, if $k \geq 2$ and $G = \square_{i=1}^{k} G_i$, where $G_i$, $i \in [k]$, are graphs, then

$$d_G(g,g') = \sum_{i=1}^{k} d_{G_i}(g_i, g_i')\,,$$

where $g = (g_1,\ldots,g_k)$ and $g' = (g_1',\ldots,g_k')$ are vertices of $G$. From here one can again more or less straightforwardly deduce the Wiener index of Cartesian products of a finite number of factors, cf. [15, p. 46].

**Proposition 3.1.** *If $k \geq 2$, and $G_i$, $i \in [k]$ are graphs, then*

$$W(G_1\,\square\,G_2\,\square\,\cdots\,\square\,G_k) = \sum_{i=1}^{k}\left(W(G_i)\cdot\prod_{j\neq i} n(G_j)^2\right).$$

The following consequences of Proposition 3.1 are needed for our purpose.

**Corollary 3.1.** *If $k \geq 2$ and $r_i \geq 2$, $i \in [k]$, are integers such that $r_1 + \cdots + r_k = r$, then the following hold.*

*(i)* $W(P_{2^{r_1}}\,\square\,\cdots\,\square\,P_{2^{r_k}}) = \dfrac{2^{2r}}{6}\left[(2^{r_1}+\cdots+2^{r_k}) - \left(\dfrac{1}{2^{r_1}}+\cdots+\dfrac{1}{2^{r_k}}\right)\right]$.

*(ii)* $W(C_{2^{r_1}}\,\square\,\cdots\,\square\,C_{2^{r_k}}) = 2^{2r-3}\cdot(2^{r_1}+\cdots+2^{r_k})$.

*(iii)* $W(P_{2^{r_1}}\,\square\,\cdots\,\square\,P_{2^{r_s}}\,\square\,C_{2^{r_{s+1}}}\,\square\,\cdots\,\square\,C_{2^{r_k}}) = \dfrac{1}{6}\displaystyle\sum_{i=1}^{s} 2^{2r-r_i}(2^{2r_i}-1) + \sum_{i=s+1}^{k} 2^{2r+r_i-3}$.

*Proof.* (i) It is well-known (and easy to see) that $W(P_n) = \binom{n+1}{3} = n(n^2-1)/6$. Combining this fact with Proposition 3.1 we get:

$$\begin{aligned}
W(P_{2^{r_1}}\,\square\,\cdots\,\square\,P_{2^{r_k}}) &= \sum_{i=1}^{k}\frac{1}{6}2^{r_i}\left(2^{2r_i}-1\right)\cdot\prod_{j\neq i}2^{2r_j} = \frac{1}{6}\sum_{i=1}^{k}2^{r_i}\left(2^{2r_i}-1\right)\cdot\frac{2^{2r}}{2^{2r_i}} \\
&= \frac{2^{2r}}{6}\sum_{i=1}^{k}\left(2^{2r_i}-1\right)\cdot\frac{1}{2^{r_i}} \\
&= \frac{2^{2r}}{6}\sum_{i=1}^{k}\left(2^{r_i}-\frac{1}{2^{r_i}}\right).
\end{aligned}$$

The proof for (ii) proceeds along the same lines as for (i). The formula (iii) is then obtained by using the associativity of the Cartesian product and writing

$$P_{2^{r_1}}\,\square\,\cdots\,\square\,P_{2^{r_s}}\,\square\,C_{2^{r_{s+1}}}\,\square\,\cdots\,\square\,C_{2^{r_k}} = (P_{2^{r_1}}\,\square\,\cdots\,\square\,P_{2^{r_s}})\,\square\,(C_{2^{r_{s+1}}}\,\square\,\cdots\,\square\,C_{2^{r_k}})\,,$$

and then applying (1) together with the already established formulas (i) and (ii). $\square$

## 4. Asymptotically largest possible wirelengths

In this section we prove an exact formula for the wirelength as a function of the Wiener index that can be viewed as an asymptotic approximation of the bound from Theorem 2.1. For this sake we first recall the following two known results.

**Theorem 4.1.** [26] Let $G$ be the complete $t$-partite graph $K_{r,r,\ldots,r}$, $r, t \geq 2$. If $f$ is an embedding of $G$ into $H$, then

$(i)$ $WL(G,H) = rt(tr^2+1)(t-1)/6$, where $H$ is a path $P_n$, $n = tr$;

$(ii)$ $WL(G,H) = 2^{2n-p-3}(2^p-1)(2^{n-1}+1)$, where $H$ is the circulant graph $G(2^n; \pm\{1,2\})$, $t = 2^p$, $r = 2^{n-p}$ and $n \geq 2$.

**Theorem 4.2.** [27] *Let $G$ be the complete $2^p$-partite graphs $K_{2^{r-p},\ldots,2^{r-p}}$ and $H$ be the Cartesian product of $n \geq 3$ factors of respective order $2^{r_i}$, $i \in [n]$, where $r_1 \leq \cdots \leq r_n$, $r_1 + \cdots + r_n = r$, and each factor is a path or a cycle, $r \geq 3$, $1 \leq p < r$. If $s \geq 0$ factors of $H$ are paths, and the remaining $n-s$ factors are cycles, then*

$$WL(G,H) = \frac{2^{2r-p}(2^p-1)}{6}\left[(2^{r_1}+\cdots+2^{r_s}) - \left(\frac{1}{2^{r_1}}+\cdots+\frac{1}{2^{r_s}}\right)\right] + 2^{2r-p-3}(2^p-1)(2^{r_{s+1}}+\cdots+2^{r_n}).$$

The main result of this section now reads as follows.

**Theorem 4.3.** *Let $G$ be the complete $2^p$-partite graphs $K_{2^{r-p},\ldots,2^{r-p}}$, where $p \geq 1$, $r \geq 3$ and $p < r$. Let $H$ be the Cartesian product of $k \geq 3$ factors of respective order $2^{r_i}$, $i \in [k]$, where $r_1 + \cdots + r_k = r$, and each factor is a path or a cycle. Then*

$$WL(G,H) = \frac{(2^p-1)}{2^p}\, W(H)\,.$$

*Proof.* If $G$, $p$, $r$, $H$, and $k$ are as stated, then, by Theorem 4.2, we have the following. If $s \geq 0$ factors of $H$ are paths and the other factors are cycles, then,

$$WL(G,H) = \frac{1}{6}\sum_{i=1}^{s} 2^{2r-r_i-p}(2^p-1)(2^{2r_i}-1) + \sum_{i=s+1}^{k} 2^{2r+r_i-p-3}(2^p-1)\,.$$

(Note that this formula also includes the cases when all the factors are paths ($s = k$) and when all the factors are cycles ($s = 0$).) The result then follows by comparing the above formula with Corollary 3.1. □

A question arises, whether the equality $WL(G,H) = \frac{(2^p-1)}{2^p}\, W(H)$ can hold for some additional host graphs $H$ with $n(G) = n(H)$, where $G$ is the complete $2^p$-partite graph $K_{2^{r-p},2^{r-p},\ldots,2^{r-p}}$, $r \geq 3$, $p \geq 1$ and $p < r$. This is not the case in the following two examples.

Let $G = K_{8,8,8,8}$ and $H = P_{32}$. It is straightforward to check that $W(H) = 5456$. On the other hand, by Theorem 4.1$(i)$, we have $WL(G,H) = 4112$. Hence,

$$WL(G,H) = 4112 > \frac{(2^p-1)}{2^p}\, W(H) = \frac{3}{4}(5456) = 4092\,.$$

In the second example let $G = K_{4,4,4,4}$ and let $H$ be the circulant graph $G(16; \pm\{1,2\})$ as shown in Figure 1.

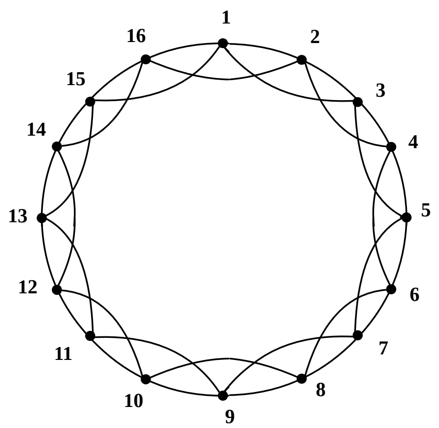


Figure 1: The circulant graph $G(16; \pm\{1,2\})$.

It is easy to verify that $W(H) = 320$, while by Theorem 4.1$(ii)$ we have $WL(G,H) = 216$. Hence,

$$WL(G,H) = 216 < \frac{(2^p-1)}{2^p}\, W(H) = \frac{3}{4}(320) = 240\,.$$

These observations lead to:

**Problem 4.1.** *Find families of (host) graphs $H$ such that*

$$WL(G,H) = \frac{(2^p-1)}{2^p}\, W(H)$$

*holds, where $G$ is the complete $2^p$-partite graph $K_{2^{r-p},\ldots,2^{r-p}}$, $r \geq 3,\ p \geq 1$ and $p < r$.*

A partial progress on Problem 4.1 has been recently reported in [21].

## 5. Conclusion

In this note we have obtained the wirelength $WL(G,H)$ of embedding $G$ onto $H$ using the Wiener index of $H$, where $G$ is the complete $2^p$-partite graph $K_{2^{r-p},\ldots,2^{r-p}}$ and $H$ is the Cartesian product of paths and cycles. Finding the wirelength of embedding complete multipartite graph into graphs such as Cayley graphs, permutation graphs, and interval graphs are under investigation.

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