**Abstract**

A red-white coloring of a nontrival connected graph \( G \) of diameter \( d \) is an assignment of red and white colors to the vertices of \( G \) where at least one vertex is colored red. Associated with each vertex \( v \) of \( G \) is a \( d \)-vector, called the code of \( v \), whose \( i \)th coordinate is the distance from \( i \) from \( v \). A red-white coloring of \( G \) for which distinct vertices have distinct codes is called an identification coloring or ID-coloring of \( G \). A graph \( G \) possessing an ID-coloring is an ID-graph. The minimum number of red vertices among all ID-colorings of an ID-graph \( G \) is the identification number or ID-number of \( G \). Necessary conditions are established for those trees that are ID-graphs. A tree \( T \) is starlike if \( T \) is obtained by subdividing the edges of a star of order 4 or more. It is shown that for every positive integer \( r \) different from 2, there exist starlike trees satisfying some prescribed properties having ID-number \( r \).

**Keywords:** distance; vertex identification; identification coloring; tree.

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1. Introduction

Over the years, many methods have been introduced with the goal of uniquely identifying the vertices of a connected graph. Often these approaches have employed distance and coloring. The oldest of these methods deal with what is referred to as the metric dimension of a connected graph. For a nontrival connected graph \( G \) of order \( n \), the goal is to locate an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of \( k \) vertices in \( G \), \( 1 \leq k \leq n \), and associate with each vertex \( v \) of \( G \) the \( k \)-vector \((a_1, a_2, \ldots, a_k)\), where \( a_i \) is the distance \( d(v, w_i) \) between \( v \) and \( w_i \), \( 1 \leq i \leq k \). If the \( n \) \( k \)-vectors produced in this manners are distinct, then the vertices of \( G \) have been uniquely identified. For each connected graph \( G \), such a set \( W \) can always be found since we can always choose \( W = V(G) \). The primary problem here is to determine the minimum size of such a set \( W \). This is referred to as the metric dimension of \( G \). Equivalently, the metric dimension of a connected graph \( G \) can be defined as the minimum number of vertices of \( G \) that can be assigned the same color, say red, such that for every two vertices \( u \) and \( v \) of \( G \), there exists a red vertex \( w \) such that \( d(u, w) \neq d(v, w) \). This parameter is defined for every connected graph.

Another method that has been studied to uniquely identify the vertices of a connected graph \( G \) has been referred to as the partition dimension of \( G \). For a nontrivial connected graph \( G \) of order \( n \), the goal is to obtain a \( k \)-coloring, \( 1 \leq k \leq n \), of the vertices of \( G \), where the coloring is not required (or expected) to be a proper coloring. This results in \( k \) color classes \( V_1, V_2, \ldots, V_k \) of \( V(G) \). For each vertex \( v \) of \( G \), we once again associate a vector, here a \( k \)-vector \((a_1, a_2, \ldots, a_k)\) where \( a_i \) denotes the distance from \( v \) to a nearest vertex in \( V_i \) for \( 1 \leq i \leq k \). If the vertices of \( G \) have distinct \( k \)-vectors, then the vertices of \( G \) have been uniquely identified. Such a coloring always exists since we can always assign distinct colors to the vertices of \( G \), thereby obtaining a procedure that has similarity to metric dimension. The minimum number of colors that accomplishes this goal is referred to as the partition dimension of \( G \). The partition dimension of a connected graph \( G \) can also be defined as the minimum number \( k \) of colors (denoted by \( 1, 2, \ldots, k \)) that can be assigned to the vertices of \( G \), one color to each vertex, so that for every two vertices \( u \) and \( v \) of \( G \), there exists a color \( i \) such that the distance between \( u \) and a nearest vertex colored \( i \) is distinct from the distance between \( v \) and a nearest vertex colored \( i \). This parameter is also defined for every connected graph.

Another method that has been introduced for the purpose of uniquely identifying the vertices of a connected graph is referred to as an identification coloring. Let \( G \) be a connected graph of diameter \( d \geq 2 \) and let there be given a red-white vertex coloring \( c \) of the graph \( G \) where at least one vertex is colored red. That is, the color \( c(v) \) of a vertex \( v \) in \( G \) is either red or white and \( c(v) \) is red for at least one vertex \( v \) of \( G \). With each vertex \( v \) of \( G \), there is associated a \( d \)-vector \( \vec{d}(v) = (a_1, a_2, \ldots, a_d) \) called the code of \( v \) corresponding to \( c \), where the \( i \)th coordinate \( a_i \) is the number of red vertices at
distance \( i \) from \( v \) for \( 1 \leq i \leq d \). If distinct vertices of \( G \) have distinct codes, then \( c \) is called an identification coloring or ID-coloring. Equivalently, an identification coloring of a connected graph \( G \) is an assignment of the color red to a nonempty subset of \( V(G) \) (with the color white assigned to the remaining vertices of \( G \)) such that for every two vertices \( u \) and \( v \) of \( G \), there is an integer \( k \) with \( 1 \leq k \leq d \) such that the number of red vertices at distance \( k \) from \( u \) is different from the number of red vertices at distance \( k \) from \( v \). A graph possessing an identification coloring is an ID-graph. A major difference here from the two methods described above is that not all connected graphs are ID-graphs.

The concept of metric dimension was introduced independently by Slater [15] and by Harary and Melter [10] and has been studied by many (see [4, 7], for example). Slater described the usefulness of these ideas when working with U.S. Coast Guard Loran (long range aids to navigation) stations in [15, 16]. Johnson [13, 14] of the former Upjohn Pharmaceutical Company applied this in attempts to develop the capability of large datasets of chemical graphs. The concept of partition dimension was introduced in [6]. These concepts as well as other methods of vertex identifications in graphs have been applied in various applications (see [2, 3, 8, 9, 11, 12, 17, 18] for example). The concepts of ID-colorings and ID-graphs were introduced and studied in [5].

All of the methods mentioned above involve constructing a vertex coloring of a connected graph \( G \) with the goal of producing a vertex labeling of \( G \) (using vectors of the same size as labels) so that distinct vertices of \( G \) have the distinct labels. Consequently, the goal of each of these methods is to obtain an irregular labeling of \( G \). The general topic of irregularity in graphs is described in [2]. There is the related topic of obtaining a labeling of \( G \) by means of colorings where exactly two vertices of \( G \) have the same label. These are called articular labeling, a topic discussed in [1].

We first present five results obtained in [5] on ID-colorings. For an integer \( t \geq 2 \), the members of a set \( S \) of \( t \) vertices of a graph \( G \) are called \( t \)-tuplets (twins if \( t = 2 \) and triplets if \( t = 3 \)) if either (1) \( S \) is an independent set in \( G \) and every two vertices in \( S \) have the same neighborhood or (2) \( S \) is a clique, that is the subgraph \( G[S] \) induced by \( S \) is complete and every two vertices in \( S \) have the same closed neighborhood.

**Proposition 1.1.** Let \( c \) be an ID-coloring of a connected graph \( G \). If \( u \) and \( v \) are twins of \( G \), then \( c(u) \neq c(v) \). Consequently, if \( G \) is an ID-graph, then \( G \) is triplet-free.

**Proposition 1.2.** Let \( c \) be a red-white coloring of a connected graph \( G \) where there is at least one vertex of each color. If \( x \) is a red vertex and \( y \) is a white vertex, then \( d(x) \neq d(y) \).

**Theorem 1.1.** A nontrivial connected graph \( G \) has \( ID(G) = 1 \) if and only if \( G \) is a path.

**Theorem 1.2.** A connected graph \( G \) of diameter 2 is an ID-graph if and only if \( G = P_3 \).

**Theorem 1.3.** For a positive integer \( r \), there exists a connected graph \( G \) with \( ID(G) = r \) if and only if \( r \neq 2 \).

The following result describes a property of ID-colorings.

**Proposition 1.3.** Let \( G \) be a connected graph with an ID-coloring \( c \). If \( H \) is a connected subgraph of \( G \) such that (i) \( H \) contains all red vertices in \( G \) and (ii) \( d_G(x, y) = d_H(x, y) \) for every two vertices \( x \) and \( y \) of \( H \), then the restriction of \( c \) to \( H \) is an ID-coloring of \( H \).

**Proof.** Let \( \text{diam}(H) = d \) and let \( c_H \) be the restriction of \( c \) to \( H \). For a vertex \( v \) of \( H \), let \( \vec{d}_{c_H}(v) = (a_1, a_2, \ldots, a_d) \) and let \( \vec{d}_c(v) = (a_1, a_2, \ldots, a_d) \). Notice that if \( d = \text{diam}(G) \), then \( \vec{d}_c(v) = (a_1, a_2, \ldots, a_d) \) while if \( d < \text{diam}(G) \), then \( a_i = 0 \) for each integer \( t \) with \( d + 1 \leq t \leq \text{diam}(G) \). Since \( H \) contains all red vertices in \( G \) and \( d_G(v, w) = d_H(v, w) \) for every vertex \( w \) of \( G \), it follows that \( a_i = a'_i \) for \( 1 \leq i \leq d \) and so the restriction of \( c \) to \( H \) is an ID-coloring of \( H \). 

Both conditions stated in the hypothesis of Proposition 1.3 for a connected subgraph \( H \) of a graph \( G \) are needed. For example, consider the ID-graph \( G \) in Figure 1. The subgraph \( H_1 \) of \( G \) does not contain all red vertices of \( G \) while the subgraph \( H_2 \) is not distance-preserving. For \( i = 1, 2 \), the restriction of the ID-coloring \( c \) of \( G \) to the subgraph \( H_i \) of \( G \) is not an ID-coloring of \( H_i \) (since there are twins both of which are colored white).

Here, our emphasis turns to trees that are ID-graphs, namely ID-trees. We investigate structural problems of ID-trees, provide necessary conditions for trees to be ID-trees, and establish a realization result on ID-numbers of ID-trees satisfying some prescribed conditions.

2. ID-colorings of trees

The only \( t \)-tuplets, \( t \geq 2 \), in a tree \( T \) are end-vertices of \( T \), all with the same neighbor. As we saw, if \( T \) contains triplets, then \( T \) is not an ID-tree. If \( T \) contains twins and possesses an ID-coloring, then the twins must be colored differently in every ID-coloring. We now see that for trees, the concepts of twins and triplets are special cases of something more general.
Observation 2.1. Suppose that a tree $T$ has twin branches $B_1$ and $B_2$ at a vertex $v$ and $c$ is a red-white coloring of $T$. For $i = 1, 2$, let $c_i$ be the restriction of $c$ to $B_i$, rooted at $v$. If $c_1 \cong c_2$, then $c$ is not an ID-coloring of $T$.

Let $T_0$ be a tree of size $k \geq 1$ rooted at a vertex $v$. If the color of $v$ is fixed, say $v$ is white, then there are at most $2^k$ distinct (non-isomorphic) red-white colorings of $T_0$ in which $v$ is colored white. Consequently, if there are more than $2^k$ copies of a particular branch of size $k$ at $v$, then $T$ is not an ID-tree by Observation 2.1. In the case when $k = 1$, this simply says that no ID-tree can contain a triplet.

Let $T$ be a tree rooted at a vertex $v$ and let $c$ be an ID-coloring of $T$. If $T_0$ is a subgraph of $T$ of minimum order rooted at $v$ such that $T_0$ contains all red vertices in $T$, then the restriction of $c$ to $T_0$ is an ID-coloring of $T_0$ by Proposition 1.3. Necessarily, all end-vertices of $T_0$ are red. In the case when $T_0$ is a path $P_{k+1}$ of size $k$ whose end-vertices are $v$ and $w$, there are at most $2^{k-1}$ distinct red-white colorings of $P_{k+1}$ in which $v$ is white and $w$ is red.

A tree $T$ is starlike if $T$ is obtained by subdividing the edges of a star of order 4 or more. Equivalently, a tree $T$ is starlike if and only if $T$ has exactly one vertex whose degree is greater than 2. This vertex is referred to as the central vertex of $T$. If the degree of the central vertex $v$ of a starlike $T$ is $k \geq 3$, then $T$ has $k$ branches (paths) at $v$, each branch containing $v$ as an end-vertex of $T$. For example, the starlike tree $T$ in Figure 3 has four branches at its central vertex. This tree is twin-free but does contain twin branches at its central vertex. This starlike tree is an ID-graph and an ID-coloring having exactly four red vertices is shown in Figure 3. In fact, $\text{ID}(T) = 4$.

Proposition 2.1. Let $T$ be a starlike tree whose largest branch at its central vertex $v$ has size $k$. If $T$ is an ID-tree, then for each integer $i$ with $1 \leq i \leq k$, there are at most $2^i$ branches of size $i$ or less at $v$. Consequently, if $T$ is an ID-tree, then $T$ has at most $2^k$ branches at $v$.

Proof. In view of Proposition 1.3, it suffices to determine the maximum number of distinct red-white colorings of all branches (paths) of $T$ such that $v$ is white and all end-vertices are red. For each integer $i$ with $1 \leq i \leq k$, there are $2^{i-1}$ distinct red-white colorings of branches of size $i$ at $v$ in which $v$ is colored white and the other end-vertex of each
branch is colored red. Thus, the minimum number of branches of size $i$ at $v$ without duplicating a red-white coloring of these branches is $2^{i-1}$. Therefore, the maximum number of all such red-white colorings of branches of all possible sizes at $v$ is $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$. Since there can be one branch of size $k$ or less at $v$ all of whose vertices are colored white, it follows that there can be $2^k$ branches at $v$ such that the red-white colorings of every two isomorphic branches at $v$ are different.

\[ \text{Corollary 2.1. Let } T \text{ be a starlike tree whose largest branch at its central vertex } v \text{ has size } k. \text{ If } T \text{ has more than } 2^k \text{ branches at } v, \text{ then } T \text{ is not an ID-tree.} \]

For example, if $T$ is a starlike ID-tree whose largest branch at its central vertex $v$ has size 3, then (1) there are at most two branches of size 1 at $v$, (2) there are at most four branches of size 2 or less at $v$, and (3) there are at most eight branches of size 3 or less at $v$. As an illustration, the three starlike trees of Figure 4 satisfy all conditions (1)–(3).

\[ \text{Theorem 2.1. If } T \text{ is a starlike tree with central vertex } v \text{ whose branches at } v \text{ have distinct sizes, then } T \text{ is an ID-tree.} \]

\[ \text{Proof. Suppose that } \deg v = k \geq 3 \text{ and let } B_1, B_2, \ldots, B_k \text{ be the branches (paths) of } T \text{ at } v, \text{ where } B_i \text{ has size } m_i \text{ and } m_i < m_{i+1} \text{ for } 1 \leq i < k. \text{ Define a red-white coloring } c \text{ of } T \text{ that assigns the color white to } v \text{ and the color red to all other vertices of } T. \text{ We show that } c \text{ is an ID-coloring of } T. \text{ By Proposition 1.2, it suffices to show that every two red vertices have distinct codes, this red-white coloring is an ID-coloring of } T. \]

\[ \text{Figure 3: An ID-coloring of a twin-free starlike tree.} \]

\[ \text{Figure 4: Three starlike trees whose largest branch at its central vertex has size 3.} \]

\[ \text{Figure 5: An ID-coloring of a starlike tree.} \]

\[ \text{Theorem 2.1. If } T \text{ is a starlike tree with central vertex } v \text{ whose branches at } v \text{ have distinct sizes, then } T \text{ is an ID-tree.} \]

\[ \text{Proof. Suppose that } \deg v = k \geq 3 \text{ and let } B_1, B_2, \ldots, B_k \text{ be the branches (paths) of } T \text{ at } v, \text{ where } B_i \text{ has size } m_i \text{ and } m_i < m_{i+1} \text{ for } 1 \leq i < k. \text{ Define a red-white coloring } c \text{ of } T \text{ that assigns the color white to } v \text{ and the color red to all other vertices of } T. \text{ We show that } c \text{ is an ID-coloring of } T. \text{ By Proposition 1.2, it suffices to show that every two red vertices have distinct codes, this red-white coloring is an ID-coloring of } T. \]
distinct codes. Let \(x, y \in V(T) - \{v\}\) and let \(\vec{d}(x) = (a_1, a_2, \ldots, a_d)\) and \(\vec{d}(y) = (b_1, b_1, \ldots, b_d)\) where \(d = \text{diam}(T) = m_{k-1} + m_k\).

Suppose that \(\vec{d}(x, v) = s\) and \(\vec{d}(y, v) = t\). We consider two cases, according to whether \(s \neq t\) or \(s = t\).

**Case 1.** \(s \neq t\), say \(s < t\). Then \(a_s \in \{0, 1\}\) and \(b_t \in \{1, 2\}\). If \(a_s \neq b_t\), then \(\vec{d}(x) \neq \vec{d}(y)\). Thus, we may assume that \(a_s = b_t = 1\). Thus, \(a_{s+1} \in \{k - 1, k\}\) and \(b_{s+1} \in \{0, 1\}\). Since \(k \geq 3\), it follows that \(a_{s+1} \geq 2\) and so \(a_{s+1} \neq b_{s+1}\), implying that \(\vec{d}(x) \neq \vec{d}(y)\).

**Case 2.** \(s = t\). Then \(x\) and \(y\) belong to different branches of \(T\) at \(v\), say \(x \in V(B_i)\) and \(y \in V(B_j)\) where \(1 \leq i < j \leq k\). Let \(B_i = (v = v_0, v_1, \ldots, v_{m_i})\) and \(B_j = (v = u_0, u_1, \ldots, u_{m_j})\), where then \(x = v_0\) and \(y = u_s\). If \(m_i - s + 1 = s\), then \(a_{m_i-s+1} = 0\)

and \(b_{m_i-s+1} \geq 1\). If \(m_i - s + 1 \neq s\), then \(b_{m_i-s+1} = a_{m_i-s+1} + 1\). In either case, \(\vec{d}(x) \neq \vec{d}(y)\). Therefore, \(c\) is an ID-coloring of \(T\).

\[\square\]

### 3. Starlike trees with prescribed ID-number

We saw in Theorem 1.3 that for every integer \(r \geq 3\), there exists a connected graph \(G\) with \(\text{ID}(G) = r\). For such a given integer \(r\), the graph \(G\) described in the proof of Theorem 1.3 contains \(r\) pairwise disjoint twins from which it follows that \(\text{ID}(G) \geq r\). It was therefore only necessary to show that \(\text{ID}(G) \leq r\). We now show that for every integer \(r \geq 3\), there exists a tree \(T\) with no twins at all such that \(\text{ID}(T) = r\). In addition, we show that there is a tree without twin branches having ID-number \(r\). In particular, we show that for every odd integer \(r \geq 5\) there is a twin-free tree \(T\) whose automorphism group contains \((r + 1)!\) elements such that \(\text{ID}(T) = r\). We also show that there is a red-white coloring \(c\) of the same class of trees \(T\) where exactly \(r - 1\) vertices are colored red such that \(\vec{d}(x) = \vec{d}(y)\) for exactly one pair \(x, y\) of vertices of \(T\). Consequently, there is a red-white coloring of these trees \(T\) with exactly two vertices having the same code. As we mentioned earlier, such a (red-white) coloring results in an antiregular labeling (see [1, 2], for example.)

For each integer \(r \geq 3\), let \(T = S_{r-1}(K_{r+1})\) be the starlike tree obtained from the star \(K_{r+1}\) of order \(r+2\) by subdividing each edge of the \(r+1\) edges in \(K_{r+1}\) exactly \(r-1\) times. Let \(v\) be the central vertex of \(T\). Then the degree of \(v\) is \(r+1\) and each of the \(r+1\) branches of \(T\) at \(v\) has length \(r\). For each integer \(i\) with \(0 \leq i \leq r\), let \(B_i = (v, v_{i,1}, v_{i,2}, \ldots, v_{i,r})\) be a branch of \(T\) at \(v\). Then \(\text{diam}(T) = 2r\) and \(T\) is twin-free.

**Theorem 3.1.** For each odd integer \(r \geq 3\), \(\text{ID}(S_{r-1}(K_{r+1})) = r\).

**Proof.** For an odd integer \(r \geq 3\), let \(T = S_{r-1}(K_{r+1})\), where \(v\) is the central vertex of \(T\) and \(B_i = (v, v_{i,1}, v_{i,2}, \ldots, v_{i,r})\) is a branch of \(T\) at \(v\) for \(0 \leq i \leq r\). First, we show that \(\text{ID}(T) \geq r\). For any red-white coloring of \(T\) that assigns the color red to at most \(r - 1\) vertices of \(T\), there are at least two branches, say \(B_0\) and \(B_1\), of \(T\) at \(v\) such that the paths \(B_0 - v\) and \(B_1 - v\) contain no red vertices of \(T\). However then, \(\vec{d}(v_{0,1}) = \vec{d}(v_{1,1})\), for example, and so this red-white coloring is not an ID-coloring of \(T\). Therefore, \(\text{ID}(T) \geq r\).

Next, we show that \(T\) has an ID-coloring with exactly \(r\) red vertices. Define a red-white coloring \(c\) of \(T\) by assigning the color red to each vertex \(v_{i,j}\) for \(1 \leq i \leq r\) and white to the remaining vertices of \(T\). Thus, \(T\) has exactly \(r\) red vertices. It remains to show that \(c\) is an ID-coloring of \(T\).

Since \(\text{diam}(T) = 2r\), the code of each vertex of \(T\) is a \((2r)\)-vector. Let \(x\) and \(y\) be two distinct vertices of \(T\). We consider two cases, according to whether \(x\) and \(y\) are both red or both white. Let \(\vec{d}(x) = (a_1, a_2, \ldots, a_{2r})\) and \(\vec{d}(y) = (b_1, b_2, \ldots, b_{2r})\).

**Case 1.** \(x\) and \(y\) are both red. Let \(x = v_{i,j}\) and \(y = v_{j,k}\) where \(1 \leq i < j \leq r\).

\* First, suppose that \(j \neq r\). Since (1) the last nonzero coordinate in \(\vec{d}(v_{i,j})\) is the \((i + r)\)th coordinate where \(i + r = d(v_{i,j}, v_{i,r})\) and the last nonzero coordinate in \(\vec{d}(v_{j,k})\) is the \((j + r)\)th coordinate where \(j + r = d(v_{j,k}, v_{k,r})\) and (2) \(i < j\), it follows that \(d_{i+r} = 0\) and \(b_{j+r} = 1\) and so \(\vec{d}(x) \neq \vec{d}(y)\).

\* Next, suppose that \(j = r\). We saw that the last nonzero coordinate in \(\vec{d}(v_{i,j})\) where \(1 \leq i \leq r - 1\) is the \((i + r)\)th coordinate. Since the last nonzero coordinate in \(\vec{d}(v_{j,r})\) is the \((2r - 1)\)th coordinate where \(2r - 1 = d(v_{r-r-1,r-i-1}, v_{i,r})\), it follows that if \(i \neq r - 1\), then \(\vec{d}(x) \neq \vec{d}(y)\). Thus, we may assume that \(x = v_{r-r-1,i-1}\). Because the first nonzero coordinate in \(\vec{d}(v_{r-r-1,r-i})\) is the \(r\)th coordinate where \(r = d(v_{i,1}, v_{r-r-1,i-1})\) and the first nonzero coordinate in \(\vec{d}(v_{j,r})\) is the \((r + 1)\)th coordinate where \(r + 1 = d(v_{r,i}, v_{i,r})\), it follows that \(a_r = 1\) and \(b_r = 0\) and so \(\vec{d}(x) \neq \vec{d}(y)\).

**Case 2.** \(x\) and \(y\) are both white. First, we make some observations on the codes of vertices on \(B_0\).

\* The vertices on \(B_0\) are the only white vertices of \(T\) whose codes contain the \(r\)-tuple \((1, 1, \ldots, 1) = 1^r\) as a subsequence. The vertex \(v\) is the only white vertex of \(T\) such that the first \(r\) coordinates of its code are 1 (that is, \(\vec{d}(v) = (1^r, 0^0)\)).

For \(1 \leq t \leq r\), the vertex \(v_{0,t}\) is the only white vertex such that in \(\vec{d}(v_{0,t})\) the first \(t\) coordinates and the last \(r - t\) coordinates are 0 while the remaining coordinates are 1 (that is, \(\vec{d}(v_{0,t}) = (0^t, 1^r, 0^{r-t})\) for \(1 \leq t \leq r\)). Thus, all codes of the vertices of \(B_0\) are distinct and they are also distinct from the codes of those white vertices that are not in \(B_0\).
Hence, we may assume that neither $x$ nor $y$ belongs to $B_0$. Let $Q_i = B_i - v = (v_{i,1}, v_{i,2}, \ldots, v_{i,r})$ be the subpath of $B_i$ for $1 \leq i \leq r$. We consider two subcases, according to the location of $x$ and $y$.

**Subcase 2.1.** $x, y \in V(Q_i)$ where $1 \leq i \leq r$. Let $x = v_{i,p}$ and $y = v_{i,q}$ where $1 \leq p < q \leq r$ and $p, q \neq i$.

* First, suppose that $i \neq r$. Since (1) the last nonzero coordinate in $\vec{d}(v_{i,p})$ is the $(p+r)$th coordinate where $p + r = d(v_{i,p}, v_{i,r})$ and the last nonzero coordinate in $\vec{d}(v_{i,q})$ is the $(q+r)$th coordinate where $q + r = d(v_{i,q}, v_{i,r})$ and (2) $p < q$, it follows that $a_{q+r-1} = 0$ and $b_{q+r-1} = 1$ and so $\vec{x}(y) \neq \vec{y}(y)$.

* Next, suppose that $i = r$. Since the last nonzero coordinate in $\vec{d}(v_{i,p})$ is the $(p+r-1)$th coordinate where $p + r - 1 = d(v_{i,p}, v_{r-1,r-1})$ and the last nonzero coordinate in $\vec{d}(v_{i,q})$ is the $(q+r-1)$th coordinate where $q + r - 1 = d(v_{i,q}, v_{r-1,r-1})$ and (2) $p < q$, it follows that $a_{q+r-1} = 0$ and $b_{q+r-1} = 1$ and so $\vec{x}(y) \neq \vec{y}(y)$.

**Subcase 2.2.** $x \in V(Q_i)$ and $y \in V(Q_j)$ where $1 \leq i < j \leq r$. Let $x = v_{i,p}$ and $y = v_{j,q}$ where $1 \leq p, q \leq r$, $p \neq i$, and $q \neq j$. We consider two subcases, according to whether $p = q$ or $p \neq q$.

**Subcase 2.2.1.** $p = q$. Then $x = v_{i,p}$ and $y = v_{j,p}$ where $1 \leq i < j \leq r$ and $p \notin \{i, j\}$.

* First, suppose that $j + 1 \leq p \leq r$. Since (1) the first nonzero coordinate in $\vec{d}(v_{i,p})$ is the $(p-i)$th coordinate where $p - i = d(v_{i,p}, v_{i,i})$ and the first nonzero coordinate in $\vec{d}(v_{j,p})$ is the $(p-j)$th coordinate where $p - j = d(v_{j,p}, v_{j,j})$ and (2) $i < j$, it follows that $a_{p-j} = 0$ and $b_{p-j} = 1$ and so $\vec{x}(y) \neq \vec{y}(y)$.

* Next suppose that $1 \leq p \leq i - 1$. Let $c_0$ be the red-white coloring of $T$ obtained by recoloring $v_{i,i}$ and $v_{j,j}$ white and all other vertices of $T$ remain the same colors as in $c$. Since $d(v_{i,p}, w) = d(v_{i,p}, w)$ for every red vertex $w$ such that $w \notin \{v_{i,i}, v_{j,j}\}$, it follows that $\vec{d}_c(x) = \vec{d}_c(y) = (i, i, \ldots, i)$. Observe that $d(v_{i,p}, v_{i,i}) = i - p, d(v_{i,p}, v_{j,j}) = j + p, d(v_{j,q}, v_{j,j}) = j - p$. Since $i - p < \min\{i + p, j - p, j + p\}$, it follows that $a_{i-p} = f_{i-p} + 1$ and $b_{i-p} = f_{i-p}$ and so $\vec{x}(y) \neq \vec{y}(y)$.

* Finally, suppose that $i + 1 \leq p \leq j - 1$. Let $c_0$ be the red-white coloring of $T$ obtained by recoloring $v_{i,i}$ and $v_{j,j}$ white and all other vertices of $T$ remain the same colors as in $c$. Since $d(v_{i,p}, w) = d(v_{i,p}, w)$ for every red vertex $w$ such that $w \notin \{v_{i,i}, v_{j,j}\}$, it follows that $\vec{d}_c(x) = \vec{d}_c(y) = (i, i, \ldots, i)$. Observe that $d(v_{i,p}, v_{i,i}) = p - i, d(v_{i,p}, v_{j,j}) = j + p, d(v_{j,q}, v_{j,j}) = j - p$. Since $j + p > \max\{i + p, j - p, j + p\}$, it follows that $a_{j+p} = f_{j+p} + 1$ and $b_{j+p} = f_{j+p}$ and so $\vec{x}(y) \neq \vec{y}(y)$.

**Subcase 2.2.2.** $p \neq q$. First, suppose that $j \neq r$. Since (1) the last nonzero coordinate in $\vec{d}(v_{i,p})$ is the $(p+r)$th coordinate where $p + r = d(v_{i,p}, v_{r,r})$ and the last nonzero coordinate in $\vec{d}(v_{i,q})$ is the $(q+r)$th coordinate where $q + r = d(v_{i,q}, v_{r,r})$ and (2) $p \neq q$, it follows that either $a_{p+r} \neq b_{p+r}$ or $a_{q+r} \neq b_{q+r}$, implying that $\vec{x}(y) \neq \vec{y}(y)$.

For simplicity, we now introduce notation where a code is expressed when no 0 coordinate is given after the final nonzero coordinate of a code. For example, if a code of a vertex is a 4-tuple $(1, 0, 2, 1, 0, 2, 1)$, we simply write this code as the 4-tuple $(1, 0, 2, 1)$.

Next, suppose that $j = r$. Thus, $x = v_{i,p}$ where $1 \leq i \leq r - 1$ and $p \neq i$ and $y = v_{r,q}$ where $1 \leq q \leq r - 1$ and $p \neq q$. We consider two possibilities.

**Subcase 2.2.2.1.** $2 \leq i \leq r - 1$. First, suppose that $p \geq i + 1$. Then $d(x) = d(v_{i,p}) = (0^{i-p-1}, 1, 0^{p-i}, 1^{r-i})$. If $d(y) = d(v_{r,q})$ contains a coordinate 2, then $\vec{x}(y) \neq \vec{y}(y)$. Thus, we may assume that $\vec{x}(y) = d(v_{r,q}) = (0^{r-q-1}, 1, 0^{p-r}, 1^{r-1})$. Since $1^{r-1}$ is a subsequence of $\vec{y}(y)$ and is not a subsequence of $\vec{x}(y)$, it follows that $\vec{x}(y) \neq \vec{y}(y)$.

Next, suppose that $1 \leq p \leq i - 1$. If $i - p \neq p + \ell$ for some $\ell \in [r] - \{i\}$, then $(1^{r-i}, 0^{r-i})$ is a subsequence of $\vec{x}(y)$ and so there is no 2 as a coordinate of $\vec{x}(y)$. If $d(y) = d(v_{r,q})$ contains a coordinate 2, then $\vec{x}(y) \neq \vec{y}(y)$ and so $d(y) = d(v_{r,q}) = (0^{r-q-1}, 1, 0^{p-r}, 1^{r-1})$. Thus, $\vec{x}(y) \neq \vec{y}(y)$. Hence, we may assume that $i - p = p + \ell$ for some $\ell \in [r] - \{i\}$ and so $i = 2p + \ell$. This implies that there is exactly one coordinate 2 of $\vec{x}(y)$, namely $a_{i-p} = 2$. If $d(y)$ has no coordinate 2, then $\vec{x}(y) \neq \vec{y}(y)$. Hence, we assume that $\vec{x}(y)$ has coordinate 2. This implies that $d(v_{r,q}, v_{r,r}) = r - q = q + t$ for some $t \in [r - 1]$ and $b_{r-q} = 2$ is the only coordinate 2 in $\vec{y}(y)$. Hence, $i - p = r - q$ or $r - i = q - p$ and so $q > p$. There are two possibilities here. If $i - p = r - q < p + q$, then the second nonzero coordinate in $\vec{x}(y)$ is $a_{p+1}$ while the second nonzero coordinate in $\vec{y}(y)$ is $b_{q+1}$. If $i - p = r - q > p + 1$, then the first nonzero coordinate in $\vec{x}(y)$ is $a_{q+1}$ while the the first nonzero coordinate in $\vec{y}(y)$ is $b_{q+1}$. In either case, $\vec{x}(y) \neq \vec{y}(y)$. Therefore, $c$ is an ID-coloring and so $ID(T) = r$.

**Subcase 2.2.2.2.** $i = 1$. Then $d(x) = d(v_{i,p}) = (0^{p-2}, 1, 0, 1^{r-1})$. If $d(y) = d(v_{r,q})$ contains a coordinate 2, then $\vec{x}(y) \neq \vec{y}(y)$. Thus, we may assume that $d(v_{r,q}, v_{r,r}) = r - q \neq q + t$ for $1 \leq t \leq r - 1$. Since $r - q < q + r - 1$, it follows that $r - q \leq q$ or $q \geq r/2$. Then $\vec{x}(y) = d(v_{r,q}) = (0^{r-q-1}, 1, 0^{p-r}, 1^{r-1})$. Thus, $p - 2 = r - q - 1$ or $p + q = r + 1, 2 = 2q - r$ (or $r = 2q - 2$ is even) and so $p = q - 1$ or $(q = p + 1)$. Since $r$ is odd, it follows that $\vec{x}(y) \neq \vec{y}(y)$.

□
The following is a consequence of Theorem 3.1.

**Corollary 3.1.** For each odd integer \( r \geq 3 \), there exist a twin-free starlike tree \( T \) such that \( \text{ID}(T) = r \).

In the statement of Theorem 3.1, the condition that \( r \geq 3 \) is an odd integer is only required in Subcase 2.2.2.2. In fact, if \( r \geq 4 \) is an even integer, then there are exactly two white vertices in the red-white coloring described in the proof of Theorem 3.1, namely \( v_{r,1} \) and \( v_{r,q} \) where \( q = p + 1 \) and \( p + q = r + 1 \), that have the same code. That is, this red-white coloring is antiregular. Therefore, we have the following.

**Proposition 3.1.** For each even integer \( r \geq 4 \), there is an antiregular red-white coloring of the starlike tree \( S_{r-1}(K_1,r+1) \) having exactly \( r \) red vertices.

By the technique used in the proof of Theorem 3.1, the following result can be verified.

**Proposition 3.2.** Let \( r \geq 4 \) be an even integer. If \( T \) is the starlike tree obtained by subdividing exactly one edge of \( S_{r-1}(K_1,r+1) \), then \( \text{ID}(T) = r \).

Theorem 3.2. For each integer \( r \geq 3 \), there is a starlike tree \( T \) of order \( 1 + \binom{r+2}{2} \) having the identity automorphism group such that \( \text{ID}(T) = r \).

**Proof.** For each integer \( r \geq 3 \), let \( K_{1,r+1} \) be the star of order \( r + 2 \) with central vertex \( v \) that is adjacent to the \( r + 1 \) end-vertices \( v_1,v_2,\ldots,v_{r+1} \). Let \( T \) be the starlike tree obtained from the star \( K_{1,r+1} \) by subdividing the edge \( v_1 \) of \( K_{1,r+1} \) exactly \( i - 1 \) times for \( 1 \leq i \leq r + 1 \). In particular, \( v_1 \) is not subdivided and \( v_{r+1} \) is subdivided exactly \( r \) times. Thus, \( T \) is twin-free, the order of \( T \) is \( 1 + \binom{r+2}{2} \) and \( \text{diam}(T) = 2r + 1 \). Since no two vertices of \( T \) are similar, it follows that \( T \) has the identity automorphism group. For each integer \( i \) with \( 1 \leq i \leq r + 1 \), let \( B_i = \{ v,v_1,v_2,\ldots,v_{i-1} \} \) be a branch of \( T \) at \( v \).

First, we show that \( \text{ID}(T) \geq r \). For any red-white coloring of \( T \) that assigns the color red to at most \( r - 1 \) vertices of \( T \), there are at least two branches \( B_1 \) and \( B_2 \) of \( T \) at \( v \) such that the paths \( B_i - v \) and \( B_j - v \) contain no red vertices of \( T \). However then, \( \vec{d}(v_1) = \vec{d}(v_j) \), for example, and so this red-white coloring is not an ID-coloring of \( T \). Therefore, \( \text{ID}(T) \geq r \).

Next, we show that \( T \) has an ID-coloring with exactly \( r \) red vertices. Define a red-white coloring \( c \) of \( T \) by assigning the color red to each vertex \( v_i \) for \( 1 \leq i \leq r \) and white to the remaining vertices of \( T \). Thus, \( T \) has exactly \( r \) red vertices. It remains to show that \( c \) is an ID-coloring of \( T \). Since \( \text{diam}(T) = 2r + 1 \), the code of each vertex of \( T \) is a \( (2r + 1) \)-vector. Let \( x \) and \( y \) be two distinct vertices of \( T \). We consider two cases, according to whether \( x \) and \( y \) are both red or both white. Let \( \vec{d}(x) = (a_1,a_2,\ldots,a_{2r+1}) \) and \( \vec{d}(y) = (b_1,b_2,\ldots,b_{2r+1}) \).

**Case 1.** \( x \) and \( y \) are both red. Let \( x = v_i,j \) and \( y = v_j,k \) where \( 1 \leq j < k \).

* First, suppose that \( j \neq r \). Since \( (1) \) the last nonzero coordinate in \( \vec{d}(v_i,j) \) is \( (i + r) \)th coordinate where \( i + r = d(v_i,j) \) and the last nonzero coordinate in \( \vec{d}(v_j,k) \) is \( (j + r) \)th coordinate where \( j + r = d(v_j,k) \). Hence, \( d(v_i,j) = d(v_j,k) \). Since \( i < j \), it follows that \( a_{i+r} = 0 \) and \( b_{j+r} = 0 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

* Next, suppose that \( j = r \). We saw that the last nonzero coordinate in \( \vec{d}(v_i,j) \) where \( 1 \leq i \leq r - 1 \) is \( (i + r) \)th coordinate. Since the last nonzero coordinate in \( \vec{d}(v_r,r) \) is \( (2r - 1) \)th coordinate where \( 2r - 1 = d(v_{r-1},v_r) \), it follows that if \( i \neq r - 1 \), then \( \vec{d}(x) \neq \vec{d}(y) \). Thus, we may assume that \( x = v_{r-1},v_r \). Because the first nonzero coordinate in \( \vec{d}(v_{r-1},r-1) \) is \( (r-1) \)th coordinate where \( r - 1 = d(v_1,v_{r-1},v_r) \) and the first nonzero coordinate in \( \vec{d}(v_r,r) \) is \( (r+1) \)th coordinate where \( r + 1 = d(v_1,v_r,v_{r-1}) \), it follows that \( a_r = 1 \) and \( b_r = 0 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

**Case 2.** \( x \) and \( y \) are both white. First, we verify the following claim.

**Claim.** If \( x \in V(B_{r+1}) \) or \( y \in V(B_{r+1}) \), then \( \vec{d}(x) \neq \vec{d}(y) \).

The vertices on \( B_{r+1} \) are the only white vertices of \( T \) whose codes contain the \( r \)-tuple \( (1,1,\ldots,1) = v_r \) as a subsequence. The vertex \( v_r \) is the only white vertex of \( T \) such that the first \( r \) coordinates of its code are \( (1,1,\ldots,1) = \vec{d}(v_r) = (1^r,0^{r+1}) \). For \( 1 \leq t \leq r + 1 \), the vertex \( v_{r+1,t} \) is the only white vertex such that in \( \vec{d}(v_{r+1,t}) \) the first \( t \) coordinates and the last \( r + 1 - t \) coordinates are \( 0 \) while the remaining \( t - 1 \) coordinates are \( 1 \) (that is, \( \vec{d}(v_{r+1,t}) = (0^t,1^r,0^{r+1-t}) \) for \( 1 \leq t \leq r \)). Thus, all codes of the vertices of \( B_{r+1} \) are distinct and they are also distinct from the codes of those white vertices that are not in \( B_{r+1} \). Hence, the claim holds.

By the claim, we may assume that \( x \notin V(B_{r+1}) \) and \( y \notin V(B_{r+1}) \). Let \( Q_i = B_i - v = (v_{i+1},v_{i+2},\ldots,v_{i+1}) \) be the subpath of \( B_i \) for \( 2 \leq i \leq r \). We consider two subcases, according to the location of \( x \) and \( y \).

**Subcase 2.1.** \( x,y \in V(Q_i) \) where \( 2 \leq i \leq r \). Let \( x = v_{i,p} \) and \( y = v_{i,q} \) where \( 1 \leq p < q \leq i - 1 \).
First, suppose that \( i \neq r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_i, p) \) is the \((p + r)\)th coordinate where \( p + r = d(v_i, p, v_r, r) \) and the last nonzero coordinate in \( \vec{d}(v_r, q) \) is the \((q + r)\)th coordinate where \( q + r = d(v_r, q, v_r, r) \) and (2) \( p < q \), it follows that \( a_{q+r-1} = 0 \) and \( b_{q+r-1} = 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

Next, suppose that \( i = r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_r, p) \) is the \((p + r - 1)\)th coordinate where \( p + r - 1 = d(v_r, p, v_r, r, r) \) and the last nonzero coordinate in \( \vec{d}(v_r, q) \) is the \((q + r - 1)\)th coordinate where \( q + r - 1 = d(v_r, q, v_r, r, r) \) and (2) \( p < q \), it follows that \( a_{q+r-1} = 0 \) and \( b_{q+r-1} = 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

Subcase 2.2. \( x \in V(Q_2) \) and \( y \in V(Q_j) \) where \( 2 \leq i \leq j \leq r \). Let \( x = v_{i,p} \) and \( y = v_{j,q} \) where \( 1 \leq p < i-1 \) and \( 1 \leq q < j-1 \). We consider two subcases, according to \( p = q \) or \( p \neq q \).

Subcase 2.2.1. \( p = q \). Then \( x = v_{i,p} \) and \( y = v_{j,p} \) where \( 2 \leq i < j \leq r \) and \( p \notin \{i, j\} \). In either case, we may assume that \( a_{p+r} \neq b_{q+r} \) or \( a_{q+r} \neq b_{p+r} \), implying that \( \vec{d}(x) \neq \vec{d}(y) \).

Subcase 2.2.2. \( p \neq q \). First, suppose that \( j \neq r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_i, p) \) is the \((p + r)\)th coordinate where \( p + r = d(v_i, p, v_r, r) \) and the last nonzero coordinate in \( \vec{d}(v_j, q) \) is the \((q + r)\)th coordinate where \( q + r = d(v_j, q, v_r, r) \) and (2) \( p \neq q \), it follows that either \( a_{p+r} \neq b_{q+r} \) or \( a_{q+r} \neq b_{p+r} \), implying that \( \vec{d}(x) \neq \vec{d}(y) \).

Next, suppose that \( j = r \). Thus, \( x = v_{i,p} \) where \( 2 \leq i \leq r - 1 \) and \( 1 \leq p < i-1 \) and \( y = v_{r,q} \) where \( 1 \leq q \leq r - 1 \) and \( p \neq q \). If \( i - p \neq p + \ell \) for some \( \ell \in [r - i] \), then \((1 - \ell, 0, 1 - \ell)\) is a subsequence of \( \vec{d}(x) \) and so there is no coordinate 2 in \( \vec{d}(y) \). If \( \vec{d}(y) = \vec{d}(v_r, q) \) contains 2 as a coordinate, then \( \vec{d}(x) \neq \vec{d}(y) \) and so \( \vec{d}(y) = \vec{d}(v_r, q) = (0^{q-r-1}, 1, 0^{r-q-1}, 1, 0^{r-1}, 0, \ldots, 0) \). Thus, \( \vec{d}(x) \neq \vec{d}(y) \). Hence, we may assume that \( i - p = p + \ell \) for some \( \ell \in [r - i] \) and so \( i = 2p + \ell \). This implies that there is exactly one coordinate of \( \vec{d}(x) \) which is 2, namely \( a_{i-p} = 2 \). If \( \vec{d}(y) \) has no coordinate 2, then \( \vec{d}(x) \neq \vec{d}(y) \). Hence, we assume that \( \vec{d}(y) \) has 2 as a coordinate. This implies that \( \vec{d}(v_r, q, v_r, r) \neq r - q = q + t \) for some \( t \in [r - 1] \) and \( b_{r-q} = 2 \) is the only coordinate 2 in \( \vec{d}(y) \). Hence, \( i - p = r - q \) or \( r - i = q - p \) and so \( q > p \). There are two possibilities here. If \( i - p = r - q < p + 1 \), then the second nonzero coordinate in \( \vec{d}(x) \) is \( a_{p+1} \) while the the second nonzero coordinate in \( \vec{d}(y) \) is \( b_{q+1} \). If \( i - p = r - q > p + 1 \), then the first nonzero coordinate in \( \vec{d}(x) \) is \( a_{p+1} \) while the the first nonzero coordinate in \( \vec{d}(y) \) is \( b_{q+1} \). In either case, \( \vec{d}(x) \neq \vec{d}(y) \).

Therefore, \( c \) is an ID-coloring and so \( \text{ID}(T) = r \).

Several problems are suggested by the results presented here.

(1) For a given integer \( r \geq 3 \), what is the smallest order of a tree \( T \) such that \( \text{ID}(T) = r \)?

(2) For a given integer \( r \geq 3 \), what is the smallest order of a twin-free tree \( T \) such that \( \text{ID}(T) = r \)?

For (2), we have seen that this smallest order is no more than \( 1 + \binom{r}{2} \).

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References