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Research Article

Hybrid convolutions on Pell and Lucas polynomials

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Abstract

By means of the generating function approach, three classes of convolution sums between the numbers of Bernoulli, Genocchi, Euler and the polynomials of Pell and Lucas are evaluated in closed form. Several identities concerning Fibonacci and Lucas numbers are deduced as consequences. One of them gives a solution to the problem proposed recently by Frontczak [Advanced problem H-860, *Fibonacci Quart.* **58** (2020) 281].

Keywords: generating function; Pell polynomial; Lucas polynomial; Fibonacci number; Lucas number.

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1. Introduction and motivation

There exist numerous summation formulae concerning Bernoulli and Euler numbers (cf. [1,4,9]) as well as Fibonacci and Lucas numbers (cf. [2,3,11]). When Bernoulli and Euler numbers are replaced by the corresponding Bernoulli and Euler polynomials, the related finite convolution identities can be found in the recent papers by Frontczak [6] and Frontczak–Goy [8]. Denote by B_n and L_n the usual Bernoulli and Lucas numbers, respectively. Frontczak [7] proposed, in a recent issue of 'Fibonacci Quarterly', a problem demanding to prove that

$$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}} 5^{\frac{n-k}{2}}\binom{n}{k}\frac{B_{n-k+2}}{n-k+2}\{2^kL_k-2\}=\frac{2^{n+2}L_{n+2}-2}{5(n+1)(n+2)}-1, \tag{1}$$

where $i\equiv_m j$ stands for "i is congruent to j modulo m" provided that $m\in\mathbb{N}$ and $i,j\in\mathbb{Z}$. During the course of resolving this problem, we find that the related results can be generalized from Fibonacci and Lucas numbers to Pell and Lucas polynomials, that motivates the authors to write the present paper.

To our knowledge, Pell and Lucas polynomials were introduced by Horadam and Mahon [10], and can be defined equivalently by the recurrence relations $\mathrm{P}_n(x)=2x\mathrm{P}_{n-1}(x)+\mathrm{P}_{n-2}(x)$, $\mathrm{Q}_n(x)=2x\mathrm{Q}_{n-1}(x)+\mathrm{Q}_{n-2}(x)$; with different initial conditions $\mathrm{P}_0(x)=0$ and $\mathrm{P}_1(x)=1$; $\mathrm{Q}_0(x)=2$ and $\mathrm{Q}_1(x)=2x$. They reduce to the following four well–known numbers:

- Fibonacci number $F_n=\mathrm{P}_n(\frac12)$: $F_n=F_{n-1}+F_{n-2}$ with $F_0=0$ and $F_1=1$.
- Lucas number $L_n=\mathrm{Q}_n(\frac12)$: $L_n=L_{n-1}+L_{n-2}$ with $L_0=2$ and $L_1=1$.
- Pell number $P_n=\mathrm{P}_n(1)$: $P_n=2P_{n-1}+P_{n-2}$ with $P_0=0$ and $P_1=1$.
- Pell–Lucas number $Q_n=\mathrm{Q}_n(1)$: $Q_n=2Q_{n-1}+Q_{n-2}$ with $Q_0=2$ and $Q_1=2$.

Both polynomials admit the explicit formulae of Binet forms $\mathrm{P}_n(x)=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $\mathrm{Q}_n(x)=\alpha^n+\beta^n$, where for brevity, we employ the following two symbols $\alpha:=x+\sqrt{x^2+1}$ and $\beta:=x-\sqrt{x^2+1}$. Consequently, we have the exponential generating functions $\sum_{k=0}^\infty\mathrm{P}_k(x)\frac{z^k}{k!}=\frac{e^{z\alpha}-e^{z\beta}}{\alpha-\beta}$ and $\sum_{k=0}^\infty\mathrm{Q}_k(x)\frac{z^k}{k!}=e^{z\alpha}+e^{z\beta}$. Furthermore, for a natural number λ, it is routine to check the following relations:

$$\mathrm{P}_n\Big(\frac{L_{2\lambda-1}}{2}\Big)=\frac{F_{2\lambda n-n}}{F_{2\lambda-1}},\qquad \mathrm{P}_n\Big(\frac{\sqrt{-1}}{2}L_{2\lambda}\Big)=\sqrt{-1}^{\,n-1}\frac{F_{2\lambda n}}{F_{2\lambda}}. \tag{2}$$

$$\mathrm{Q}_n\Big(\frac{L_{2\lambda-1}}{2}\Big)=L_{2\lambda n-n},\qquad \mathrm{Q}_n\Big(\frac{\sqrt{-1}}{2}L_{2\lambda}\Big)=\sqrt{-1}^{\,n}L_{2\lambda n}. \tag{3}$$

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$$\mathrm{P}_n\Big(\frac{\sqrt{5}}{2}F_{2\lambda}\Big)=\begin{cases}\frac{F_{2\lambda n}}{L_{2\lambda}}\sqrt{5}, & n\equiv_2 0;\\ \frac{L_{2\lambda n}}{L_{2\lambda}}, & n\equiv_2 1;\end{cases}\qquad \mathrm{P}_n\Big(\frac{\sqrt{-5}}{2}F_{2\lambda-1}\Big)=\sqrt{-1}^{\,n-1}\begin{cases}\frac{F_{2\lambda n-n}}{L_{2\lambda-1}}\sqrt{5}, & n\equiv_2 0;\\ \frac{L_{2\lambda n-n}}{L_{2\lambda-1}}, & n\equiv_2 1.\end{cases}\tag{4}$$

$$\mathrm{Q}_n\Big(\frac{\sqrt{5}}{2}F_{2\lambda}\Big)=\begin{cases}L_{2\lambda n}, & n\equiv_2 0;\\ F_{2\lambda n}\sqrt{5}, & n\equiv_2 1;\end{cases}\qquad \mathrm{Q}_n\Big(\frac{\sqrt{-5}}{2}F_{2\lambda-1}\Big)=\sqrt{-1}^{\,n}\begin{cases}L_{2\lambda n-n}, & n\equiv_2 0;\\ F_{2\lambda n-n}\sqrt{5}, & n\equiv_2 1.\end{cases}\tag{5}$$

They will frequently be utilized as 'bridges', throughout the paper, to pass from summation formulae about Pell and Lucas polynomials to identities involving Fibonacci and Lucas numbers.

By means of the generating function approach, we shall prove, in the next section, three main theorems that evaluate in closed form the convolution sums between Bernoulli numbers, and Pell and Lucas polynomials, including Frontczak's identity (1) as a particular case. Then in Section 3, three summation formulae will be established about Genocchi numbers and Lucas polynomials. Finally, the paper will end up with section 4, where two interesting identities invoving Euler numbers and Lucas polynomials will be presented. Throughout the paper, the coefficient of z^m in the formal power series $\phi(x)$ will be denoted by $[z^m]\phi(x)$, and χ will stand for the logical function defined by $\chi(\text{true})=1$ and $\chi(\text{false})=0$.

2. Convolutions with Bernoulli numbers

Recall that the Bernoulli numbers (cf. [5, §1.14]) have the generating function $\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}$. Their convolutions with Pell and Lucas polynomials $\mathrm{P}_k(x)$ and $\mathrm{Q}_k(x)$ will be examined.

§**2.1.** Rewriting the generating function $\frac{(\alpha-\beta)z}{e^{(\alpha-\beta)z}-1}=\sum_{k=0}^{\infty}(\alpha-\beta)^kB_k\frac{z^k}{k!}$ and multiplying it with $\frac{e^{z\alpha}-e^{z\beta}}{\alpha-\beta}=\sum_{k=0}^{\infty}\mathrm{P}_k(x)\frac{z^k}{k!}$, we can express the following binomial convolution as

$$\sum_{k=0}^{m}(\alpha-\beta)^{m-k}\binom{m}{k}B_{m-k}\mathrm{P}_k(x)=m![z^m]\frac{(\alpha-\beta)z}{e^{(\alpha-\beta)z}-1}\times\frac{e^{z\alpha}-e^{z\beta}}{\alpha-\beta}=m![z^{m-1}]e^{z\beta}.$$

This consequently leads us to the following identity.

Theorem 2.1 ($m\in\mathbb{N}$). $\displaystyle\sum_{k=0}^{m}(\alpha-\beta)^{m-k}\binom{m}{k}B_{m-k}\mathrm{P}_k(x)=m\beta^{m-1}.$

Because $B_{2k+1}=0$ except for $B_1=-1/2$, the last identity can equivalently be stated as

$$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}(\alpha-\beta)^{m-k}\binom{m}{k}B_{m-k}\mathrm{P}_k(x)=\frac{m}{2}\mathrm{Q}_{m-1}(x).\tag{6}$$

Proposition 2.1 ($m,\ \lambda\in\mathbb{N}$).

$$\text{(a)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}}\Big(F_\lambda\sqrt{5}\Big)^{m-k}\binom{m}{k}B_{m-k}F_{k\lambda}=\frac{m}{2}F_\lambda L_{m\lambda-\lambda}.$$

$$\text{(b)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}}L_\lambda^{m-k}\binom{m}{k}B_{m-k}F_{k\lambda}=\frac{m}{2}L_\lambda F_{m\lambda-\lambda}. \qquad \boxed{m\equiv_2 0}$$

$$\text{(c)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}}L_\lambda^{m-k}\binom{m}{k}B_{m-k}L_{k\lambda}=\frac{m}{2}L_\lambda L_{m\lambda-\lambda}. \qquad \boxed{m\equiv_2 1}$$

Proof. The first identity (a) has been obtained by Frontczak and Goy [8], which can be deduced from (6) by letting

$$x=\frac{L_\lambda}{2}\quad\text{for odd }\lambda\quad\text{and}\quad x=\frac{\sqrt{-1}}{2}L_\lambda\quad\text{for even }\lambda\tag{7}$$

and subsequently applying (2) and (3) for simplifications.

Similarly, for the two remaining identities (b) and (c), we can derive them from (6) by taking

$$x=\frac{\sqrt{5}}{2}F_\lambda\quad\text{for even }\lambda\quad\text{and}\quad x=\frac{\sqrt{-5}}{2}F_\lambda\quad\text{for odd }\lambda\tag{8}$$

and then making use of (4) and (5). □

When $\lambda=1$, the above identity (a) reduces to the first formula below, which is due to Frontczak [6].

Corollary 2.1 ($m\in\mathbb{N}$).

$$\text{(a)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 5^{\frac{m-k}{2}}\binom{m}{k}B_{m-k}F_k=\frac{m}{2}L_{m-1}.$$

$$\text{(b)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 8^{\frac{m-k}{2}}\binom{m}{k}B_{m-k}P_k=\frac{m}{2}Q_{m-1}.$$

$$\text{(c)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}} 3^{m-k}\binom{m}{k}B_{m-k}F_{2k}=\frac{3m}{2}F_{2m-2}. \qquad \boxed{m\equiv_2 0}$$

$$\text{(d)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}} 3^{m-k}\binom{m}{k}B_{m-k}L_{2k}=\frac{3m}{2}L_{2m-2}. \qquad \boxed{m\equiv_2 1}$$

Proof. The first identity (a) can also be derived by letting $x=1/2$ in Theorem 2.1. Instead, letting $x=1$ in the same theorem gives the second identity (b). The two remaining identities (c) and (d) are the $\lambda=2$ cases of (b) and (c) in Proposition 2.1. □

§**2.2.** The generating function of Bernoulli numbers can be rewritten as $\frac{(\alpha-\beta)z}{2}+\frac{(\alpha-\beta)z}{e^{(\alpha-\beta)z}-1}=\sum_{k=0}^{\infty}(\alpha-\beta)^{2k}B_{2k}\frac{z^{2k}}{(2k)!}$. Multiplying it with $\left(e^{z\alpha}-e^{z\beta}\right)^2=\sum_{k=0}^{\infty}\left\{2^k\mathrm{Q}_k(x)-2(2x)^k\right\}\frac{z^k}{k!}$ and then extracting the coefficient of x^m, we find the following convolution formula.

Theorem 2.2 ($m\in\mathbb{N}$). $\displaystyle\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}(\alpha-\beta)^{m-k}\binom{m}{k}B_{m-k}\left\{2^k\mathrm{Q}_k(x)-2(2x)^k\right\}=2^{m-2}m(\alpha-\beta)^2\mathrm{P}_{m-1}(x).$

Analogous to the derivation for Proposition 2.1, by specifying in the above theorem by (7) and (8), and then appealing to the transformations (2–5), we can prove the following three summation formulae.

Proposition 2.2 ($m,\ \lambda\in\mathbb{N}$).

$$\text{(a)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}}\left(F_\lambda\sqrt{5}\right)^{m-k}\binom{m}{k}B_{m-k}\left\{2^kL_{k\lambda}-2L_\lambda^k\right\}=2^{m-2}(5m)F_\lambda F_{m\lambda-\lambda}.$$

$$\text{(b)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}}L_\lambda^{m-k}\binom{m}{k}B_{m-k}\left\{2^kL_{k\lambda}-2(F_\lambda\sqrt{5})^k\right\}=2^{m-2}mL_\lambda L_{m\lambda-\lambda}. \qquad \boxed{m\equiv_2 0}$$

$$\text{(c)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}}L_\lambda^{m-k}\binom{m}{k}B_{m-k}\left\{2^kF_{k\lambda}\sqrt{5}-2(F_\lambda\sqrt{5})^k\right\}=2^{m-2}mL_\lambda F_{m\lambda-\lambda}\sqrt{5}. \qquad \boxed{m\equiv_2 1}$$

The above identities contain further the following interesting particular cases.

Corollary 2.2 ($m\in\mathbb{N}$).

$$\text{(a)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 5^{\frac{m-k}{2}}\binom{m}{k}B_{m-k}\{2^kL_k-2\}=2^{m-2}(5m)F_{m-1}.$$

$$\text{(b)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 2^{\frac{m-k}{2}}\binom{m}{k}B_{m-k}\{Q_k-2\}=2mP_{m-1}.$$

$$\text{(c)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}} 3^{m-k}\binom{m}{k}B_{m-k}\left\{2^kL_{2k}-2(\sqrt{5})^k\right\}=2^{m-2}(3m)L_{2m-2}. \qquad \boxed{m\equiv_2 0}$$

$$\text{(d)}\quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}} 3^{m-k}\binom{m}{k}B_{m-k}\left\{2^kF_{2k}-2(\sqrt{5})^{k-1}\right\}=2^{m-2}(3m)F_{2m-2}. \qquad \boxed{m\equiv_2 1}$$

Proof. The first two identities (a) and (b) are deduced by letting $x=1/2$ and $x=1$ in Theorem 2.2. Other two formulae (c) and (d) correspond to the cases $\lambda=2$ of (b) and (c) displayed in Proposition 2.2. □

§**2.3.** For the generating function of Bernoulli numbers, dividing by z and then differentiating it with respect to z, we get, after having multiplied the resultant equation by z^2, the expression $\frac{-z^2e^z}{(e^z-1)^2}=\sum_{k=0}^{\infty}(k-1)B_k\frac{z^k}{k!}$. Under the replacement

$z \to (\alpha-\beta)z$, this becomes the equality $\frac{-(\alpha-\beta)^2 z^2 e^{(\alpha-\beta)z}}{(e^{(\alpha-\beta)z}-1)^2} = \sum_{k=0}^{\infty}(k-1)(\alpha-\beta)^k B_k \frac{z^k}{k!}$. Now by multiplying this with another generating function $\left(e^{z\alpha}-e^{z\beta}\right)^2 = \sum_{k=0}^{\infty}\left\{2^k Q_k(x) - 2(2x)^k\right\}\frac{z^k}{k!}$, we can evaluate the convolution

$$\begin{aligned}\sum_{k=0}^{m}(\alpha-\beta)^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k Q_k(x) - 2(2x)^k\Big\} &= m![x^m]\frac{-(\alpha-\beta)^2 z^2 e^{(\alpha-\beta)z}}{(e^{(\alpha-\beta)z}-1)^2}\times\Big(e^{x\alpha}-e^{x\beta}\Big)^2 \\ &= -(\alpha-\beta)^2 m![z^{m-2}]e^{2xz}.\end{aligned}$$

This gives rise to the following summation formula.

Theorem 2.3 ($m \in \mathbb{N}$). $\sum_{k=0}^{m}(\alpha-\beta)^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k Q_k(x) - 2(2x)^k\Big\} = 4m(1-m)(x^2+1)(2x)^{m-2}.$

Specializing this theorem by (7) and (8), and then invoking (3) and (5), we get the three summation formulae below.

Proposition 2.3 ($m,\ \lambda \in \mathbb{N}$).

(a) $$\sum_{k=0}^{m}\Big(F_\lambda\sqrt{5}\Big)^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k L_{k\lambda} - 2L_\lambda^k\Big\} = 5m(1-m)F_\lambda^2 L_\lambda^{m-2}.$$

(b) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 0}} L_\lambda^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k L_{k\lambda} - 2(F_\lambda\sqrt{5})^k\Big\} = m(1-m)L_\lambda^2(F_\lambda\sqrt{5})^{m-2}.$$ $\boxed{m\equiv_2 0}$

(c) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 1}} L_\lambda^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k F_{k\lambda}\sqrt{5} - 2(F_\lambda\sqrt{5})^k\Big\} = m(1-m)L_\lambda^2(F_\lambda\sqrt{5})^{m-2}.$$ $\boxed{m\equiv_2 1}$

The above formulae can be reduced further as in the following corollary.

Corollary 2.3 ($m \in \mathbb{N}$).

(a) $$\sum_{k=0}^{m}5^{\frac{m-k}{2}}\binom{m}{k}(m-k-1)B_{m-k}\{2^k L_k - 2\} = 5m(1-m).$$

(b) $$\sum_{k=0}^{m}2^{\frac{m-k}{2}}\binom{m}{k}(m-k-1)B_{m-k}\{Q_k - 2\} = 2m(1-m).$$

(c) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 0}} 3^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k L_{2k} - 2(\sqrt{5})^k\Big\} = 9m(1-m)(\sqrt{5})^{m-2}.$$ $\boxed{m\equiv_2 0}$

(d) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 1}} 3^{m-k}\binom{m}{k}(m-k-1)B_{m-k}\Big\{2^k F_{2k}\sqrt{5} - 2(\sqrt{5})^k\Big\} = 9m(1-m)(\sqrt{5})^{m-2}.$$ $\boxed{m\equiv_2 1}$

Proof. The first two convolution identities follow by letting $x = 1/2$ and $x = 1$ in Theorem 2.3. The last two formulae (c) and (d) result from the cases $\lambda = 2$ of identities (b) and (c) in Proposition 2.3. □

Observe that when $m = n+2$, it holds $\binom{m}{k}(m-k-1) = \binom{n}{k}\frac{(n+1)(n+2)}{n-k+2}$. Then switching the two terms corresponding to $k = n+1$ and $k = n+2$ to the right hand sides from the first two sums in the above corollary, we can check without difficulty that the resultant formulae are equivalent to the following ones:

$$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}} 5^{\frac{n-k}{2}}\binom{n}{k}\frac{B_{n-k+2}}{n-k+2}\{2^k L_k - 2\} = \frac{2^{n+2}L_{n+2}-2}{5(n+1)(n+2)} - 1, \tag{9}$$

$$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}} 2^{\frac{n-k}{2}}\binom{n}{k}\frac{B_{n-k+2}}{n-k+2}\{Q_k - 2\} = \frac{Q_{n+2}-2}{2(n+1)(n+2)} - 1. \tag{10}$$

Among these two identities, the first one resolves a problem proposed recently by Frontczak [7], which has been the primary motivation for the authors to carry out this research.

By performing the above procedure for the formulae in Proposition 2.3, we can show the following three further identities, where the first one generalizes (9), which corresponding to the case $\lambda = 1$:

$$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}} \Big(F_\lambda\sqrt{5}\Big)^{n-k}\binom{n}{k}\frac{B_{n-k+2}}{n-k+2}\Big\{2^k L_{k\lambda} - 2L_\lambda^k\Big\} = \frac{2^{n+2}L_{n\lambda+2\lambda} - 2L_\lambda^{n+2}}{5(n+1)(n+2)F_\lambda^2} - L_\lambda^n.$$

$$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}} L_\lambda^{n-k}\binom{n}{k}\frac{B_{n-k+2}}{n-k+2}\Big\{2^kL_{k\lambda}-2(F_\lambda\sqrt{5})^k\Big\}=\frac{2^{n+2}L_{n\lambda+2\lambda}-2(F_\lambda\sqrt{5})^{n+2}}{(n+1)(n+2)L_\lambda^2}-(F_\lambda\sqrt{5})^n. \qquad \boxed{n\equiv_2 0}$$

$$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}} L_\lambda^{n-k}\binom{n}{k}\frac{B_{n-k+2}}{n-k+2}\Big\{2^kF_{k\lambda}\sqrt{5}-2(F_\lambda\sqrt{5})^k\Big\}=\frac{2^{n+2}F_{n\lambda+2\lambda}\sqrt{5}-2(F_\lambda\sqrt{5})^{n+2}}{(n+1)(n+2)L_\lambda^2}-(F_\lambda\sqrt{5})^n. \qquad \boxed{n\equiv_2 1}$$

3. Convolutions with Genocchi numbers

In this section, we shall evaluate convolutions of Lucas polynomials $\mathrm{Q}_k(x)$ with Genocchi numbers which are defined by the generating function (cf. [5, §1.14]): $\frac{2z}{e^z+1}=\sum_{k=0}^\infty G_k\frac{z^k}{k!}$.

§**3.1.** Replacing z by $z(\alpha-\beta)$ in the above equation $\frac{2(\alpha-\beta)z}{e^{(\alpha-\beta)z}+1}=\sum_{k=0}^\infty(\alpha-\beta)^kG_k\frac{z^k}{k!}$ and then multiplying it with the generating function $e^{z\alpha}+e^{z\beta}=\sum_{k=0}^\infty \mathrm{Q}_k(x)\frac{z^k}{k!}$, we can evaluate the convolution sum

$$\sum_{k=0}^m(\alpha-\beta)^{m-k}\binom{m}{k}G_{m-k}\mathrm{Q}_k(x)=m![z^m]\frac{2(\alpha-\beta)z}{e^{(\alpha-\beta)z}+1}\times\left(e^{z\alpha}+e^{z\beta}\right)=2(\alpha-\beta)m![z^{m-1}]e^{z\beta}.$$

This yields the following summation formula: $\sum_{k=0}^m(\alpha-\beta)^{m-k}\binom{m}{k}G_{m-k}\mathrm{Q}_k(x)=2(\alpha-\beta)m\beta^{m-1}$. Since $G_{2k+1}=0$ except for $G_1=1$, the last identity can equivalently be reformulated, by shifting the term corresponding to $k=m-1$ to the right hand side, as in the theorem below.

Theorem 3.1 ($m\in\mathbb{N}$). $\displaystyle\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}(\alpha-\beta)^{m-k}\binom{m}{k}G_{m-k}\mathrm{Q}_k(x)=-4(x^2+1)m\mathrm{P}_{m-1}(x).$

Similar to the proof of Proposition 2.1, by specifying in the above theorem with (7) and (8), and then applying the transformations in (2–5), we get the following three summation formulae.

Proposition 3.1 ($m,\ \lambda\in\mathbb{N}$).

(a) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}\left(F_\lambda\sqrt{5}\right)^{m-k}\binom{m}{k}G_{m-k}L_{k\lambda}=-5mF_\lambda F_{m\lambda-\lambda}.$$

(b) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 0}}L_\lambda^{m-k}\binom{m}{k}G_{m-k}L_{k\lambda}=-mL_\lambda L_{m\lambda-\lambda}. \qquad \boxed{m\equiv_2 0}$$

(c) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 1}}L_\lambda^{m-k}\binom{m}{k}G_{m-k}F_{k\lambda}=-mL_\lambda F_{m\lambda-\lambda}. \qquad \boxed{m\equiv_2 1}$$

Corollary 3.1 ($m\in\mathbb{N}$).

(a) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}5^{\frac{m-k}{2}}\binom{m}{k}G_{m-k}L_k=-5mF_{m-1}.$$

(b) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}8^{\frac{m-k}{2}}\binom{m}{k}G_{m-k}Q_k=-8mP_{m-1}.$$

(c) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 0}}3^{m-k}\binom{m}{k}G_{m-k}L_{2k}=-3mL_{2m-2}. \qquad \boxed{m\equiv_2 0}$$

(d) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 1}}3^{m-k}\binom{m}{k}G_{m-k}F_{2k}=-3mF_{2m-2}. \qquad \boxed{m\equiv_2 1}$$

Proof. The formulae (c) and (d) are special cases $\lambda=2$ of (b) and (c) displayed in Proposition 3.1. The two other identities (a) and (b) are obtained by putting $x=1/2$ and $x=1$ in Theorem 3.1. □

§**3.2.** Analogously, by extracting the coefficient of z^m from the product between $\frac{2(\alpha-\beta)z}{e^{(\alpha-\beta)z}+1}-(\alpha-\beta)z=\sum_{k=0}^\infty(\alpha-\beta)^{2k}G_{2k}\frac{z^{2k}}{(2k)!}$ and

$$\left(e^{z\alpha}+e^{z\beta}\right)^2=\sum_{k=0}^\infty\Big\{2^k\mathrm{Q}_k(x)+2(2x)^k\Big\}\frac{z^k}{k!}, \qquad (11)$$

we can show the next convolution formula

$$\sum_{k=0}^{m}(\alpha-\beta)^{m-k}\binom{m}{k}G_{m-k}\Big\{2^k\mathrm{Q}_k(x)+2(2x)^k\Big\}=2(\alpha-\beta)m\Big\{(2\beta)^{m-1}+(2x)^{m-1}\Big\}.$$

This can be restated equivalently in the following theorem.

Theorem 3.2 ($m\in\mathbb{N}$). $\displaystyle\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}(\alpha-\beta)^{m-k}\binom{m}{k}G_{m-k}\Big\{2^k\mathrm{Q}_k(x)+2(2x)^k\Big\}=-2^{m+1}(x^2+1)m\mathrm{P}_{m-1}(x).$

Assigning x in Theorem 3.2 by (7) and (8), and then simplifying the resultant equations by (2–5), we find the following three identities.

Proposition 3.2 ($m,\ \lambda\in\mathbb{N}$).

(a) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}\Big(F_\lambda\sqrt{5}\Big)^{m-k}\binom{m}{k}G_{m-k}\Big\{2^kL_{k\lambda}+2L_\lambda^k\Big\}=-2^{m-1}(5m)F_\lambda F_{m\lambda-\lambda}.$$

(b) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 0}}L_\lambda^{m-k}\binom{m}{k}G_{m-k}\Big\{2^kL_{k\lambda}+2(F_\lambda\sqrt{5})^k\Big\}=-2^{m-1}mL_\lambda L_{m\lambda-\lambda}.\qquad \boxed{m\equiv_2 0}$$

(c) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 1}}L_\lambda^{m-k}\binom{m}{k}G_{m-k}\Big\{2^kF_{k\lambda}\sqrt{5}+2(F_\lambda\sqrt{5})^k\Big\}=-2^{m-1}m\sqrt{5}L_\lambda F_{m\lambda-\lambda}.\qquad \boxed{m\equiv_2 1}$$

Corollary 3.2 ($m\in\mathbb{N}$).

(a) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}5^{\frac{m-k}{2}}\binom{m}{k}G_{m-k}\{2^kL_k+2\}=-2^{m-1}(5m)F_{m-1}.$$

(b) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 m}}2^{\frac{m-k}{2}}\binom{m}{k}G_{m-k}\{Q_k+2\}=-4mP_{m-1}.$$

(c) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 0}}3^{m-k}\binom{m}{k}G_{m-k}\Big\{2^kL_{2k}+2(\sqrt{5})^k\Big\}=-2^{m-1}(3m)L_{2m-2}.\qquad \boxed{m\equiv_2 0}$$

(d) $$\sum_{\substack{0\le k\le m\\ k\equiv_2 1}}3^{m-k}\binom{m}{k}G_{m-k}\Big\{2^kF_{2k}\sqrt{5}+2(\sqrt{5})^k\Big\}=-2^{m-1}(3m)\sqrt{5}F_{2m-2}.\qquad \boxed{m\equiv_2 1}$$

Proof. When $x=1/2$ and $x=1$ in Theorem 3.2, we get identities (a) and (b). The remaining two formulae (c) and (d) are done by letting $\lambda=2$ for the corresponding (b) and (c) in Proposition 3.2 . □

§**3.3.** Finally, dividing by z the generating function of Genocchi numbers, and then differentiating with respect to z, we find, after having multiplied the resultant expression by z^2, the equality $\frac{-2z^2e^z}{(e^z+1)^2}=\sum_{k=0}^{\infty}(k-1)G_k\frac{z^k}{k!}$. Replacing further z by $z(\alpha-\beta)$, we have $\frac{-2(\alpha-\beta)^2z^2e^{(\alpha-\beta)z}}{(e^{(\alpha-\beta)z}+1)^2}=\sum_{k=0}^{\infty}(k-1)(\alpha-\beta)^kG_k\frac{z^k}{k!}$. Multiplying it with another generating function (11), we arrive at the convolution formula

$$\sum_{k=0}^{m}(\alpha-\beta)^{m-k}\binom{m}{k}(m-k-1)G_{m-k}\Big\{2^k\mathrm{Q}_k(x)+2(2x)^k\Big\}=8m(1-m)(x^2+1)(2x)^{m-2}.$$

Letting $m=n+2$ in the above equation and then singling out the terms with $k=n+1$ and $k=n+2$, we can simplify the resultant equation into the following one.

Theorem 3.3 ($n\in\mathbb{N}$). $\displaystyle\sum_{\substack{0\le k\le n\\ k\equiv_2 n}}(\alpha-\beta)^{n-k}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\Big\{2^k\mathrm{Q}_k(x)+2(2x)^k\Big\}=-2(2x)^n.$

Specifying this theorem by (7) and (8), and then taking into account (3) and (5), we find the three identities below.

Proposition 3.3 ($n,\ \lambda\in\mathbb{N}$).

(a) $$\sum_{\substack{0\le k\le n\\ k\equiv_2 n}}\Big(F_\lambda\sqrt{5}\Big)^{n-k}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\Big\{2^kL_{k\lambda}+2L_\lambda^k\Big\}=-2L_\lambda^n.$$

$$\text{(b)} \sum_{\substack{0\le k\le n\\ k\equiv_2 0}} L_\lambda^{n-k}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\Big\{2^k L_{k\lambda}+2(F_\lambda\sqrt{5})^k\Big\} = -2(F_\lambda\sqrt{5})^n. \qquad \boxed{n\equiv_2 0}$$

$$\text{(c)} \sum_{\substack{0\le k\le n\\ k\equiv_2 1}} L_\lambda^{n-k}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\Big\{2^k F_{k\lambda}\sqrt{5}+2(F_\lambda\sqrt{5})^k\Big\} = -2(F_\lambda\sqrt{5})^n. \qquad \boxed{n\equiv_2 1}$$

As applications, four further elegant formulae are highlighted as in the corollary below, where (a) and (b), obtained by letting $x = 1/2$ and $x = 1$ in Theorem 3.3, resemble somewhat the two identities displayed in (9) and (10).

Corollary 3.3 ($n \in \mathbb{N}$)**.**

$$\text{(a)} \sum_{\substack{0\le k\le n\\ k\equiv_2 n}} 5^{\frac{n-k}{2}}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\{2^k L_k+2\} = -2.$$

$$\text{(b)} \sum_{\substack{0\le k\le n\\ k\equiv_2 n}} 2^{\frac{n-k}{2}}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\{Q_k+2\} = -2.$$

$$\text{(c)} \sum_{\substack{0\le k\le n\\ k\equiv_2 0}} 3^{n-k}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\Big\{2^k L_{2k}+2(\sqrt{5})^k\Big\} = -2(\sqrt{5})^n. \qquad \boxed{n\equiv_2 0}$$

$$\text{(d)} \sum_{\substack{0\le k\le n\\ k\equiv_2 1}} 3^{n-k}\binom{n}{k}\frac{G_{n+2-k}}{n+2-k}\Big\{2^k F_{2k}\sqrt{5}+2(\sqrt{5})^k\Big\} = -2(\sqrt{5})^n. \qquad \boxed{n\equiv_2 1}$$

Proof. We only need to show (c) and (d). They are just the $\lambda = 2$ cases of (b) and (c) in Proposition 3.3 . □

4. Convolutions with Euler numbers

For Euler numbers E_k, their generating function reads (cf. [5, §1.14]) as $\frac{2e^z}{e^{2z}+1} = \sum_{k=0}^\infty E_k \frac{z^k}{k!}$. We are going to show two identities about Euler numbers and Lucas polynomials.

§**4.1.** Replacing z by $z(\alpha-\beta)$, we can reformulate the above generating function as

$$\frac{2e^{(\alpha-\beta)z}}{e^{2(\alpha-\beta)z}+1} = \sum_{k=0}^\infty (\alpha-\beta)^k E_k \frac{z^k}{k!}. \tag{12}$$

Its product with the generating function $e^{2z\alpha}+e^{2z\beta} = \sum_{k=0}^\infty 2^k \mathrm{Q}_k(x)\frac{z^k}{k!}$ gives rise to the convolution formula.

Theorem 4.1 ($m \in \mathbb{N}$)**.** $\displaystyle\sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 2^k(\alpha-\beta)^{m-k}\binom{m}{k}E_{m-k}\mathrm{Q}_k(x) = 2(2x)^m.$

Assigning x in this theorem by (7) and (8), and then invoking (3) and (5), we prove the next three identities.

Proposition 4.1 ($m,\ \lambda \in \mathbb{N}$)**.**

$$\text{(a)} \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 2^k\Big(F_\lambda\sqrt{5}\Big)^{m-k}\binom{m}{k}E_{m-k}L_{k\lambda} = 2L_\lambda^m.$$

$$\text{(b)} \sum_{\substack{0\le k\le m\\ k\equiv_2 0}} 2^k L_\lambda^{m-k}\binom{m}{k}E_{m-k}L_{k\lambda} = 2(F_\lambda\sqrt{5})^m. \qquad \boxed{m\equiv_2 0}$$

$$\text{(c)} \sum_{\substack{0\le k\le m\\ k\equiv_2 1}} 2^k L_\lambda^{m-k}\binom{m}{k}E_{m-k}F_{k\lambda}\sqrt{5} = 2(F_\lambda\sqrt{5})^m. \qquad \boxed{m\equiv_2 1}$$

By letting $x = \frac{1}{2}$ and $x = 1$ in Theorem 4.1 as well as $\lambda = 2$ in the above (b) and (c), we deduce four further identities.

Corollary 4.1 ($m \in \mathbb{N}$)**.**

$$\text{(a)} \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 5^{\frac{m-k}{2}}\binom{m}{k}2^k E_{m-k}L_k = 2.$$

$$\text{(b)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 2^{\frac{m-k}{2}}\binom{m}{k}E_{m-k}Q_k = 2.$$

$$\text{(c)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}} 2^k 3^{m-k}\binom{m}{k}E_{m-k}L_{2k} = 2(\sqrt{5})^m. \qquad \boxed{m\equiv_2 0}$$

$$\text{(d)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}} 2^k 3^{m-k}\binom{m}{k}E_{m-k}F_{2k} = 2(\sqrt{5})^{m-1}. \qquad \boxed{m\equiv_2 1}$$

§**4.2.** Alternatively, the product between (12) and the generating function $\left(e^{2z\alpha}+e^{2z\beta}\right)^2 = \sum_{k=0}^{\infty}\left\{4^k Q_k(x)+2(4x)^k\right\}\frac{z^k}{k!}$ will lead us to another convolution formula.

Theorem 4.2 ($m\in\mathbb{N}$). $\displaystyle\sum_{\substack{0\le k\le m\\ k\equiv_2 m}} (\alpha-\beta)^{m-k}\binom{m}{k}E_{m-k}\Big\{4^k Q_k(x)+2(4x)^k\Big\} = 2^{m+1}\Big\{(x+\alpha)^m+(x+\beta)^m\Big\}.$

Finally, in a similar manner as that from Theorem 4.1 to Proposition 4.1, the following three formulae can be confirmed.

Proposition 4.2 ($m,\ \lambda\in\mathbb{N}$).

$$\text{(a)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} \left(F_\lambda\sqrt{5}\right)^{m-k}\binom{m}{k}E_{m-k}\Big\{4^k L_{k\lambda}+2^{k+1}L_\lambda^k\Big\} = 2\Big\{(2L_\lambda+F_\lambda\sqrt{5})^m+(2L_\lambda-F_\lambda\sqrt{5})^m\Big\}.$$

$$\text{(b)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}} L_\lambda^{m-k}\binom{m}{k}E_{m-k}\Big\{4^k L_{k\lambda}+2(2F_\lambda\sqrt{5})^k\Big\} = 2\Big\{(L_\lambda+2F_\lambda\sqrt{5})^m+(2F_\lambda\sqrt{5}-L_\lambda)^m\Big\}. \qquad \boxed{m\equiv_2 0}$$

$$\text{(c)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}} L_\lambda^{m-k}\binom{m}{k}E_{m-k}\Big\{4^k F_{k\lambda}\sqrt{5}+2(2F_\lambda\sqrt{5})^k\Big\} = 2\Big\{(L_\lambda+2F_\lambda\sqrt{5})^m+(2F_\lambda\sqrt{5}-L_\lambda)^m\Big\}. \qquad \boxed{m\equiv_2 1}$$

Analogously, four further examples can be shown by specializing in Theorem 4.2 and Proposition 4.2.

Corollary 4.2 ($m\in\mathbb{N}$).

$$\text{(a)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 5^{\frac{m-k}{2}}\binom{m}{k}E_{m-k}\Big\{4^k L_k+2^{k+1}\Big\} = 2\Big\{(2+\sqrt{5})^m+(2-\sqrt{5})^m\Big\}.$$

$$\text{(b)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 m}} 2^{\frac{m+k}{2}}\binom{m}{k}E_{m-k}\{Q_k+2\} = 2\Big\{(2+\sqrt{2})^m+(2-\sqrt{2})^m\Big\}.$$

$$\text{(c)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 0}} 3^{m-k}\binom{m}{k}E_{m-k}\Big\{4^k L_{2k}+2(2\sqrt{5})^k\Big\} = 2\Big\{(2\sqrt{5}+3)^m+(2\sqrt{5}-3)^m\Big\}. \qquad \boxed{m\equiv_2 0}$$

$$\text{(d)} \quad \sum_{\substack{0\le k\le m\\ k\equiv_2 1}} 3^{m-k}\binom{m}{k}E_{m-k}\Big\{4^k F_{2k}\sqrt{5}+2(2\sqrt{5})^k\Big\} = 2\Big\{(2\sqrt{5}+3)^m+(2\sqrt{5}-3)^m\Big\}. \qquad \boxed{m\equiv_2 1}$$

Besides Bernoulli, Genocchi and Euler numbers as well as Pell and Lucas polynomials, it is possible to derive, by following the same scheme, more convolution formulae. The interested reader is encouraged to make further exploration.

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