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Research Article

Minimum 2-vertex strongly biconnected spanning directed subgraph problem

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Abstract

A directed graph $G = (V, E)$ is strongly biconnected if G is strongly connected and its underlying graph is biconnected. A strongly biconnected directed graph $G = (V, E)$ is called 2-vertex-strongly biconnected if $|V| \geq 3$ and the induced subgraph on $V \setminus \{w\}$ is strongly biconnected for every vertex $w \in V$. In this paper, the following problem is studied: Given a 2-vertex-strongly biconnected directed graph $G = (V, E)$, compute an edge subset $E^{2sb} \subseteq E$ of minimum size such that the subgraph (V, E^{2sb}) is 2-vertex-strongly biconnected.

Keywords: directed graphs; approximation algorithms; graph algorithms; strongly connected graphs; strongly biconnected directed graphs.

2020 Mathematics Subject Classification: 05C85, 05C20.

1. Introduction

The underlying graph of a directed graph $G = (V, E)$ is the undirected graph $G_1 = (V_1, E_1)$, where $V_1 = V$ and $E_1 = \{(v, w) \mid (v, w) \in E \text{ or } (w, v) \in E\}$. A directed graph $G = (V, E)$ is strongly biconnected if G is strongly connected and its underlying graph is biconnected. A strongly biconnected directed graph $G = (V, E)$ is called k-vertex-strongly biconnected if $|V| > k$ and for each $L \subset V$ with $|L| < k$, the induced subgraph on $V \setminus L$ is strongly biconnected. The minimum k-vertex-strongly biconnected spanning subgraph problem (denoted by MKVSBSS) is formulated as follows. Given a k-vertex-strongly biconnected directed graph $G = (V, E)$, compute an edge subset $E^{ksb} \subseteq E$ of minimum size such that the subgraph (V, E^{ksb}) is k-vertex-strongly biconnected. In this paper, we consider the MKVSBSS problem for $k = 2$. Note that each 2-vertex-strongly-biconnected directed graph is 2-vertex-connected, but the converse is not necessarily true.

Thus, optimal solutions for minimum 2-vertex-connected spanning subgraph (M2VCSS) problem are not necessarily feasible solutions for the 2-vertex strongly biconnnected spanning subgraph problem, as shown in Figure 1.

The problem of finding a k-vertex-connected spanning subgraph of a k-vertex-connected directed graph is NP-hard for $k \geq 1$ [4]. Results of Edmonds [2] and Mader [16] imply that the number of edges in each minimal k-vertex-connected directed graph is at most $2kn$ [1]. Cheriyan and Thurimella [1] gave a $(1+1/k)$-approximation algorithm for the minimum k-vertex-connected spanning subgraph problem. Georgiadis [6] improved the running time of this algorithm for the M2VCSS problem and presented a linear time approximation algorithm that achieves an approximation factor of 3 for the M2VCSS problem. Georgiadis et al. [7] provided efficient approximation algorithms based on the results of [3,5,9,10] for the M2VCSS problem. Strongly connected components of a directed graph and blocks of an undirected graphs can be found in linear time using Tarjan's algorithm [17, 18]. Wu and Grumbach [19] introduced the concept of strongly biconnected directed graph and strongly biconnected components. Clearly, the MKVSBSS problem is NP-hard for $k \geq 1$. In this paper, we study the MKVSBSS problem when $k = 2$ (denoted by M2VSBSS).

2. Approximation algorithms for the M2VSBSS problem

In this section, we present approximation algorithms for the M2VSBSS Problem. A vertex w in a strongly biconnected directed graph $G = (V, E)$ is a b-articulation point if $G \setminus \{w\}$ is not strongly biconnected. Algorithm 2.1 is based on b-articulation points, minimal 2-vertex-connected subgraphs, and Lemma 2.1.

Lemma 2.1. *Let $G_s = (V, E_s)$ be a subgraph of a strongly biconnected directed graph $G = (V, E)$ such that G_s is strongly connected and G_s has $t > 0$ strongly biconnected components. Let (u, w) be an edge in $E \setminus E_s$ such that u, w are not in the*

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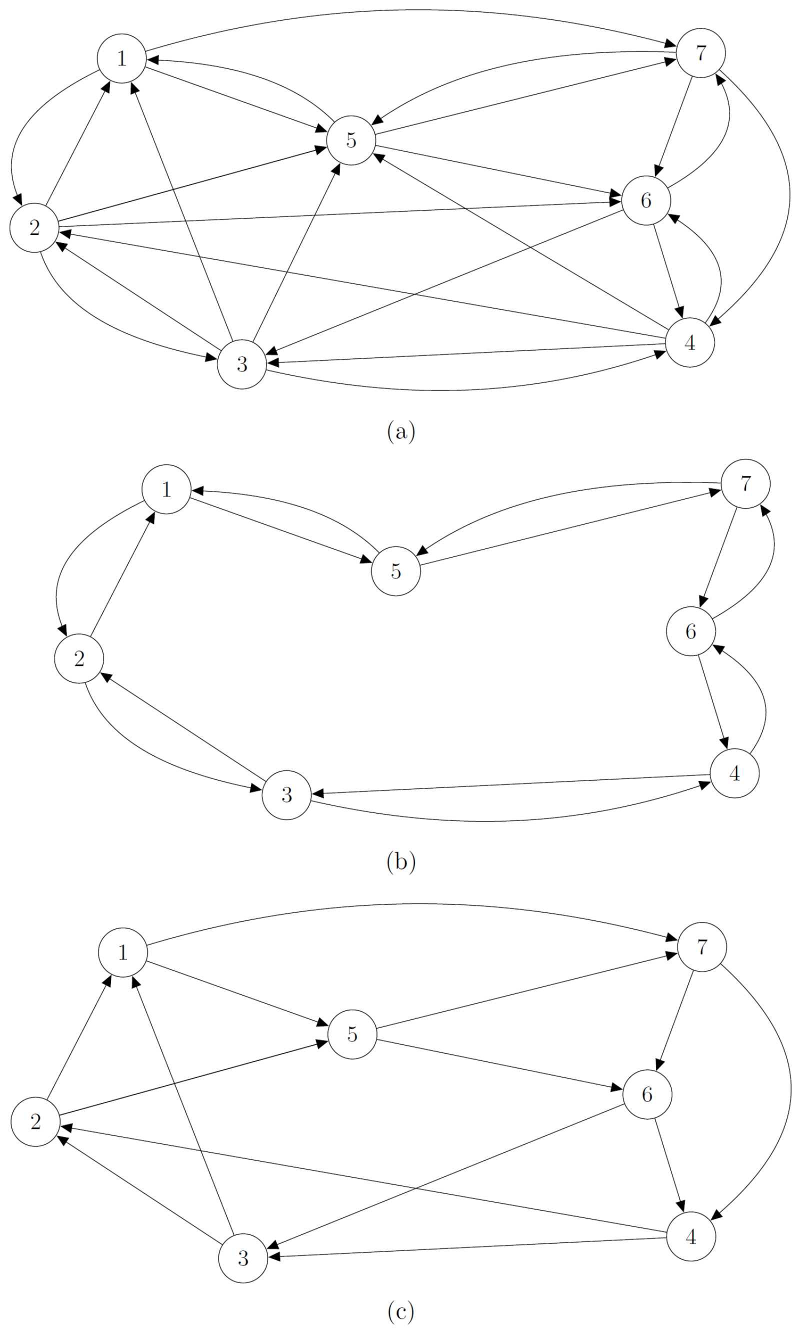

Figure 1: (a) A 2-vertex strongly biconnected graph. (b) An optimal solution for the minimum 2-vertex-connected spanning subgraph problem. But note that this subgraph is not 2-vertex strongly biconnected because the underlying graph of the subgraph obtained by removing vertex 1 is not biconnected. (c) An optimal solution for the minimum 2-vertex strongly biconnected spanning subgraph problem.

same strongly biconnected component of G_s. Then the directed subgraph $(V, E_s \cup \{(u, w)\})$ contains at most $t-1$ strongly biconnected components.

Proof. Since G_s is strongly connected, there exists a simple path p from w to u in G_s. Since the edge (u, w) does not belong to the path p, the path p and edge (u, w) form a simple directed cycle c in the directed subgraph $(V, E_s \cup \{(u, w)\})$. Moreover, the cycle c is also a simple undirected cycle in the underlying undirected graph of the directed graph $(V, E_s \cup \{(u, w)\})$. Consequently, the vertices u, w are in the same strongly biconnected component of the subgraph $(V, E_s \cup \{(u, w)\})$. □

Lemma 2.2. *Algorithm 2.1 returns a 2-vertex strongly biconnected directed subgraph.*

Proof. It follows from Lemma 2.1. □

The following lemma shows that each optimal solution for the M2VSBSS problem has at least $2n$ edges.

Lemma 2.3. *Let $G = (V, E)$ be a 2-vertex-strongly biconnected directed graph. Let $O \subseteq E$ be an optimal solution for the M2VSBSS problem. Then $|O| \geq 2n$.*

Algorithm 2.1.
Input: A 2-vertex strongly biconnected directed graph $G = (V, E)$
Output: a 2-vertex strongly biconnected subgraph $G_{2s} = (V, E_{2s})$
1 find a minimal 2-vertex-connected subgraph $G_1 = (V, E_1)$ of G
2 **if** G_1 is 2-vertex strongly biconnected **then**
3 output G_1
4 **else**
5 $E_{2s} \leftarrow E_1$
6 $G_{2s} \leftarrow (V, E_{2s})$
7 identify the b-articulation points of G_1
8 **for** every b-articulation point $b \in V$ **do**
9 **while** $G_{2s} \setminus \{b\}$ is not strongly biconnected **do**
10 calculate the strongly biconnected components of $G_{2s} \setminus \{b\}$
11 find an edge $(u, w) \in E \setminus E_{2s}$ such that u, w are not in
12 the same strongly biconnected component of $G_{2s} \setminus \{b\}$.
13 $E_{2s} \leftarrow E_{2s} \cup \{(u, w)\}$
14 output G_{2s}.

Proof. for any vertex $x \in V$, the removal of x from the subgraph (V, O) leaves a strongly biconnected directed subgraph. Since each strongly biconnected directed graph is strongly connected, the subgraph (V, O) has no strong articulation points. Therefore, the directed subgraph (V, O) is 2-vertex-connected. □

Let l be the number of b-articulation points in G_1. The following lemma shows that Algorithm 2.1 has an approximation factor of $(2 + l/2)$.

Theorem 2.1. *Let l be the number of b-articulation points in G_1. Then, $|E_{2s}| \leq l(n-1) + 4n$.*

Proof. Results of Edmonds [2] and Mader [16] imply that $|E_1| \leq 4n$ [1,6]. Moreover, by Lemma 2.3, every optimal solution for the M2VSBSS problem has size at least $2n$. For every b-articulation point in line 8, Algorithm 2.1 adds at most $n - 1$ edges to E_{2s} in while loop. Therefore, $|E_{2s}| \leq l(n-1) + 4n$ □

Theorem 2.1. *The running time of Algorithm 2.1 is $O(n^2 m)$.*

Proof. A minimal 2-vertex-connected subgraph can be found in time $O(n^2)$ [6,7]. B-articulation points can be computed in $O(nm)$ time. The strongly biconnected components of a directed graph can be identified in linear time [19]. Furthermore, by Lemma 2.1, lines 9–13 take $O(nm)$ time. □

Results of Mader [14, 15] imply that the number of edges in each minimal k-vertex-connected undirected graph is at most kn [1]. Results of Edmonds [2] and Mader [16] imply that the number of edges in each minimal k-vertex-connected directed graph is at most $2kn$ [1]. These results imply a 2-approximation algorithm [1] for minimum k-vertex-connected spanning subgraph problem for undirected and directed graphs [1] because every vertex in a k-vertex-connected undirected graphs has degree at least k and every vertex in a k-vertex-connected directed graph has outdegree at least k [1]. Note that these results imply a 7/2 approximation algorithm for the M2VSBSS problem by calculating a minimal 2-vertex-connected directed subgraph of a 2-vertex strongly biconnected directed graph $G = (V, E)$ and a minimal 3-vertex connected undirected subgraph of the underlying graph of G. The running time of this algorithm is $O(m^2)$.

Lemma 2.4. *Let $G = (V, E)$ be a 2-vertex strongly biconnected directed graph. Let $G_1 = (V, L)$ be a minimal 2-vertex-connected subgraph of G and let $G_2 = (V, U)$ be a minimal 3-vertex-connected subgraph of the underlying graph of G. Then the directed subgraph $G_s = (V, L \cup A)$ is 2-vertex strongly biconnected, where $A = \{(v, w) \mid (v, w) \in E$ and $(v, w) \in U\}$. Moreover, $|L \cup A| \leq 7n$*

Proof. Let w be any vertex of the subgraph G_s. Since the $G_1 = (V, L)$ is 2-vertex-connected, the directed subgraph G_s has no strong articulation points. Therefore, $G_s \setminus \{w\}$ is strongly connected. Moreover, the underlying graph of $G_s \setminus \{w\}$ is biconnected because G_2 is 3-vertex-connected and G_2 is a subgraph of the underlying graph of G_s. Results of Edmonds [2] and Mader [16] imply that $|L| \leq 4n$. Furthermore, Results of Mader [14, 15] imply that $|U| \leq 3n$. □

3. Open problems

We leave as an open problem whether each minimal k-vertex strongly biconnected directed graph has at most $2kn$ edges.

Cheriyan and Thurimella [1] presented a $(1+1/k)$-approximation algorithm for the minimum k-vertex-connected spanning subgraph problem for directed and undirected graphs. The algorithm of Cheriyan and Thurimella [1] has an approximation factor of $3/2$ for the minimum 2-vertex-connected directed subgraph problem. Let $G=(V,E)$ be a 2-vertex strongly biconnected directed graph and let E^{CT} be the output of the algorithm of Cheriyan and Thurimella [1]. The directed subgraph (V,E^{CT}) is not necessarily 2-vertex strongly biconnected. But a 2-vertex strongly biconnected subgraph can be obtained by performing the following third phase. For each edge $e \in E \setminus E^{CT}$, if the underlying graph of $G \setminus \{e\}$ is 3-vertex-connected, delete e from G. We leave as as open problem whether this algorithm has an approximation factor of $3/2$ for the M2VSBSS problem.

The present author [11–13] studied twinless articulation points and some related problems. Georgiadis and Kosinas [8] presented linear time algorithms for computing twinless articulation points and twinless bridges. An important question is whether there is a connection between twinless articulation points and the M2VSBSS problem.

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