On spectral radius of the generalized distance matrix of a graph

Shariefuddin Pirzada*

Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India

Abstract

If $Tr(G)$ and $D(G)$ are respectively the diagonal matrix of vertex transmission degrees and distance matrix of a connected graph $G$, the generalized distance matrix $D_\alpha(G)$ is defined as $D_\alpha(G) = \alpha Tr(G) + (1 - \alpha) D(G)$, where $0 \leq \alpha \leq 1$. We obtain an upper bound for the spectral radius $\partial(G)$ (largest eigenvalue) of $D_\alpha(G)$ as

$$\partial(G) \leq \max_{1 \leq i, j \leq n} \frac{1}{2} \left[ \alpha t_i + t_j - (1 - \alpha)d_{ij} + \sqrt{(\alpha t_i - t_j)^2 + (1 - \alpha)(1 - \alpha - 2t_j - 4t_i - 2\alpha t_i)d_{ij}} \right],$$

where $t_{\text{max}} = \max_{1 \leq i, j \leq n} \left( (1 - \alpha)t_i + t_j \right)$ is the vertex transmission degrees of $G$ and $d_{ij}$ is the distance between the vertices $v_i, v_j \in G$. Further, we show the existence of graphs for which equality holds.

Keywords: distance matrix, generalized distance matrix; spectral radius; transmission degree.

2020 Mathematics Subject Classification: 05C12, 05C50, 15A18.

1. Introduction

Let $G(V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and order $|V(G)| = n$. The degree $d(v_i)$ or $d_i$ of a vertex $v_i$ is the number of edges incident on $v_i$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the neighborhood of $v$. A graph is regular if each of its vertices has the same degree. A graph is said to be $(r, s)$-semi-regular, denoted by $G(r, s)$, if degree of each vertex is either $r$ or $s$. In $G$, the distance between two vertices $u, v \in V(G)$, denoted by $d_{uv}$, is defined as the length of a shortest path between $u$ and $v$. The distance matrix of $G$ is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The transmission $t_G(v)$ of a vertex $v$ is defined as the sum of the distances from $v$ to all other vertices in $G$, that is, $t_G(v) = \sum_{u \in V(G)} d_{uv}$. A graph $G$ is said to be $k$-transmission regular if $t_G(v) = k$, for each $v \in V(G)$. For any vertex $v_i \in V(G)$, the transmission $t_G(v_i)$ is also called the transmission degree, shortly denoted by $t_i$ and the sequence $\{t_1, t_2, \ldots, t_n\}$ is called the transmission degree sequence of the graph $G$. The matrix $Tr(G) = \text{diag}(t_1, t_2, \ldots, t_n)$ is the diagonal matrix of vertex transmissions. The generalized distance matrix $D_\alpha(G)$ is defined as $D_\alpha(G) = \alpha Tr(G) + (1 - \alpha) D(G)$, for $0 \leq \alpha \leq 1$. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ be the eigenvalues of $D_\alpha(G)$. We will denote the largest eigenvalue (generalized distance spectral radius) $\partial_1$ by $\partial(G)$. As $D_\alpha(G)$ is non-negative and irreducible, by the Perron-Frobenius theorem, $\partial(G)$ is unique and there is a unique positive unit eigenvector $X$ corresponding to $\partial(G)$, which is called the generalized distance Perron vector of $G$. For some recent results, we refer to [1, 3–7, 9] and the references therein. For standard definitions, we refer to [2, 8].

Consider a graph in which there is a vertex $v_i$ having transmission degree equal to $t_1 = t_{\text{max}}$, transmission degree of every neighbor of $v_i$ equal to $t_2$ and transmission degree of every vertex non-adjacent to $v_i$ equal to $t_{\text{min}}$. We name such a graph as $H$.

The following is the main result.

Theorem 1.1. If $\partial(G)$ is the spectral radius of $D_\alpha(G)$, then

$$\partial(G) \leq \max_{1 \leq i, j \leq n} \frac{1}{2} \left[ \alpha t_i + t_j - (1 - \alpha)d_{ij} + \sqrt{(\alpha t_i - t_j)^2 + (1 - \alpha)(1 - \alpha - 2t_j - 4t_i - 2\alpha t_i)d_{ij}} \right],$$

where $t_{\text{max}} = \max_{1 \leq i, j \leq n}(\alpha t_i + t_j)$ are the vertex transmission degrees of $G$. Further, equality holds if and only if $G$ is a transmission regular graph or $G$ is a semi-regular graph $G(n - 1, s)$ or $G$ is isomorphic to $H$, where $H$ is defined above.

*E-mail address: pirzadasd@kashmiruniversity.ac.in
2. Proof of the main theorem

We consider a column vector \( X = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) to be a function defined on \( V(G) \) which maps vertex \( v_i \) to \( x_i \), that is, \( X(v_i) = x_i \) for \( i, 1 \leq i \leq n \). The quadratic form \( X^T D_\alpha(G)X \) can be described as

\[
X^T D_\alpha(G)X = \alpha \sum_{i=1}^{n} t(v_i)x_i^2 + 2(1 - \alpha) \sum_{1 \leq i < j \leq n} d(v_i, v_j)x_ix_j,
\]

and

\[
X^T D_\alpha(G)X = (2\alpha - 1) \sum_{i=1}^{n} t(v_i)x_i^2 + (1 - \alpha) \sum_{1 \leq i < j \leq n} d(v_i, v_j)(x_i + x_j)^2.
\]

Also, \( \partial \) is an eigenvalue of \( D_\alpha(G) \) corresponding to the eigenvector \( X \) if and only if \( X \neq 0 \) and

\[
\partial x_i = \alpha \text{Tr}(v_i)x_i + (1 - \alpha) \sum_{j=1}^{n} d(v_i, v_j)x_j.
\]

These equations are called the \((\partial, x)\)-eignequations of \( G \). For a normalized column vector \( X \in \mathbb{R}^n \) with at least one non-negative component, by the Rayleigh’s principle, we have

\[
\partial(G) \geq X^T D_\alpha(G)X,
\]

with equality if and only if \( X \) is the generalized distance Perron vector of \( G \).

**Proof of Theorem 1.1.** Corresponding to the eigenvalue \( \partial \) of \( D_\alpha(G) \), let \( X = [x_1, x_2, \ldots, x_n]^T \) be the eigenvector, with \( x_i = 1 \) and \( x_t \geq 1 \) for all \( t \neq i \). Let \( x_j = \max \{ x_t : 1 \leq t \leq n \text{ and } t \neq i \} \). Therefore, from the \( i \)-th and \( j \)-th equations of \( \partial X = D_\alpha(G)X \), we have

\[
\partial x_i = \alpha t_i x_i + (1 - \alpha) \sum_{t=1, t \neq i}^{n} d_{it}x_t,
\]

and

\[
\partial x_j = \alpha t_j x_j + (1 - \alpha) \sum_{t=1, t \neq j}^{n} d_{jt}x_t.
\]

(2) implies that \( \partial \leq \alpha t_i + (1 - \alpha)t_i x_j \), so that

\[
\partial - \alpha t_i \leq (1 - \alpha)t_i x_j.
\]

Also, (3) implies that \( \partial x_j \leq \alpha t_j x_j + (1 - \alpha)d_{ij} + (1 - \alpha)(t_j - d_{ij})x_j \), which on simplification gives

\[
[\partial - t_j + (1 - \alpha)d_{ij}]x_j \leq (1 - \alpha)d_{ij}.
\]

Combining (4) and (5), we get

\[
(\partial - \alpha t_i)[\partial - t_j + (1 - \alpha)d_{ij}]x_j \leq [(1 - \alpha)t_j x_j][(1 - \alpha)d_{ij}].
\]

Since \( x_j > 0 \), therefore we have

\[
(\partial - \alpha t_i)[\partial - t_j + (1 - \alpha)d_{ij}] \leq (1 - \alpha)^2 t_i d_{ij},
\]

which on simplification yields

\[
\partial^2 - [\alpha t_i + t_j - (1 - \alpha)d_{ij}] \partial + \alpha t_i t_j - (1 - \alpha)t_i d_{ij} \leq 0.
\]

The solution of (6) is given by

\[
\partial \leq \frac{1}{2} \left[ \alpha t_i + t_j - (1 - \alpha)d_{ij} + \sqrt{[\alpha t_i + t_j - (1 - \alpha)d_{ij}]^2 - 4t_i [\alpha t_j + (1 - \alpha)d_{ij}]} \right]
\]

and

\[
\partial \geq \frac{1}{2} \left[ \alpha t_i + t_j - (1 - \alpha)d_{ij} - \sqrt{[\alpha t_i + t_j - (1 - \alpha)d_{ij}]^2 - 4t_i [\alpha t_j + (1 - \alpha)d_{ij}]} \right].
\]

Therefore, in all cases, it follows that

\[
\partial(G) \leq \max_{1 \leq i, j \leq n} \frac{1}{2} \left[ \alpha t_i + t_j - (1 - \alpha)d_{ij} + \sqrt{[\alpha t_i + t_j - (1 - \alpha)d_{ij}]^2 - 4t_i [\alpha t_j + (1 - \alpha)d_{ij}]} \right].
\]
Now, we characterize the graphs for which equality holds in (1). In this regard, first assume that equality holds in (1). Therefore, equality cases in (4) and (5) imply that \( x_t = x_j \) for all \( t \), with \( t \neq i \). As \( x_j \leq x_i = 1 \), we have the following two possibilities to consider.

**Case 1.** \( x_j = x_i = 1 \). Here, \( \partial = t_i \) for all \( i, 1 \leq i \leq n \). So \( G \) is a transmission regular graph.

**Case 2.** \( x_j < x_i = 1 \). Therefore, either \( d_i = n - 1 \) or \( d_i < n - 1 \). We look at these two cases separately as follows.

**Case 2.1.** If \( d_i = n - 1 \), then clearly \( v_i \in N(v_i) \) for all those \( v_i \) which are in \( V(G) - \{v_i\} \). Now, for any \( v_i \in V(G) - \{v_i\} \), we have \( \partial x_j = (t_2 + \alpha - 1)x_j + (1 - \alpha) \). This clearly indicates that transmission degree of every vertex \( v_i \in V(G) - \{v_i\} \) is equal to \( t_i \), with of course \( t_i > n - 1 \). So, evidently \( t_2 = t_3 = \cdots = t_n = t_{\min} \). Further, \( d_i = n - 1 \) implies that \( G \) is of diameter 2 and thus \( t_2 = 2n - 2 - d_i \). Therefore, the vertex degrees of \( G \) are \( n - 1, d_2 = d_3 = \cdots = d_n = s \) (say) with clearly \( n - 1 > s \). Hence \( G \) is isomorphic to \( G(n - 1, s) \).

**Case 2.2.** Now, for the case \( d_i < n - 1 \), we consider the vertex partition of \( V(G) - \{v_i\} \) as \( U = N(v_i) \) and \( W = V(G) - (U \cup \{v_i\}) \). Then

\[
\partial = t_i \left[ (1 - \alpha)x_j \right] \text{ for } v_i, \\
\partial x_j = (t_i - 1 - \alpha)x_j + (1 - \alpha) \text{ for } v_i \in N(v_i), \\
\partial = t_k \text{ for } v_k \notin (N(v_i) \cup \{v_i\}).
\]

Simplifying (8) implies that \( t_i = \partial + (1 - \alpha)(x_j - 1) \frac{1}{x_j} \), for all \( v_i \in N(v_i) \). Therefore, transmission degree of every vertex in \( N(v_i) \) is equal to \( t_i \), and transmission degree of every vertex outside \( N(v_i) \cup \{v_i\} \) is equal to \( t_i \). Since \( (1 - \alpha)x_j \leq 1 \), from (7) and (9), we have \( t_k > t_i \). In a similar way, \( t_k > t_i \) follows from (8) and (9). Thus, \( t_i = t_{\max} \), and \( t_k = t_2 \) for all \( v_k \in W \). Also, \( t_i = t_{\min} \), for all vertices \( v_i \in U \).

From the arguments given above, we conclude that the given connected graph contains a vertex \( v_i \) with transmission degree \( t_i = t_{\max} \), transmission degree of every neighbor of \( v_i \) is equal to \( t_2 \) and transmission degree of every vertex non-adjacent to \( v_i \) is equal to \( t_{\min} \). This is clearly the graph \( H \) defined above.

Conversely, for the regular graph, or the graph isomorphic to \( G(n - 1, s) \), or graph \( H \), it is easy to verify that equality holds.

\[\square\]

**Acknowledgment**

This research is supported by SERB-DST, New Delhi under the research project number MTR/2017/000084.

**References**


