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Research Article

On spectral radius of the generalized distance matrix of a graph

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Abstract

If $Tr(G)$ and $D(G)$ are respectively the diagonal matrix of vertex transmission degrees and distance matrix of a connected graph G, the generalized distance matrix $D_\alpha(G)$ is defined as $D_\alpha(G) = \alpha\, Tr(G) + (1-\alpha)\, D(G)$, where $0 \le \alpha \le 1$. We obtain an upper bound for the spectral radius $\partial(G)$ (largest eigenvalue) of $D_\alpha(G)$ as

$$\partial(G) \le \max_{1\le i,\ j\le n} \frac{1}{2}\left[\alpha t_i + t_j - (1-\alpha)d_{ij} + \sqrt{(\alpha t_i - t_j)^2 + (1-\alpha)(1-\alpha-2t_j-4t_i-2\alpha t_i)d_{ij}}\right],$$

where $t_{max} = t_1 \ge t_2 \ge \cdots \ge t_n = t_{min}$ are the vertex transmission degrees of G and d_{ij} is the distance between the vertices $v_i, v_j \in G$. Further, we show the existence of graphs for which equality holds.

Keywords: distance matrix, generalized distance matrix; spectral radius; transmission degree.

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1. Introduction

Let $G(V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and order $|V(G)| = n$. The degree $d(v_i)$ or d_i of a vertex v_i is the number of edges incident on v_i. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. A graph is regular if each of its vertices has the same degree. A graph is said to be (r,s)-semi-regular, denoted by $G(r,s)$, if degree of each vertex is either r or s. In G, the *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v\in V(G)}$. The *transmission* $t_G(v)$ of a vertex v is defined as the sum of the distances from v to all other vertices in G, that is, $t_G(v) = \sum_{u\in V(G)} d_{uv}$. A graph G is said to be k-*transmission regular* if $t_G(v) = k$, for each $v \in V(G)$. For any vertex $v_i \in V(G)$, the transmission $t_G(v_i)$ is also called the *transmission degree*, shortly denoted by t_i and the sequence $\{t_1, t_2, \ldots, t_n\}$ is called the *transmission degree sequence* of the graph G. The matrix $Tr(G) = diag\ (t_1, t_2, \ldots, t_n$ is the diagonal matrix of vertex transmissions. The *generalized distance matrix* [3] is defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, for $0 \le \alpha \le 1$. Let $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$ be the eigenvalues of $D_\alpha(G)$. We will denote the largest eigenvalue (generalized distance spectral radius) ∂_1 by $\partial(G)$. As $D_\alpha(G)$ is non-negative and irreducible, by the Perron-Frobenius theorem, $\partial(G)$ is unique and there is a unique positive unit eigenvector X corresponding to $\partial(G)$, which is called the *generalized distance Perron vector* of G. For some recent results, we refer to [1, 3–7, 9] and the references therein. For standard definitions, we refer to [2, 8].

Consider a graph in which there is a vertex v_i having transmission degree equal to $t_1 = t_{max}$, transmission degree of every neighbor of v_i equal to t_2 and transmission degree of every vertex non-adjacent to v_i equal to t_{min}. We name such a graph as H.

The following is the main result.

Theorem 1.1. *If $\partial(G)$ is the spectral radius of $D_\alpha(G)$, then*

$$\partial(G) \le \max_{1\le i,\ j\le n} \frac{1}{2}\left[\alpha t_i + t_j - (1-\alpha)d_{ij} + \sqrt{(\alpha t_i - t_j)^2 + (1-\alpha)(1-\alpha-2t_j-4t_i-2\alpha t_i)d_{ij}}\right], \tag{1}$$

where $t_{max} = t_1 \ge t_2 \ge \cdots \ge t_n = t_{min}$ are the vertex transmission degrees of G. Further, equality holds if and only if G is a transmission regular graph or G is a semi-regular graph $G(n-1,s)$ or G is isomorphic to H, where H is defined above.

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2. Proof of the main theorem

We consider a column vector $X = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ to be a function defined on $V(G)$ which maps vertex v_i to x_i, that is, $X(v_i) = x_i$ for i, $1 \le i \le n$. The quadratic form $X^T D_\alpha(G) X$ can be described as

$$X^T D_\alpha(G) X = \alpha \sum_{i=1}^{n} t(v_i) x_i^2 + 2(1-\alpha) \sum_{1 \le i < j \le n} d(v_i, v_j) x_i x_j,$$

and

$$X^T D_\alpha(G) X = (2\alpha - 1) \sum_{i=1}^{n} t(v_i) x_i^2 + (1-\alpha) \sum_{1 \le i < j \le n} d(v_i, v_j)(x_i + x_j)^2.$$

Also, ∂ is an eigenvalue of $D_\alpha(G)$ corresponding to the eigenvector X if and only if $X \neq \mathbf{0}$ and

$$\partial x_v = \alpha Tr(v_i) x_i + (1-\alpha) \sum_{j=1}^{n} d(v_i, v_j) x_j.$$

These equations are called the (∂, x)-*eigenequations* of G. For a normalized column vector $X \in \mathbb{R}^n$ with at least one non-negative component, by the Rayleigh's principle, we have

$$\partial(G) \ge X^T D_\alpha(G) X,$$

with equality if and only if X is the generalized distance Perron vector of G.

Proof of Theorem 1.1. Corresponding to the eigenvalue ∂ of $D_\alpha(G)$, let $X = [x_1, x_2, \dots, x_n]^T$ be the eigenvector, with $x_i = 1$ and $x_t \ge 1$ for all $t \neq i$. Let $x_j = \max\{x_t : \ 1 \le t \le n \ and \ t \neq i\}$. Therefore, from the i-th and j-th equations of $\partial\, X = D_\alpha(G)\, X$, we have

$$\partial\, x_i = \alpha t_i x_i + (1-\alpha) \sum_{t=1, t\neq i}^{n} d_{it} x_t \tag{2}$$

and

$$\partial\, x_j = \alpha t_j x_j + (1-\alpha) \sum_{t=1, t\neq j}^{n} d_{jt} x_t. \tag{3}$$

(2) implies that $\partial \le \alpha t_i + (1-\alpha) t_i x_j$, so that

$$\partial - \alpha t_i \le (1-\alpha) t_i x_j \tag{4}$$

Also, (3) implies that $\partial\, x_j \le \alpha t_j x_j + (1-\alpha) d_{ij} + (1-\alpha)(t_j - d_{ij}) x_j$, which on simplification gives

$$[\partial - t_j + (1-\alpha) d_{ij}]\, x_j \le (1-\alpha) d_{ij}. \tag{5}$$

Combining (4) and (5), we get

$$(\partial - \alpha t_i)[\partial - t_j + (1-\alpha) d_{ij}] x_j \le [(1-\alpha) t_i x_j][(1-\alpha) d_{ij}].$$

Since $x_j > 0$, therefore we have

$$(\partial - \alpha t_i)[\partial - t_j + (1-\alpha) d_{ij}] \le (1-\alpha)^2 t_i d_{ij},$$

which on simplification yields

$$\partial^2 - [\alpha t_i + t_j - (1-\alpha) d_{ij}]\, \partial + \alpha t_i t_j - (1-\alpha) t_i d_{ij} \le 0. \tag{6}$$

The solution of (6) is given by

$$\partial \le \frac{1}{2} \left[\alpha t_i + t_j - (1-\alpha) d_{ij} + \sqrt{[\alpha t_i + t_j - (1-\alpha) d_{ij}]^2 - 4 t_i\, [\alpha t_j + (1-\alpha) d_{ij}]} \right]$$

and

$$\partial \ge \frac{1}{2} \left[\alpha t_i + t_j - (1-\alpha) d_{ij} - \sqrt{[\alpha t_i + t_j - (1-\alpha) d_{ij}]^2 - 4 t_i\, [\alpha t_j + (1-\alpha) d_{ij}]} \right].$$

Therefore, in all cases, it follows that

$$\partial(G) \le \max_{1 \le i,\ j \le n} \frac{1}{2} \left[\alpha t_i + t_j - (1-\alpha) d_{ij} + \sqrt{[\alpha t_i + t_j - (1-\alpha) d_{ij}]^2 - 4 t_i\, [\alpha t_j + (1-\alpha) d_{ij}]} \right].$$

Now, we characterize the graphs for which equality holds in (1). In this regard, first assume that equality holds in (1). Therefore, equality cases in (4) and (5) imply that $x_t = x_j$ for all t, with $t \neq i$. As $x_j \leq x_i = 1$, we have the following two possibilities to consider.

Case 1. $x_j = x_i = 1$. Here, $\partial = t_i$ for all i, $1 \leq i \leq n$. So G is a transmission regular graph.

Case 2. $x_j < x_i = 1$. Therefore, either $d_i = n-1$ or $d_i < n-1$. We look at these two cases separately as follows.

Case 2.1. If $d_i = n-1$, then clearly $v_t \in N(v_i)$ for all those v_t which are in $V(G) - \{v_i\}$. Now, for any $v_t \in V(G) - \{v_i\}$, we have $\partial x_j = (t_t + \alpha - 1)x_j + (1-\alpha)$. This clearly indicates that transmission degree of every vertex $v_t \in V(G) - \{v_i\}$ is equal to t_t, with of course $t_t > n-1$. So, evidently $t_2 = t_3 = \cdots = t_n = t_{min}$. Further, $d_i = n-1$ implies that G is of diameter 2 and thus $t_i = 2n - 2 - d_i$. Therefore, the vertex degrees of G are $n-1$, $d_2 = d_3 = \cdots = d_n = s$ (say) with clearly $n-1 > s$. Hence G is isomorphic to $G(n-1, s)$.

Case 2.2. Now, for the case $d_i < n-1$, we consider the vertex partition of $V(G) - \{v_i\}$ as $U = N(v_i)$ and $W = V(G) - (U \cup \{v_i\})$. Then

$$\partial = t_i\left[\alpha + (1-\alpha)x_j\right] \quad for \quad v_i, \tag{7}$$

$$\partial x_j = \left[t_t - (1-\alpha)\right]x_j + (1-\alpha) \quad for \quad v_t \in N(v_i), \tag{8}$$

$$\partial = t_k \qquad for \qquad v_k \notin \left[N(v_i) \cup \{v_i\}\right]. \tag{9}$$

Simplifying (8) implies that $t_t = \partial + (1-\alpha)(x_j - 1)\frac{1}{x_j}$, for all $v_t \in N(v_i)$. Therefore, transmission degree of every vertex in $N(v_i)$ is equal to t_t, and transmission degree of every vertex outside $N(v_i) \cup \{v_i\}$ is equal to t_i. Since $\alpha + (1-\alpha)x_j \leq 1$, from (7) and (9), we have $t_k > t_i$. In a similar way, $t_k > t_t$ follows from (8) and (9). Thus, $t_i = t_{max}$, and $t_k = t_2$ for all $v_k \in W$. Also, $t_t = t_{min}$ for all vertices in U.

From the arguments given above, we conclude that the given connected graph contains a vertex v_i with transmission degree $t_1 = t_{max}$, transmission degree of every neighbor of v_i is equal to t_2 and transmission degree of every vertex non-adjacent to v_i is equal to t_{min}. This is clearly the graph H defined above.

Conversely, for the regular graph, or the graph isomorphic to $G(n-1, s)$, or graph H, it is easy to verify that equality holds. □

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