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Research Article

Several closed and determinantal forms for convolved Fibonacci numbers*

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Abstract

In this paper, with the aid of the Faà di Bruno formula, some properties of the Bell polynomials of the second kind, and a general derivative formula for a ratio of two differentiable functions, the authors establish several closed and determinantal forms for convolved Fibonacci numbers.

Keywords: Bell polynomial of the second kind; closed form; convolved Fibonacci number; determinantal form; Faà di Bruno formula; Fibonacci number; Hessenberg determinant; tridiagonal determinant.

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1. Background and motivations

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$ subject to initial values $F_0 = 0$ and $F_1 = 1$. All the Fibonacci numbers F_n for $n \ge 1$ can be generated by

$$\frac{1}{1-t-t^2} = \sum_{n=0}^{\infty} F_{n+1} t^n = 1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5 + \cdots \tag{1}$$

Numerous types of generalizations of these numbers have been offered in the literature and they have a plenty of interesting properties and applications; for more information, please refer to [15, 31] and closely related references cited therein.

For $r \in \mathbb{R}$, convolved Fibonacci numbers $F_n(r)$ can be defined by

$$\frac{1}{(1-t-t^2)^r} = \sum_{n=0}^{\infty} F_{n+1}(r) t^n. \tag{2}$$

For detail, see [1, 2, 7, 18, 22] and closely related references cited therein. It is easy to see that $F_n(1) = F_n$ for $n \ge 1$. Convolved Fibonacci numbers $F_n(r)$ for $n \in \mathbb{N}$ and their generalizations have been considered in the papers [12, 14, 16, 18, 22, 34–36] in a variety of context. For instance, Ramirez defined in [34] the convolved $h(x)$-Fibonacci numbers as an extension of classical convolved Fibonacci numbers and offered several combinatorial formulas.

A tridiagonal determinant is a determinant whose nonzero elements locate only on the diagonal and slots horizontally or vertically adjacent the diagonal. In other words, a determinant $H = |h_{ij}|_{n\times n}$ is called a tridiagonal determinant if $h_{ij} = 0$ for all pairs (i, j) such that $|i - j| > 1$; see [11].

A determinant $H = |h_{ij}|_{n\times n}$ is called a lower (or an upper, respectively) Hessenberg determinant if $h_{ij} = 0$ for all pairs (i, j) such that $i + 1 < j$ (or $j + 1 < i$, respectively); see [17].

In analytic combinatorics and analytic number theory, explaining a sequence of numbers or a sequence of polynomials in terms of a special and simple determinant is an interesting and important direction and sometimes produces significant results. Accordingly, some sequences and polynomials, such as the Bernoulli numbers and polynomials [25, 28], the Euler numbers and polynomials [37], the Delannoy numbers [19, 23, 24, 26], the Horadam polynomials [29], and the Fibonacci numbers and polynomial [23, 27], have been considered in this framework. Interested readers can also refer to the studies in [13, 27, 32, 33] and closely related references listed therein.

In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations and may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit. For systematic explanation of closed forms, please refer to the article [3] and closely related references therein.

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In this paper, we find several closed and determinantal forms for convolved Fibonacci numbers $F_{n+1}(r)$ by means of the Faà di Bruno formula, some properties and special values for the Bell polynomials of the second kind, and the generating function method. We also give a determinantal formula for convolved Fibonacci numbers $F_{n+1}(r)$ in terms of tridiagonal determinants by aid of a general derivative formula, which is fundamental but not extensively circulated yet, for the ratio of two differentiable functions.

2. Lemmas

In order to prove our main results, we recall several lemmas below.

Lemma 2.1 ([6, pp. 134 and 139]). *The Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *for* $n\ge k\ge 0$*, is defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n} i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}^{\infty}\frac{n!}{\prod_{i=1}^{\ell-k+1}\ell_i!}\prod_{i=1}^{\ell-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big). \tag{3}$$

Lemma 2.2 ([6, p. 135]). *For* $n\ge k\ge 0$*, we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{4}$$

where a *and* b *are any complex number.*

Lemma 2.3 ([30, Section 1.4]). *For* $n\ge k\ge 0$*, we have*

$$\mathrm{B}_{n,k}(x,1,0,\dots,0)=\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{5}$$

Lemma 2.4 ([25, Lemma 2.4]). *Let* $f(t)=1+\sum_{k=1}^{\infty}a_kt^k$ *and* $g(t)=1+\sum_{k=1}^{\infty}b_kt^k$ *be formal power series such that* $f(t)g(t)=1$*. Then*

$$b_n=(-1)^n\begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0\\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0\\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 & 0\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & 1\\ a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}.$$

Lemma 2.5 ([4, p. 40, Entry 5]). *For two differentiable functions* $p(x)$ *and* $q(x)\ne 0$*, we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^k}{q^{k+1}}\begin{vmatrix} p & q & 0 & \cdots & 0 & 0\\ p' & q' & q & \cdots & 0 & 0\\ p'' & q'' & \binom{2}{1}q' & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & 0 & 0\\ p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1}q^{(k-3)} & \cdots & q & 0\\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \cdots & \binom{k-1}{k-2}q' & q\\ p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \cdots & \binom{k}{k-2}q'' & \binom{k}{k-1}q' \end{vmatrix}. \tag{6}$$

In other words,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^k}{q^{k+1}(x)}\left|U_{(k+1)\times1}(x)\quad V_{(k+1)\times k}(x)\right|,$$

where the matrix $U_{(k+1)\times1}(x)=(u_{\ell,1}(x))$ *has the elements* $u_{\ell,1}(x)=p^{(\ell-1)}(x)$ *for* $1\le\ell\le k+1$ *and the matrix* $V_{(k+1)\times k}(x)$ *has the entries of the form*

$$v_{ij}(x)=\begin{cases}\dbinom{i-1}{j-1}q^{(i-j)}(x), & i-j\ge 0\\ 0, & i-j<0\end{cases}$$

for $1\le i\le k+1$ *and* $1\le j\le k$*.*

3. Closed and determinantal forms for convolved Fibonacci numbers

In this section, we state and prove our main results, several closed and determinantal forms for convolved Fibonacci numbers $F_{k+1}(r)$.

Theorem 3.1. *For $k \ge 0$ and $r \in \mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r) = \sum_{\ell=0}^{k} \binom{\ell}{k-\ell} \frac{(r)_\ell}{\ell!},$$

where

$$(x)_n = \prod_{k=0}^{n-1}(x+k) = \begin{cases} x(x+1)\cdots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

denotes the rising factorial, or say, Pochhammer symbol, of $x \in \mathbb{C}$. Consequently, the Fibonacci numbers F_{k+1} for $k \ge 0$ can be computed by

$$F_{k+1} = \sum_{\ell=0}^{k} \binom{\ell}{k-\ell}.$$

Proof. By virtue of (3), (4), and (5), we have

$$\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{1}{(1-t-t^2)^r}\right] &= \sum_{\ell=0}^{k} \langle -r\rangle_\ell \big(1-t-t^2\big)^{-r-\ell} \mathrm{B}_{k,\ell}(-1-2t,-2,0,\dots,0)\\
&= \sum_{\ell=0}^{k} \langle -r\rangle_\ell \big(1-t-t^2\big)^{-r-\ell} (-2)^\ell \mathrm{B}_{k,\ell}\left(t+\frac{1}{2},1,0,\dots,0\right)\\
&\to \sum_{\ell=0}^{k} \langle -r\rangle_\ell (-2)^\ell \mathrm{B}_{k,\ell}\left(\frac{1}{2},1,0,\dots,0\right), \quad t \to 0\\
&= \sum_{\ell=0}^{k} \langle -r\rangle_\ell (-2)^\ell \frac{(k-\ell)!}{2^{k-\ell}} \binom{k}{\ell} \binom{\ell}{k-\ell} \left(\frac{1}{2}\right)^{2\ell-k}\\
&= k! \sum_{\ell=0}^{k} \langle -r\rangle_\ell (-1)^\ell \frac{1}{\ell!} \binom{\ell}{k-\ell}\\
&= k! \sum_{\ell=0}^{k} (r)_\ell \frac{1}{\ell!} \binom{\ell}{k-\ell},
\end{aligned}$$

where

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

denotes the falling factorial of $x \in \mathbb{C}$. So, it follows from (2) that

$$F_{k+1}(r) = \frac{1}{k!} \lim_{t\to 0} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{1}{(1-t-t^2)^r}\right] = \sum_{l=0}^{k} \frac{(r)_\ell}{\ell!} \binom{\ell}{k-\ell}. \tag{7}$$

The proof of Theorem 3.1 is complete. □

Theorem 3.2. *For $k \ge 0$ and $r \in \mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r) = (-1)^k \begin{vmatrix}
a_1 & 1 & 0 & \cdots & 0 & 0 & 0\\
a_2 & a_1 & 1 & \cdots & 0 & 0 & 0\\
a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & 1 & 0\\
a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_2 & a_1 & 1\\
a_k & a_{k-1} & a_{k-2} & \cdots & a_3 & a_2 & a_1
\end{vmatrix}$$

and satisfy the identity

$$\sum_{\ell=0}^{k}(-1)^{\ell}\frac{\langle r\rangle_\ell}{\ell!}\binom{\ell}{k-\ell}=(-1)^k\begin{vmatrix}F_1(r) & 1 & 0 & \cdots & 0 & 0 & 0\\ F_2(r) & F_1(r) & 1 & \cdots & 0 & 0 & 0\\ F_3(r) & F_2(r) & F_1(r) & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ F_{k-2}(r) & F_{k-3}(r) & F_{k-4}(r) & \cdots & F_1(r) & 1 & 0\\ F_{k-1}(r) & F_{k-2}(r) & F_{k-3}(r) & \cdots & F_2(r) & F_1(r) & 1\\ F_k(r) & F_{k-1}(r) & F_{k-2}(r) & \cdots & F_3(r) & F_2(r) & F_1(r)\end{vmatrix}.$$

where

$$a_k=\sum_{\ell=0}^{k}(-1)^{\ell}\frac{\langle r\rangle_\ell}{\ell!}\binom{\ell}{k-\ell}.$$

Proof. For $|t^2+t|<1$, using the binomial theorem leads to

$$\left(1-t-t^2\right)^r=\sum_{k=0}^{\infty}\langle r\rangle_k\frac{(-1)^k}{k!}(t^2+t)^k=\sum_{k=0}^{\infty}\langle r\rangle_k\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}t^{k+\ell}$$
$$=\sum_{k=0}^{\infty}\sum_{m=k}^{2k}\langle r\rangle_k\frac{(-1)^k}{k!}\binom{k}{m-k}t^m=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\langle r\rangle_k\frac{(-1)^k}{k!}\binom{k}{m-k}t^m=\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\langle r\rangle_\ell\frac{(-1)^\ell}{\ell!}\binom{\ell}{k-\ell}t^k.$$

From the equation (2), it follows that

$$1=\left(1-t-t^2\right)^r\sum_{k=0}^{\infty}F_{k+1}(r)t^k=\left(\sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k}\langle r\rangle_\ell\frac{(-1)^\ell}{\ell!}\binom{\ell}{k-\ell}\right]t^k\right)\sum_{k=0}^{\infty}F_{k+1}(r)t^k.$$

Further regarding coefficients in the above bracket as a_k and $F_{k+1}(r)$ as b_k in Lemma 2.4, interchanging the roles of a_k and b_k, and simplifying arrive at desired expressions. □

Theorem 3.3. *For $k\ge 0$ and $r\in\mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r)=(-1)^k\begin{vmatrix}1 & \gamma_0 & 0 & 0 & \cdots & 0 & 0\\ 0 & \gamma_1 & \gamma_0 & 0 & \cdots & 0 & 0\\ 0 & \gamma_2 & \binom{2}{1}\gamma_1 & \gamma_0 & \cdots & 0 & 0\\ 0 & \gamma_3 & \binom{3}{1}\gamma_2 & \binom{3}{2}\gamma_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \gamma_{k-2} & \binom{k-2}{1}\gamma_{k-3} & \binom{k-2}{2}\gamma_{k-4} & \cdots & \gamma_0 & 0\\ 0 & \gamma_{k-1} & \binom{k-1}{1}\gamma_{k-2} & \binom{k-1}{2}\gamma_{k-3} & \cdots & \binom{k-1}{k-2}\gamma_1 & \gamma_0\\ 0 & \gamma_k & \binom{k}{1}\gamma_{k-1} & \binom{k}{2}\gamma_{k-2} & \cdots & \binom{k}{k-2}\gamma_2 & \binom{k}{k-1}\gamma_1\end{vmatrix},$$

where

$$\gamma_k=\gamma_k(r)=\sum_{\ell=0}^{k}\frac{\langle r\rangle_\ell}{\ell!}\binom{\ell}{k-\ell}.$$

Proof. Proceeding as in the proof of Theorem 3.1 reveals

$$\lim_{t\to0}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\left(1-t-t^2\right)^r\right]=k!\sum_{\ell=0}^{k}\frac{\langle r\rangle_\ell}{\ell!}\binom{\ell}{k-\ell}=k!\gamma_k(r).$$

By the aid of Lemma 2.5 for $p(t)=1$ and $q(t)=\left(1-t-t^2\right)^r$, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{1}{(1-t-t^2)^r}\right]=\frac{(-1)^k}{\left(1-t-t^2\right)^{r(k+1)}}\begin{vmatrix}1 & q(t) & 0 & \cdots & 0 & 0\\ 0 & q'(t) & q(t) & \cdots & 0 & 0\\ 0 & q''(t) & \binom{2}{1}q'(t) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & q^{(k-2)}(t) & \binom{k-2}{1}q^{(k-3)}(t) & \cdots & q(t) & 0\\ 0 & q^{(k-1)}(t) & \binom{k-1}{1}q^{(k-2)}(t) & \cdots & \binom{k-1}{k-2}q'(t) & q(t)\\ 0 & q^{(k)}(t) & \binom{k}{1}q^{(k-1)}(t) & \cdots & \binom{k}{k-2}q''(t) & \binom{k}{k-1}q'(t)\end{vmatrix}$$

$$\to(-1)^k\begin{vmatrix}1 & \gamma_0 & 0 & \cdots & 0 & 0\\ 0 & \gamma_1 & \gamma_0 & \cdots & 0 & 0\\ 0 & \gamma_2 & \binom{2}{1}\gamma_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \gamma_{k-2} & \binom{k-2}{1}\gamma_{k-3} & \cdots & \gamma_0 & 0\\ 0 & \gamma_{k-1} & \binom{k-1}{1}\gamma_{k-2} & \cdots & \binom{k-1}{k-2}\gamma_1 & \gamma_0\\ 0 & \gamma_k & \binom{k}{1}\gamma_{k-1} & \cdots & \binom{k}{k-2}\gamma_2 & \binom{k}{k-1}\gamma_1\end{vmatrix}$$

as $t\to0$. Using the first equality in (7), we find that

$$F_{k+1}(r)=\frac{1}{k!}\lim_{t\to0}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{1}{(1-t-t^2)^r}\right]=(-1)^k\begin{vmatrix}1 & \gamma_0 & 0 & \cdots & 0 & 0\\ 0 & \gamma_1 & \gamma_0 & \cdots & 0 & 0\\ 0 & \gamma_2 & \binom{2}{1}\gamma_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \gamma_{k-2} & \binom{k-2}{1}\gamma_{k-3} & \cdots & \gamma_0 & 0\\ 0 & \gamma_{k-1} & \binom{k-1}{1}\gamma_{k-2} & \cdots & \binom{k-1}{k-2}\gamma_1 & \gamma_0\\ 0 & \gamma_k & \binom{k}{1}\gamma_{k-1} & \cdots & \binom{k}{k-2}\gamma_2 & \binom{k}{k-1}\gamma_1\end{vmatrix}.$$

The proof of Theorem 3.3 is complete. □

Now we supply an alternative and technically different proof of [22, Theorem 1].

Theorem 3.4 ([22, Theorem 1]). *For $k\ge0$ and $r\in\mathbb{R}$, convolved Fibonacci numbers $F_{k+1}(r)$ can be computed by*

$$F_{k+1}(r)=\frac{1}{k!}\frac{1}{\left(\frac{\sqrt5-1}{2}\right)^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\left(\frac{\sqrt5-1}{\sqrt5+1}\right)^\ell(r)_\ell(r)_{k-\ell}. \tag{8}$$

Consequently, the Fibonacci numbers F_{k+1} for $k\ge0$ can be computed by

$$F_{k+1}=\frac{1}{\left(\frac{\sqrt5-1}{2}\right)^k}\sum_{\ell=0}^{k}(-1)^\ell\left(\frac{\sqrt5-1}{\sqrt5+1}\right)^\ell.$$

Proof. It is easy to see that

$$\frac{1}{1-t-t^2}=\frac{1}{\left(t+\frac{\sqrt5+1}{2}\right)\left(\frac{\sqrt5-1}{2}-t\right)}.$$

It is not difficult to see that

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{1}{(t+a)^r(b-t)^r}\right]&=\sum_{\ell=0}^{k}\binom{k}{\ell}\left[\frac{1}{(t+a)^r}\right]^{(\ell)}\left[\frac{1}{(b-t)^r}\right]^{(k-\ell)}\\&=\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{\langle-r\rangle_\ell}{(t+a)^{r+\ell}}\frac{\langle-r\rangle_{k-\ell}(-1)^{k-\ell}}{(b-t)^{r+k-\ell}}\end{aligned}$$

$$= \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{(r)_\ell}{(t+a)^{r+\ell}}\frac{(r)_{k-\ell}}{(b-t)^{r+k-\ell}}$$
$$\to \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{(r)_\ell}{a^{r+\ell}}\frac{(r)_{k-\ell}}{b^{r+k-\ell}}, \quad t\to 0.$$

Accordingly, considering the first equality in (7), taking $a=\frac{\sqrt{5}+1}{2}$ and $b=\frac{\sqrt{5}-1}{2}$, and simplifying yield the formula (8). The proof of Theorem 3.4 is complete. □

4. Remarks

Finally, we give several remarks on something in this paper.

Remark 4.1. *The general formula* (6) *in Lemma 2.5 for derivatives of the ratio of two differentiable functions, which appeared earlier than 1950s, was re-discovered in [9, 10] and closely related references therein in terms of different forms.*

Remark 4.2. *In the paper [1, 2, 7, 18, 22] and two preprints [20, 21] of [22], convolved Fibonacci numbers were denoted by $\mathcal{F}_n(x)$ which are defined by*

$$F(t,x)=\frac{1}{(1-t-t^2)^x}=\sum_{n=0}^{\infty}\mathcal{F}_n(x)\frac{t^n}{n!}, \quad x\in\mathbb{R} \tag{9}$$

Comparing (9) *with* (2) *reveals that $n!F_{n+1}(r)=\mathcal{F}_n(r)$. For coinciding with the Fibonacci numbers F_n generated by* (1)*, we think that the definition and notation $F_{n+1}(r)$ adopted in* (2) *is more convenient and reasonable.*

Remark 4.3. *In the paper [5], the polynomial sequence $\{g_n(x;a,b)\}_{n\ge0}$, defined by*

$$\frac{1}{(1+at+bt^2)^x}=\sum_{n=0}^{\infty}g_n(x;a,b)\frac{t^n}{n!},$$

and its special cases $g_n(x;-x,-1)=n!\mathscr{F}_n(x)$ and $g_n(x;-x,-x^2)=n!\mathbf{F}_n(x)$ were investigated in details.

The kind of polynomials $g_n(x;a,b)$ were also mentioned in [20, Remark 5], [22, Remark 5], and [21, Remark 5] in the forms

$$\frac{1}{(1-sz-z^2)^x}=\sum_{n=0}^{\infty}\mathcal{F}_{n+1}(s;x)z^n$$

and

$$\frac{1}{(1-sz-tz^2)^x}=\sum_{n=0}^{\infty}\mathcal{F}_{n+1}(s,t;x)z^n.$$

In [20, Remark 5], [22, Remark 5], and [21, Remark 5], the quantities $\mathcal{F}_{n+1}(s;x)$ and $\mathcal{F}_{n+1}(s,t;x)$ are called convolved Fibonacci polynomials and convolved generalized Fibonacci polynomials respectively.

Remark 4.4. *This paper has an electronic preprint [8] and can be considered as a companion of the paper [22] and its preprints [20, 21].*

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