On hyper-Hamiltonian Cartesian product of undirected cycles

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(Received: 10 March 2021. Received in revised form: 14 April 2021. Accepted: 26 April 2021. Published online: 28 April 2021.)

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Abstract

A Hamiltonian graph $G$ is hyper-Hamiltonian if every vertex-deleted subgraph $G \setminus \{v\}$ is Hamiltonian for any $v \in V(G)$. We prove that a Cartesian product of undirected cycles is hyper-Hamiltonian if and only if at least one of these cycles is of odd length.

Keywords: hyper-Hamiltonian graph; Hamilton cycle; Cartesian product of cycles; Hamiltonian graph.

2020 Mathematics Subject Classification: 05C38, 05C45.

1. Introduction

A Hamiltonian graph $G$ is hyper-Hamiltonian if every vertex-deleted subgraph $G \setminus \{v\}$ is Hamiltonian, and non-Hamiltonian $G$ is hypo-Hamiltonian if every vertex-deleted subgraph $G \setminus \{v\}$ is Hamiltonian. Hyper-Hamiltonian and hypo-Hamiltonian properties have been studied for special families of graphs [1,3,5–7]. In particular, more focus in recent studies has been on a hyper-Hamiltonian property [1,3,5] since it has high relevance to network survivability in the case of a single node failure (e.g., survivability of a token-ring network, where a token passes through every node of such a network according to a corresponding Hamilton cycle).

In this note we derive the necessary and sufficient conditions for hyper-Hamiltonian property of a well-known special family of graphs called a Cartesian product of undirected cycles. Let $C_n$ denote an undirected cycle of length $n$, where $C_2 \cong K_2$ represents a single edge between two vertices. A Cartesian product $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ of $k$ undirected cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ is the graph such that the vertex set $V(G)$ equals the Cartesian product $V(C_{n_1}) \times V(C_{n_2}) \times \cdots \times V(C_{n_k})$ and there is an edge in $G$ between vertices $u = (u_1, u_2, \ldots, u_k)$ and $v = (v_1, v_2, \ldots, v_k)$ if and only if there exists $1 \leq r \leq k$ such that there is an edge $(u_r, v_r)$ in $C_{n_r}$ and $u_i = v_i$ for all $i \neq r$. Related work for the Cartesian product of two directed cycles has been done by Penn and Witte [6], which focused on hypo-Hamiltonicity instead.

In 1973, Kotzig proved the following:

**Theorem 1.1.** [4] Every Cartesian product of two undirected cycles is decomposable into two Hamilton cycles.

Later, Trotter and Erdős proved the following result for the Cartesian product of two directed cycles.

**Theorem 1.2.** [8] The Cartesian product of two directed cycles $C_{n_1} \square C_{n_2}$ contains a Hamilton cycle if and only if there exist positive integers $t_1$ and $t_2$ such that $\gcd(n_1, n_2) = t_1 + t_2$ and $\gcd(n_1, t_1) = \gcd(n_2, t_2) = 1$.

Subsequently, Curran and Witte extended result of Trotter and Erdős to the Cartesian product of $k$ directed cycles for $k \geq 3$.

**Theorem 1.3.** [2] There is a Hamilton circuit in the Cartesian product of any three or more nontrivial directed cycles.

Clearly, Theorem 1.3 also implies that every Cartesian product of at least three undirected cycles is Hamiltonian. In the main theorem presented in the next section, we will leverage both Theorems 1.1 and 1.3 to prove the necessary and sufficient conditions for $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ to be hyper-Hamiltonian.

2. Main result

We now prove the necessary and sufficient conditions for an arbitrary undirected Cartesian product of cycles to be hyper-Hamiltonian.

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Theorem 2.1. Let \( G = C_{n_1} \square C_{n_2} \cdots \square C_{n_k} \) be a Cartesian product of \( k \geq 2 \) undirected cycles \( C_{n_1}, C_{n_2}, \ldots, C_{n_k} \). \( G \) is hyper-Hamiltonian if and only if \( n_i \) is odd for some \( i \), where \( k \geq i \geq 1 \).

Proof. By Theorems 1.1 and 1.3, \( G \) has a Hamilton cycle. Furthermore, since \( G \) is vertex-transitive, then it suffices to prove that for some arbitrary vertex \( v \in V(G) \), \( G \backslash \{v\} \) is Hamiltonian if and only if \( n_i \) is odd for some \( i, k \geq i \geq 1 \). Let \( V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\} \).

First, consider a necessary condition for \( H = G \backslash \{v_i\} \) to be Hamiltonian. Suppose \( H \) has a Hamilton cycle and every \( n_i \) is even for \( k \geq i \geq 1 \). So, \( |V(G)| \) is even and \( |V(H)| \) is odd. Let \( t \) be the number of cycles \( C_{n_i} \) such that \( n_i = 2 \). Label vertex \( v_1 \) with \( l(v_1) = A \). Label remaining vertices in \( G \) according to the following procedure: while there exists vertex \( v \) such that \( l(v) = A \) (or \( l(v) = B \), respectively) and the number of labeled neighbors to \( v \) is less than \( 2k - t \) then label each unlabeled neighbor \( w \) of \( v \) with \( l(w) = B \) (or with \( l(w) = A \), respectively). Since \( n_i, n_2, \ldots, n_k \) are even then after completing above procedure for every vertex \( v_1 \) labeled with \( l(v_i) = A \) all its neighbors \( v_j \) are labeled with \( l(v_j) = B \), and for every vertex \( v_i \) labeled with \( l(v_i) = B \) all its neighbors \( v_j \) are labeled with \( l(v_j) = A \).

Let \( w_i \in \{v_j | l(v_j) = A \} \) and \( u_i \in \{v_j | l(v_j) = B \} \) for \( \frac{|V(G)|}{2} \leq i \leq 1 \). Furthermore, let \( w_i \neq w_j \) if \( i \neq j \), and let \( u_i \neq u_j \) if \( i \neq j \). Every walk \( W \) of odd length starting from \( w_1 \) is of form \( W = w_1 u_{i_1} w_{i_2} u_{i_3} \cdots u_{i_r} w_{i_1} \) for some natural numbers \( a, b, c, \ldots, x, y \). Since \( |V(H)| \) is odd then Hamilton cycles in \( H \) must be of the same form as \( W \) with \( w_1 = u_y \), implying \( A = l(w_1) = l(u_y) = B \) -- a contradiction. This contradiction proves our necessary condition.

For a sufficient condition, let \( v_1^1 v_2^1 \cdots v_n^1 \) denote \( j \)th consecutive cycle induced by \( C_{n_1} \) in \( G \). Let \( v_i = v_i^1 \). Without loss of generality also assume \( n_1 \) to be odd. Since \( C_{n_2} \square C_{n_3} \square \cdots \square C_{n_k} \) has a Hamilton cycle for \( k \geq 2 \), there is a simple cycle \( C = v_1^1 v_2^1 \cdots v_n^1 \) visiting exactly once all \( n_2 \cdot n_3 \cdots n_k \) cycles formed by \( C_{n_1} \) in \( G \), where \( n_1 \geq i \geq 1 \). Let \( v_i^j \) denote \( j \)th vertex on \( j \)th \( C_{n_1} \) cycle corresponding to \( C \). Then we can identify a Hamilton cycle \( C_H \) in \( H = G \backslash \{v_i^1\} \) in the following six cases:

**Case 1:** \( n_1 = 3 \) and \( r = 2 \).

It implies \( G = C_3 \square C_2 \), so a Hamilton cycle \( C_H \) is identified by:

\[
v_2^1 v_3^1 v_2^2 v_1^1.
\]

**Case 2:** \( n_1 = 3 \) and \( r \geq 3 \).

A Hamilton cycle \( C_H \) is identified by:

\[
v_2^1 v_3^2 \cdots v_n^1 v_1^r v_2^1 v_3^1 v_2^2 v_1^2.
\]

**Case 3:** \( n_1 = 5 \) and \( r = 2 \).

It implies \( G = C_5 \square C_2 \), so a Hamilton cycle \( C_H \) is identified by:

\[
v_2^1 v_3^2 v_2^1 v_3^2 v_2^2 v_1^1 v_2^1 v_3^1 v_2^2.
\]

**Case 4:** \( n_1 = 5 \) and \( r \geq 3 \).

A Hamilton cycle \( C_H \) is identified by:

\[
v_2^1 v_3^2 v_2^1 v_3^2 v_2^2 v_1^3 v_2^1 v_3^1 v_2^2 v_1^2.
\]

**Case 5:** \( n_1 \geq 7 \) and \( r = 2 \).

It implies \( G = C_{n_1} \square C_2 \), so a Hamilton cycle \( C_H \) is identified by:

\[
v_2^1 v_3^2 v_2^2 v_1^4
\]
\[
v_4^5 v_6^2 v_5^2 v_6^2
\]
\[
\ldots
\]
\[
v_{n_1-3} v_{n_1-2} v_{n_1-2} v_{n_1-1}
\]
\[
v_{n_1-4} v_{n_1-1} v_{n_1-1} v_{n_1-2} v_{n_1-2} v_{n_1-2} v_{n_1-1}.
\]

**Case 6:** \( n_1 \geq 7 \) and \( r \geq 3 \).

A Hamilton cycle \( C_H \) is identified by:

\[
v_2^1 v_3^2 v_3^2 v_2^r v_1^3 v_2^4 v_1^2 v_2^2 v_1^2 v_2^2.
\]
Consequently, existence of the Hamilton cycles in above Cases 1-6 for $H = G \setminus \{v_1\}$ proves our sufficient condition.

As a direct consequence of Theorem 2.1, we can state the following:

**Corollary 2.1.** Every Cartesian product $G$ of undirected cycles with $|V(G)|$ odd is hyper-Hamiltonian.

**References**


