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Research Article

On hyper-Hamiltonian Cartesian product of undirected cycles

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Abstract

A Hamiltonian graph $G = (V, E)$ is hyper-Hamiltonian if $G\backslash\{v\}$ is Hamiltonian for any $v \in V(G)$. We prove that a Cartesian product of undirected cycles is hyper-Hamiltonian if and only if at least one of these cycles is of odd length.

Keywords: hyper-Hamiltonian graph; Hamilton cycle; Cartesian product of cycles; Hamiltonian graph.

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1. Introduction

A Hamiltonian graph G is *hyper-Hamiltonian* if every vertex-deleted subgraph $G\backslash\{v\}$ is Hamiltonian, and non-Hamiltonian G is *hypo-Hamiltonian* if every vertex-deleted subgraph $G\backslash\{v\}$ is Hamiltonian. Hyper-Hamiltonian and hypo-Hamiltonian properties have been studied for special families of graphs [1, 3, 5–7]. In particular, more focus in recent studies has been on a hyper-Hamiltonian property [1, 3, 5] since it has high relevance to network survivability in the case of a single node failure (e.g., survivability of a token-ring network, where a token passes through every node of such a network according to a corresponding Hamilton cycle).

In this note we derive the necessary and sufficient conditions for hyper-Hamiltonian property of a well-known special family of graphs called a *Cartesian product of undirected cycles*. Let C_{n_i} denote an undirected cycle of length n_i, where $C_2 \simeq K_2$ represents a single edge between two vertices. A Cartesian product $G = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$ of k undirected cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ is the graph such that the vertex set $V(G)$ equals the Cartesian product $V(C_{n_1}) \times V(C_{n_2}) \times \cdots \times V(C_{n_k})$ and there is an edge in G between vertices $u = (u_1, u_2, \ldots, u_k)$ and $v = (v_1, v_2, \ldots, v_k)$ if and only if there exists $1 \le r \le k$ such that there is an edge (u_r, v_r) in C_{n_r} and $u_i = v_i$ for all $i \ne r$. Related work for the Cartesian product of two directed cycles has been done by Penn and Witte [6], which focused on hypo-Hamiltonicity instead.

In 1973, Kotzig proved the following:

Theorem 1.1. [4] *Every Cartesian product of two undirected cycles is decomposable into two Hamilton cycles.*

Later, Trotter and Erdös proved the following result for the Cartesian product of two directed cycles.

Theorem 1.2. [8] *The Cartesian product of two directed cycles $C_{n_1} \Box C_{n_2}$ contains a Hamilton cycle if and only if there exist positive integers t_1 and t_2 such that* $\gcd(n_1, n_2) = t_1 + t_2$ *and* $\gcd(n_1, t_1) = \gcd(n_2, t_2) = 1$.

Subsequently, Curran and Witte extended result of Trotter and Erdős to the Cartesian product of k directed cycles for $k \ge 3$.

Theorem 1.3. [2] *There is a Hamilton circuit in the Cartesian product of any three or more nontrivial directed cycles.*

Clearly, Theorem 1.3 also implies that every Cartesian product of at least three undirected cycles is Hamiltonian. In the main theorem presented in the next section, we will leverage both Theorems 1.1 and 1.3 to prove the necessary and sufficient conditions for $G = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$ to be hyper-Hamiltonian.

2. Main result

We now prove the necessary and sufficient conditions for an arbitrary undirected Cartesian product of cycles to be hyper-Hamiltonian.

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Theorem 2.1. *Let $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ be a Cartesian product of $k \geq 2$ undirected cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$. G is hyper-Hamiltonian if and only if n_i is odd for some i, where $k \geq i \geq 1$.*

Proof. By Theorems 1.1 and 1.3, G has a Hamilton cycle. Furthermore, since G is vertex-transitive, then it suffices to prove that for some arbitrary vertex $v \in V(G)$, $G \backslash \{v\}$ is Hamiltonian if and only if n_i is odd for some i, $k \geq i \geq 1$. Let $V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\}$.

First, consider a necessary condition for $H = G \backslash \{v_1\}$ to be Hamiltonian. Suppose H has a Hamilton cycle and every n_i is even for $k \geq i \geq 1$. So, $|V(G)|$ is even and $|V(H)|$ is odd. Let t be the number of cycles C_{n_i} such that $n_i = 2$. Label vertex v_1 with $l(v_1) = A$. Label remaining vertices in G according to the following procedure: while there exists vertex v such that $l(v) = A$ (or $l(v) = B$, respectively) and the number of labeled neighbors to v is less than $2k - t$ then label each unlabeled neighbor w of v with $l(w) = B$ (or with $l(w) = A$, respectively). Since $n_1, n_2, \ldots, n_k$ are even then after completing above procedure for every vertex v_i labeled with $l(v_i) = A$ all its neighbors v_j are labeled with $l(v_j) = B$, and for every vertex v_i labeled with $l(v_i) = B$ all its neighbors v_j are labeled with $l(v_j) = A$.

Let $w_i \in \{v_j | l(v_j) = A\}$ and $u_i \in \{v_j | l(v_j) = B\}$ for $\frac{|V(G)|}{2} \geq i \geq 1$. Furthermore, let $w_i \neq w_j$ if $i \neq j$, and let $u_i \neq u_j$ if $i \neq j$. Every walk W of odd length starting from w_1 is of form $W = w_1 u_a w_b u_c \cdots w_x u_y$ for some natural numbers $a, b, c, \ldots, x, y$. Since $|V(H)|$ is odd then Hamilton cycles in H must be of the same form as W with $w_1 = u_y$, implying $A = l(w_1) = l(u_y) = B$ – a contradiction. This contradiction proves our necessary condition.

For a sufficient condition, let $v_1^j v_2^j \cdots v_{n_i}^j v_1^j$ denote jth consecutive cycle induced by C_{n_i} in G. Let $v_1 = v_1^1$. Without loss of generality also assume n_1 to be odd. Since $C_{n_2} \square C_{n_3} \square \cdots \square C_{n_k}$ has a Hamilton cycle for $k \geq 2$, there is a simple cycle $C = v_i^1 v_i^2 \cdots v_i^r v_i^1$ visiting exactly once all $r = n_2 \cdot n_3 \cdots n_k$ cycles formed by C_{n_1} in G, where $n_1 \geq i \geq 1$. Let v_i^j denote ith vertex on jth C_{n_1} cycle corresponding to C. Then we can identify a Hamilton cycle C_H in $H = G \backslash \{v_1^1\}$ in the following six cases:

Case 1: $n_1 = 3$ and $r = 2$.

It implies $G = C_3 \square C_2$, so a Hamilton cycle C_H is identified by:

$$v_2^1 v_3^1 v_3^2 v_1^2 v_2^2 v_2^1.$$

Case 2: $n_1 = 3$ and $r \geq 3$.

A Hamilton cycle C_H is identified by:

$$v_2^1 v_3^1 v_3^2 \cdots v_3^r v_1^r v_1^{r-1} \cdots v_1^2 v_2^2 v_2^3 \cdots v_2^r v_2^1.$$

Case 3: $n_1 = 5$ and $r = 2$.

It implies $G = C_5 \square C_2$, so a Hamilton cycle C_H is identified by:

$$v_2^1 v_3^1 v_3^2 v_4^2 v_4^1 v_5^1 v_5^2 v_1^2 v_2^2 v_2^1.$$

Case 4: $n_1 = 5$ and $r \geq 3$.

A Hamilton cycle C_H is identified by:

$$v_2^1 v_3^1 v_3^2 \cdots v_3^{r-1} \; v_3^r v_4^r v_4^{r-1} \cdots v_4^2$$

$$v_4^1 v_5^1 v_5^2 \cdots v_5^{r-1} \; v_5^r v_1^r v_1^{r-1} \cdots v_1^2$$

$$v_2^2 v_2^3 v_2^4 \cdots v_2^r v_2^1.$$

Case 5: $n_1 \geq 7$ and $r = 2$.

It implies $G = C_{n_1} \square C_2$, so a Hamilton cycle C_H is identified by:

$$v_2^1 v_3^1 v_3^2 v_4^2$$

$$v_4^1 v_5^1 v_5^2 v_6^2$$

$$\cdots$$

$$v_{n_1-3}^1 v_{n_1-2}^1 v_{n_1-2}^2 v_{n_1-1}^2$$

$$v_{n_1-1}^1 v_{n_1}^1 v_{n_1}^2 v_1^2 v_2^2 v_2^1.$$

Case 6: $n_1 \geq 7$ and $r \geq 3$.

A Hamilton cycle C_H is identified by:

$$v_2^1 v_3^1 v_3^2 \cdots v_3^{r-1} \; v_3^r v_4^r v_4^{r-1} \cdots v_4^2$$

$$v_4^1 v_5^1 v_5^2 \cdots v_5^{r-1}\ v_5^r v_6^r v_6^{r-1} \cdots v_6^2$$

$$\cdots$$

$$v_{n_1-3}^1 v_{n_1-2}^1 v_{n_1-2}^2 \cdots v_{n_1-2}^{r-1}\ v_{n_1-2}^r v_{n_1-1}^r v_{n_1-1}^{r-1} \cdots v_{n_1-1}^2$$

$$v_{n_1-1}^1 v_{n_1}^1 v_{n_1}^2 \cdots v_{n_1}^{r-1}\ v_{n_1}^r v_1^r v_1^{r-1} \cdots v_1^2$$

$$v_2^2 v_2^3 v_2^4 \cdots v_2^r v_2^1.$$

Consequently, existence of the Hamilton cycles in above Cases 1-6 for $H = G \backslash \{v_1\}$ proves our sufficient condition. □

As a direct consequence of Theorem 2.1, we can state the following:

Corollary 2.1. *Every Cartesian product G of undirected cycles with $|V(G)|$ odd is hyper-Hamiltonian.*

References

[1] Z. R. Bogdanowicz, Hyper-Hamiltonian circulants, *Electron. J. Graph Theory Appl.* **9** (2021) 185–193.

[2] S. J. Curran, D. Witte, Hamilton paths in Cartesian products of directed cycles, *Ann. Discrete Math.* **27** (1985) 35–74.

[3] R. R. Del-Vecchio, C. T. M. Vinagre and G. B. Pereira, Hyper-Hamiltonicity in graphs: some sufficient conditions, *Electron. Notes Discrete Math.* **62** (2017) 165–170.

[4] A. Kotzig, *Every Cartesian Product of Two Circuits is Decomposable Into Two Hamiltonian Cicrcuts*, Rapport 233, Centre de Recherches Mathematique, Montreal, 1973.

[5] T. Mai, J. Wang, L. Hsu, Hyper-Hamiltonian generalized Petersen graphs, *Comput. Math. Appl.* **55** (2008) 2076–2085.

[6] L. E. Penn, W. Witte, When the Cartesian product of two directed cycles is hypo-Hamiltonian, *J. Graph Theory* **7** (1983) 441–443.

[7] C. Thomassen, Hypohamiltonian graphs and digraphs, In: Y. Alavi, D. R. Lick (Eds.) *Theory and Applications of Graphs*, Lecture Notes in Mathematics, Volume 642, Springer, Berlin, 1976, pp. 557–571.

[8] W. T. Trotter, P. Erdős, When the Cartesian product of directed cycles is Hamiltonian, *J. Graph Theory* **2** (1978) 137–142.