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Review Article

New directions in Ramsey theory*

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Abstract

New developments in the study of Ramsey theory for graphs are described. In particular, it is discussed how Ramsey theory has evolved from classical Ramsey numbers to more general Ramsey numbers, bipartite Ramsey numbers, k-Ramsey numbers, s-bipartite Ramsey numbers, Ramsey sequences of graphs, and ascending Ramsey indices.

Keywords: Ramsey number; bipartite Ramsey number; k-Ramsey number; s-bipartite Ramsey number; Ramsey sequence; monochromatic ascending subgraph sequence; ascending Ramsey index.

2020 Mathematics Subject Classification: 05C35, 05C55.

1. Introduction

During the summer of 1980, Frank Harary was in England where he had arranged to visit Lettice Ramsey, who was perhaps best known for the photography studio Ramsey & Muspratt that she operated with Helen Muspratt. The purpose of Harary's visit with Lettice Ramsey was not photography, however. Mrs. Ramsey had invited Harary to view the numerous papers and files of her late husband Frank Ramsey, who had died some 50 years earlier. Frank Ramsey, who passed away at age 26, had left a large collection of material he had been working on. Ramsey had numerous interests, including philosophy, economics, politics, and mathematics. Indeed, his brother Michael once said about Frank Ramsey: *He was interested in almost everything*. It was Ramsey's mathematical notes in which Frank Harary was primarily interested, however.

In 1930, the year that Ramsey died, a paper of his titled "On a Problem of Formal Logic" was published in the *Proceedings of the London Mathematical Society* [28]. This paper contained a result, a restricted version of which is stated below.

Theorem 1.1. (**Ramsey's Theorem**) *For any $k+1 \geq 3$ positive integers t, n_1, n_2, $\ldots$, n_k, there exists a positive integer N such that if each of the t-element subsets of the set $\{1, 2, \ldots, N\}$ is colored with one of the k colors $1, 2, \ldots, k$, then for some integer i with $1 \leq i \leq k$, there is a subset S of $\{1, 2, \ldots, N\}$ containing n_i elements such that every t-element subset of S is colored i.*

Ramsey's Theorem can be looked at as a theorem in graph theory

(1) by interpreting the set $\{1, 2, \ldots, N\}$ as the vertex set of the complete graph K_N,

(2) by taking $t = 2$, and

(3) by assigning one of the colors $1, 2, \ldots, k$ to each 2-element subset of $\{1, 2, \ldots, N\}$.

Each 2-element subset of $\{1, 2, \ldots, N\}$ can then be considered as an edge of the complete graph K_N. The most studied case of Ramsey's theorem is the one that occurs by taking $k = 2$. In this case, only two colors are involved, usually taken to be red and blue. Here, each edge of K_N is colored either red or blue, resulting in a *red-blue coloring* of K_N. Now, writing s for n_1 and t for n_2, Ramsey's theorem becomes the following result.

Theorem 1.2. (**Ramsey's Theorem**) *For every two positive integers s and t, there exists a positive integer N such that for every red-blue coloring of K_N, there is a complete subgraph K_s all of whose edges are colored red (resulting in a red K_s) or a complete subgraph K_t all of whose edges are colored blue (resulting in a blue K_t).*

Although Ramsey theory relates to many areas of mathematics, as described in the books [19, 23], for example, it is within graph theory that we are interested here.

*To the memory of Frank Harary on the occasion of the 100th anniversary of his birth.

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2. Classical Ramsey numbers

It is a consequence of Ramsey's Theorem 1.2 that for every two positive integers s and t, there is a smallest positive integer n for which every red-blue coloring of K_n results in a red K_s or a blue K_t. This positive integer n is called the *Ramsey number* of K_s and K_t, denoted by $R(K_s, K_t)$ or, often more simply, by $R(s,t)$. There is a familiar question associated with the Ramsey number $R(3,3)$.

> *What is the smallest number of people in a gathering, every two of whom are either friends or strangers, that will guarantee that there are either three mutual friends or three mutual strangers in the gathering?*

The answer to this question is $R(3,3)$ and it turns out that $R(3,3) = 6$. To see that $R(3,3) = 6$ is quite easy, for in any red-blue coloring of the complete graph K_6, every vertex is incident with at least three edges of the same color, say vv_1, vv_2, and vv_3 are three red edges. If any edge joining two vertices of $\{v_1, v_2, v_3\}$ is red, there is a red K_3; otherwise, there is a blue K_3. This says that $R(3,3) \le 6$. Because K_5 can be decomposed into two 5-cycles and one of these can be colored red and the other blue, there is neither a red K_3 nor a blue K_3 and so $R(3,3) \ge 6$. Therefore, $R(3,3) = 6$.

Because $R(3,3) = 6$, it therefore follows that (1) among any six people, every two of whom are friends or strangers, there are three mutual friends or three mutual strangers and (2) there exists some group of five people for which there is neither three mutual friends nor three mutual strangers.

The Ramsey number $R(3,3)$ came up (indirectly) in the 1953 Putnam exam. The William Lowell Putnam mathematical competition for undergraduates, first given in 1938, was designed to stimulate a healthy rivalry in colleges and universities in the United States and Canada. The 1953 exam contained the following problem (suggested by Frank Harary):

Problem A2. *The complete graph with* 6 *points* (*vertices*) *and* 15 *edges has each edge colored red or blue. Show that we can find* 3 *points such that the* 3 *edges joining them are the same color.*

Inspired by this problem, Robert Greenwood and Andrew Gleason [20] not only showed that $R(3,3) = 6$, but showed as well that $R(3,4) = 9$, $R(3,5) = 14$, and $R(4,4) = 18$. In fact, they established an upper bound for the Ramsey numbers $R(s,t)$ for any two positive integers s and t.

Theorem 2.1. [20] *For every two positive integers s and t,*

$$R(s,t) \le \binom{s+t-2}{s-1}.$$

The only known Ramsey numbers $R(s,t)$ with $3 \le s \le t$ are those stated in the following table.

t	3	4	5	6	7	8	9
$R(3,t)$	6	9	14	18	23	28	36

t	4	5
$R(4,t)$	18	25

t	5
$R(5,t)$	?

In particular, the Ramsey number $R(5,5)$ is not known. All that is known about this Ramsey number is $43 \le R(5,5) \le 48$ (see [4,25]). This means that every red-blue coloring of K_{48} results in a red K_5 or a blue K_5 (that is, a monochromatic K_5) and there is a red-blue coloring of K_{42} for which there is neither a red K_5 nor a blue K_5.

The Ramsey numbers $R(s,t) = R(K_s, K_t)$ have become known as the *classical Ramsey numbers*. Classical Ramsey numbers are not limited to two positive integers (and two colors), however. For example, for positive integers r, s, and t, the Ramsey number $R(r,s,t)$ is the smallest positive integer n for which every red-blue-green coloring of K_n results in either a red K_r, a blue K_s, or a green K_t. As expected, very few such Ramsey numbers $R(r,s,t)$ have been determined for $3 \le r \le s \le t$. In fact, only two of these numbers have been found. In 2016, Michael Codish, Michael Frank, Avraham Itzhakov, and Alice Miller [13] showed that $R(3,3,4) = 30$. The Ramsey number $R(3,3,3)$ was determined by Greenwood and Gleason in their 1955 paper [20]. The proof we present, however, is due to Sun and Cohen [31].

Theorem 2.2. [20] $R(3,3,3) = 17$.

Proof. First, we show that $R(3,3,3) \le 17$. Let there be given a red-blue-green coloring of K_{17}. Since the degree of every vertex of K_{17} is 16, every vertex is incident with at least six edges of the same color, say the vertex v of K_{17} is incident with six green edges $vv_1, vv_2, \ldots, vv_6$. If any two vertices in the set $S = \{v_1, v_2, \ldots, v_6\}$ are joined by a green edge, then K_{17} has a green K_3. On the other hand, if no edge joining two vertices of S is colored green, then every such edge is colored red or blue. Since $R(3,3) = 6$, there is a red K_3 or a blue K_3. Consequently, $R(3,3,3) \le 17$.

Next, we show that $R(3,3,3) \geq 17$. Consider the complete graph $G = K_{16}$ whose 16 vertices are labeled with the 16 elements of the additive group $\mathbb{Z}_2^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, every element of which is self-inverse. The 15 non-identity elements of $\mathbb{Z}_2^4$ are partitioned into the following three sets

$$\begin{aligned} V_r &= \{0001, 0010, 0111, 1011, 1111\} \\ V_b &= \{0100, 0110, 0101, 1010, 1101\} \\ V_g &= \{1000, 0011, 1001, 1100, 1110\}, \end{aligned}$$

where r, b, g represent the colors red, blue, green, respectively. These three sets have the property that the sum of any two distinct elements of V_x, $x \in \{r, b, g\}$, do not belong to V_x. Now, an edge uv of G is colored x if $u + v \in V_x$. Clearly, there is no monochromatic K_3 in G containing 0000. Suppose that there are three distinct non-identity vertices u, v, w that form a monochromatic K_3 in G. Then $u + v, u + w, v + w \in V_x$ for some $x \in \{r, b, g\}$. Since $u + w, v + w \in V_x$, it follows that $(u + w) + (v + w) = u + v \notin V_x$, a contradiction. Hence, this red-blue-green coloring of K_{16} contains no monochromatic K_3 and so $R(3,3,3) \geq 17$. Therefore, $R(3,3,3) = 17$. □

3. More general Ramsey numbers

Later, Ramsey numbers more general than the classical Ramsey numbers began to be investigated. For two graphs F and H, not necessarily complete, the *Ramsey number* $R(F, H)$ is defined as the minimum positive integer n for which every red-blue coloring of K_n results in either a subgraph isomorphic to F, all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H, all of whose edges are colored blue (a *blue* H). Frank Harary always liked this notation! All such Ramsey numbers $R(F, H)$ exist as well, for if F has order s and H has order t, then $R(F, H) \leq R(s, t)$. The dynamic survey "Small Ramsey numbers" by Stanislaw Radziszowski [27] provides a host of information on such Ramsey numbers.

While determining $R(F, H)$ is challenging in most instances, in a paper appearing in the first volume of the *Journal of Graph Theory* (founded by Frank Harary), Vašek Chvátal [12] found the exact value of $R(F, H)$ whenever F is any tree of a fixed order and H is the complete graph of a fixed order. For the proof of this result, we first present the following known lemma (see [10], for example).

Lemma 3.1. *Let T be a tree of order k. If G is a graph for which $\delta(G) \geq k - 1$, then G contains a subgraph isomorphic to T.*

Proof. We proceed by induction. The result is obvious for $k = 1, 2, 3$. Assume for an integer k where $k \geq 4$ that for every tree T' of order $k - 1$ and every graph G' with $\delta(G') \geq k - 2$ that G' contains a subgraph isomorphic to T'. Now, let T be a tree of order k and let G be a graph with $\delta(G) \geq k - 1$. Let v be an end-vertex of T where u is the neighbor of v. Then $T - v$ is a tree of order $k - 1$. Let x be a vertex of G and let $H = G - x$. Since $\delta(G) \geq k - 1$, we have $\delta(H) \geq k - 2$. By the induction hypothesis, H contains a subgraph T' isomorphic to $T - v$. Let u' be the vertex of T' corresponding to u in $T - v$ in an isomorphism. Since $\deg_G u' \geq k - 1$, the vertex u' is adjacent to a vertex $y \in V(G) - V(T')$. Adding the vertex y and the edge xy to T' produces a tree in G isomorphic to T. □

Theorem 3.1. [12] *Let T be a tree of order $p \geq 2$. For every integer $n \geq 2$,*

$$R(T, K_n) = (p-1)(n-1) + 1.$$

Proof. First, we show that $R(T, K_n) \geq (p-1)(n-1) + 1$. Let there be given a red-blue coloring of the complete graph $K_{(p-1)(n-1)}$ such that the resulting red subgraph is $(n-1)K_{p-1}$; that is, the red subgraph consists of $n - 1$ copies of K_{p-1}. Since each component of the red subgraph has order $p - 1$, it contains no connected subgraph of order greater than $p - 1$. In particular, there is no red tree of order p. The blue subgraph is then the complete $(n-1)$-partite graph $K_{p-1,p-1,\dots,p-1}$, where every partite set contains exactly $p - 1$ vertices. Hence, there is no blue K_n either. Since this red-blue coloring avoids both a red tree T and a blue K_n, it follows that $R(T, K_n) \geq (p-1)(n-1) + 1$.

We now show that $R(T, K_n) \leq (p-1)(n-1) + 1$ for an arbitrary but fixed tree T of order $p \geq 2$ and an integer $n \geq 2$. We verify this inequality by induction on n. For $n = 2$, we show that $R(T, K_2) \leq (p-1)(2-1) + 1 = p$. Let there be given a red-blue coloring of K_p. If any edge of K_p is colored blue, then a blue K_2 is produced. Otherwise, every edge of K_p is colored red and a red T is produced. Thus, $R(T, K_2) \leq p$. Therefore, the inequality $R(T, K_n) \leq (p-1)(n-1)+1$ holds when $n = 2$. Assume for an integer $k \geq 2$ that $R(T, K_k) \leq (p-1)(k-1)+1$. Consequently, every red-blue coloring of $K_{(p-1)(k-1)+1}$ contains either a red T or a blue K_k. We now show that $R(T, K_{k+1}) \leq (p-1)k + 1$. Let there be given a red-blue coloring of $K_{(p-1)k+1}$. We show that there is either a red tree T or a blue K_{k+1}. We consider two cases.

Case 1. *There exists a vertex v in $K_{(p-1)k+1}$ that is incident with at least $(p-1)(k-1)+1$ blue edges.* Suppose that vv_i is a blue edge for $1 \leq i \leq (p-1)(k-1)+1$. Consider the subgraph H induced by the set $\{v_i : 1 \leq i \leq (p-1)(k-1)+1\}$. Thus,

$H = K_{(p-1)(k-1)+1}$. By the induction hypothesis, H contains either a red T or a blue K_k. If H contains a red T, so does $K_{(p-1)k+1}$. On the other hand, if H contains a blue K_k, then, since v is joined to every vertex of H by a blue edge, there is a blue K_{k+1} in $K_{(p-1)k+1}$.

Case 2. *Every vertex of* $K_{(p-1)k+1}$ *is incident with at most* $(p-1)(k-1)$ *blue edges*. So, every vertex of $K_{(p-1)k+1}$ is incident with at least $p-1$ red edges. Thus, the red subgraph of $K_{(p-1)k+1}$ has minimum degree at least $p-1$. By Lemma 3.1, this red subgraph contains a red T. Therefore, $K_{(p-1)k+1}$ contains a red T as well. □

Since $R(3,3) = R(K_3, K_3) = 6$, it follows that $R(C_3, C_3) = 6$, that is, the complete graphs K_3 could be looked at in terms of 3-cycles. This suggests investigating the Ramsey numbers of other cycles, say $R(C_4, C_4)$ for example. As we mentioned earlier, the complete graph K_5 can be decomposed into two 5-cycles. Once again, coloring one 5-cycle red and the other blue shows that there is a red-blue coloring of K_5 where there is neither a red C_4 nor a blue C_4 and so $R(C_4, C_4) \geq 6$. We now show that $R(C_4, C_4) \leq 6$ (and so $R(C_4, C_4) = 6$).

Let there be given a red-blue coloring of K_6. Since $R(3,3) = 6$, there is either a red C_3 or a blue C_3, say the former. Let C be a red C_3 in K_6 with vertices u, v, w and let x, y, z be the remaining three vertices of K_6. If any of x, y, z is joined to C by two red edges, then a red C_4 is produced. Thus, we may assume that each of x, y, z is joined to C by at least two blue edges. If any two of x, y, z are joined to the same two vertices of C by blue edges, then a blue C_4 is produced. Thus, we may further assume that the red-blue coloring of K_6 contains the red-blue subgraph shown in Figure 1, where bold edges represent red edges and dashed edges are blue edges.

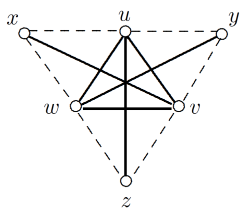


Figure 1: A red-blue subgraph in K_6.

If any of the uncolored edges xy, xz, yz is red, then a red C_4 is produced; if any two of these edges are blue, then a blue C_4 is produced. Hence, in any case, a monochromatic C_4 is produced, which says that $R(C_4, C_4) \leq 6$. Consequently, $R(C_4, C_4) = 6$.

The Ramsey number $R(F, H)$ has been determined when F and H are both paths and when F and H are both cycles. In the case of two paths, the Ramsey number was determined by Gerencsér and Gyárfás [17].

Theorem 3.2. [17] *For integers r and s with $2 \leq r \leq s$, $R(P_r, P_s) = s + \left\lfloor \frac{r}{2} \right\rfloor - 1$.*

When F and H are both cycles, the Ramsey number was determined by Faudree and Schelp [16] and, independently, by Károlyi and Rosta (see [24, 29, 30]). When one of F and H is a path and the other a cycle, the Ramsey number was determined by Faudree et al. in [15].

Theorem 3.3. [16, 24, 29, 30] *Let p and q be integers with $3 \leq p \leq q$.*

(i) $R(C_3, C_3) = R(C_4, C_4) = 6$;

(ii) *If p is odd and $(p, q) \neq (3,3)$, then* $R(C_p, C_q) = 2q - 1$;

(iii) *If p and q are even and $(p, q) \neq (4,4)$, then* $R(C_p, C_q) = q + \frac{p}{2} - 1$;

(iv) *If p is even and q is odd, then* $R(C_p, C_q) = \max\{q + \frac{p}{2} - 1, 2p - 1\}$.

Theorem 3.4. [15] *Let m and n be integers with $m, n \geq 2$.*

$$R(P_n, C_m) = \begin{cases} 2n-1 & \text{if } 3 \leq m \leq n \text{ and } m \text{ is odd} \\ n - 1 + \frac{m}{2} & \text{if } 4 \leq m \leq n \text{ and } m \text{ is even} \\ \max\left\{m - 1 + \left\lfloor \frac{n}{2} \right\rfloor, 2n-1\right\} & \text{if } 2 \leq n \leq m \text{ and } m \text{ is odd} \\ m - 1 + \left\lfloor \frac{n}{2} \right\rfloor & \text{if } 2 \leq n \leq m \text{ and } m \text{ is even.} \end{cases}$$

4. Bipartite Ramsey numbers

In 1975 Lowell Beineke and Allen Schwenk [5] introduced a new class of Ramsey numbers by considering for a pair F, H of graphs, a red-blue coloring of the regular complete bipartite graph $K_{r,r}$ rather than the complete graph K_n, with the goal of obtaining either a red F or a blue H in $K_{r,r}$. Of course, since $K_{r,r}$ is a bipartite graph, the only graphs F and H for which this is possible are bipartite graphs. Precisely, for two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F, H)$ of F and H is the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. We saw in the preceding section that the standard Ramsey number $R(C_4, C_4)$ is 6. Since $C_4 = K_{2,2}$ is bipartite, it is reasonable to consider the bipartite Ramsey number $BR(C_4, C_4)$. In particular, we show that $BR(C_4, C_4) = 5$.

Example 4.1. $BR(C_4, C_4) = 5$.

Proof. Since the graph $K_{4,4}$ can be decomposed into two copies of C_8 with one copy colored red and the other colored blue (see Figure 2), there is a red-blue coloring of $K_{4,4}$ that avoids a monochromatic C_4. Therefore, $BR(C_4, C_4) \geq 5$.

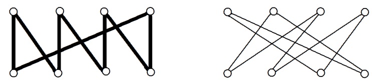


Figure 2: A red-blue coloring of $K_{4,4}$ in Example 4.1.

It remains to show that $BR(C_4, C_4) \leq 5$. Let there be given a red-blue coloring of $K_{5,5}$ whose partite sets are denoted by U and W. The partite set U, for example, contains three vertices, each incident with three or more edges of the same color, say red edges. Then two of these vertices are joined to two vertices of W by red edges, producing a red C_4. Thus, $BR(C_4, C_4) \leq 5$ and so $BR(C_4, C_4) = 5$. □

While $BR(C_4, C_4) = 5$, the primary question here is that of determining those bipartite graphs F and H for which $BR(F, H)$ exists. It turns out in fact that $BR(F, H)$ exists for every pair F, H of bipartite graphs. Although this was stated in [5], an upper bound for $BR(F, H)$ was obtained by Johann Hattingh and Michael Henning [21], thereby establishing this existence result. In order to state this bound, it is useful to introduce some additional terminology.

For positive integers s and t, bipartite Ramsey numbers of the type $BR(K_{s,s}, K_{t,t})$ are referred to as *classical bipartite Ramsey numbers*. These numbers are also denoted by $BR(s, t)$. Showing that $BR(s, t)$ exists for every pair s, t of positive integers shows that $BR(F, H)$ exists for every pair F, H of bipartite graphs. To see this, let F and H be two bipartite graphs, where the largest partite set of F has s vertices and the largest partite set of H has t vertices. Then $BR(F, H) \leq BR(s, t)$. The following result of Hattingh and Henning [21] is analogous to Theorem 2.1.

Theorem 4.1. [21] *For every two positive integers s and t,*

$$BR(s, t) \leq \binom{s+t}{s} - 1.$$

According to the bound for $BR(s, t)$ given in Theorem 4.1, $BR(K_{2,2}, K_{2,2}) = BR(2, 2) \leq \binom{4}{2} - 1 = 5$. However, this is $BR(C_4, C_4) \leq 5$. We have already seen in Example 4.1 that $BR(C_4, C_4) = 5$, so this bound is attained when $s = t = 2$. It was shown in [5] that $BR(2, 3) = 9$ and $BR(2, 4) = 14$, so this bound is also attained when $s = 2$ and $t \in \{3, 4\}$. However, it was also shown in [5] that $BR(3, 3) = 17$ and so the bound is not attained when $s = t = 3$. This last bipartite Ramsey number then gives the answer to the question asked in the following "party problem":

> *Suppose, for some positive integer r, that r girls and r boys are invited to a party where each girl-boy pair are either acquainted or are strangers. What is the smallest such r that guarantees that there exists a group of six people, three girls and three boys, such that either* (1) *every one of the three girls is acquainted with every one of the three boys or* (2) *every one of the three girls is a stranger of every one of the three boys?*

Bipartite Ramsey numbers can be defined for more than two bipartite graphs. For example, $BR(s, t, p)$ is the smallest positive integer r for which any red-blue-green coloring of $K_{r,r}$ results in either a red $K_{s,s}$, a blue $K_{t,t}$, or a green $K_{p,p}$. That these numbers exist (as well as for any prescribed number k of bipartite graphs and colors) is a consequence of a theorem of Paul Erdős and Richard Rado [14]. The only nontrivial bipartite Ramsey number that has been determined for $k \geq 3$ is $BR(2, 2, 2) = BR(K_{2,2}, K_{2,2}, K_{2,2}) = 11$, a result due to Goddard, Henning, and Oellermann [18].

5. k-Ramsey numbers

If, for two bipartite graphs F and H, we have $BR(F, H) = r$, then it follows that for every red-blue coloring of $K_{r,r}$, there is either a red F or a blue H; while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. This brings up the question of which situation can occur for the graph $K_{r-1,r}$. This question led to a concept introduced in [3].

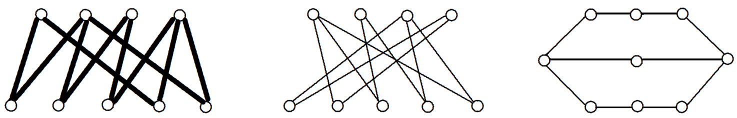
For bipartite graphs F and H, the 2-*Ramsey number* $R_2(F, H)$ is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ of order n results in a red F or a blue H. In particular, if $BR(F, H) = r$ and there is a red-blue coloring of $K_{r-1,r}$ that avoids both a red F and a blue H, then $R_2(F, H) = 2r$; while if every red-blue coloring of $K_{r-1,r}$ produces either a red F or a blue H, then $R_2(F, H) = 2r - 1$. In general then, either

$$R_2(F, H) = 2BR(F, H) \text{ or } R_2(F, H) = 2BR(F, H) - 1. \quad (1)$$

We saw in Example 4.1 that $BR(C_4, C_4) = 5$. This implies that either $R_2(C_4, C_4) = 10$ or $R_2(C_4, C_4) = 9$.

Example 5.1. $R_2(C_4, C_4) = 10$.

Proof. Let $K_{2,3}$ be the complete bipartite graph where u_1, u_2, u_3 are the three vertices of degree 2 and let H be the graph obtained from $K_{2,3}$ by subdividing each edge incident with u_1 or u_2 exactly once. The graph H is shown Figure 3(c). Then H does not contain C_4 as a subgraph. Since $K_{4,5}$ can be decomposed into two copies of H, with one copy colored red (shown in Figure 3(a)) and the other copy colored blue (shown in Figure 3(b)), it follows that $R_2(C_4, C_4) \neq 9$. Therefore, $R_2(C_4, C_4) = 10$ by (1). □



(a) (b) (c)

Figure 3: A red-blue coloring of $K_{4,5}$ in Example 5.1.

There is a concept even more general than the 2-Ramsey number of bipartite graphs. For an integer $k \geq 2$, a *balanced complete k-partite graph* of order $n \geq k$ is the complete k-partite graph in which every partite set has either $\lfloor n/k\rfloor$ or $\lceil n/k\rceil$ vertices. In particular, if $n = kq + r$ (by the Division Algorithm) where $q \geq 1$ and $0 \leq r \leq k - 1$, then the balanced complete k-partite graph G of order n has r partite sets with $q + 1$ vertices and $k - r$ partite sets with q vertices. If $r = 0$, then G is a $(k - 1)q$-regular graph.

For bipartite graphs F and H and an integer k with $2 \leq k \leq R(F, H)$, the *k-Ramsey number* $R_k(F, H)$ is defined as the smallest positive integer n such that every red-blue coloring of a balanced complete k-partite graph of order n results in a red F or a blue H. That is, $R_k(F, H)$ is the minimum order of a balanced complete k-partite graph every red-blue coloring of which results in a red F or a blue H.

If F and H are two bipartite graphs for which $R(F, H) = n \geq 3$, then every red-blue coloring of K_n produces either a red F or a blue H. However, such is not the case for the smaller complete graphs $K_2, K_3, \ldots, K_{n-1}$. Equivalently, for every red-blue coloring of the complete n-partite graph K_n where each partite set consists of a single vertex, there is either a red F or a blue H. However, for each complete k-partite graph K_k, where $2 \leq k \leq n - 1$ such that every partite set consists of a single vertex, there exists a red-blue coloring that produces neither a red F nor a blue H. On the other hand, for each of the graphs $K_2, K_3, \ldots, K_{n-1}$, we can continue to add vertices to each partite set, resulting in a balanced complete k-partite graph at each step where $2 \leq k \leq n - 1$ until eventually arriving at the balanced complete k-partite graph of smallest order $R_k(F, H)$ having the property that every red-blue coloring of this graph produces a red F or a blue H. Consequently, for every two bipartite graphs F and H and every integer k with $2 \leq k \leq R(F, H)$, the k-Ramsey number $R_k(F, H)$ exists. Furthermore, if $R(F, H) = n$, then

$$R_n(F, H) \leq R_{n-1}(F, H) \leq \cdots \leq R_3(F, H) \leq R_2(F, H).$$

In Example 5.1, we saw that $R_2(C_4, C_4) = 10$. The following result was obtained in [3].

Theorem 5.1. [3] *For every integer k with $2 \leq k \leq 6$, $R_k(C_4, C_4) = 12 - k$.*

While the k-Ramsey number $R_k(F, H)$ exists for every two bipartite graphs F and H when $2 \leq k \leq R(F, H)$, such is not the case when F and H are not bipartite. For graphs F and H that are not bipartite, not only does $R_2(F, H)$ fail to exist but $R_3(F,H)$ and $R_4(F,H)$ also do not exist. To see this, let G be any balanced complete 3-partite graph with partite sets V_1, V_2, and V_3. Assigning the color red to every edge of $[V_1, V_2]$, the set of edges joining a vertex of V_1 and a vertex of V_2, and blue to all other edges of G results in a red subgraph G_R and a blue subgraph G_B both being bipartite. Similarly, if G is a balanced complete 4-partite graph with partite sets V_1, V_2, V_3, and V_4, where the color red is assigned to every edge of $[V_1,V_2] \cup [V_2,V_3] \cup [V_3,V_4]$ and the color blue to all other edges of G, then G_R and G_B are both bipartite. Indeed, even if $\chi(F) = \chi(H) = 3$, $R_5(F,H)$ need not exist. For example, $R_5(K_3,K_3)$ does not exist. To see this, let G be a balanced complete 5-partite graph with partite sets V_i for $1 \le i \le 5$. If the edges in $[V_1,V_2] \cup [V_2,V_3] \cup [V_3,V_4] \cup [V_4,V_5] \cup [V_5,V_1]$ are colored red and all other edges are colored blue, then G does not contain a monochromatic K_3. Consequently, $R_k(K_3,K_3)$ exists only when $k = R(K_3,K_3) = 6$. Even if a red-blue coloring of K_5 does not contain a monochromatic K_3, there is another monochromatic graph it must contain.

Observation 5.2. *Every red-blue coloring of K_5 produces either a monochromatic C_3 or a monochromatic C_5.*

From our preceding discussion, the k-Ramsey number of two odd cycles does not exist when $k = 2, 3, 4$. Furthermore, the 5-Ramsey number of two triangles does not exist. However, if neither of the two odd cycles is a triangle, then the situation is different. The following was shown in [2].

Theorem 5.3. [2] *For every pair k, ℓ of integers with $k, \ell \ge 2$, $R_5(C_{2\ell+1}, C_{2k+1})$ exists.*

We have seen that Ramsey numbers are defined for three or more graphs. In particular, for three graphs F_1, F_2, and F_3, the *Ramsey number* $R(F_1,F_2,F_3)$ is the smallest positive integer n for which every red-blue-green coloring (in which every edge is colored red, blue, or green) of the complete graph K_n results in a red F_1, a blue F_2, or a green F_3. This gives rise to the concept of the k-Ramsey number of three (or more) graphs. For three graphs F_1, F_2, and F_3 and an integer k with $2 \le k \le R(F_1,F_2,F_3)$, the k-Ramsey number $R_k(F_1,F_2,F_3)$, if it exists, is the smallest order of a balanced complete k-partite graph G for which every red-blue-green coloring of G results in a red F_1, a blue F_2, or a green F_3. In particular, if $k = 2$ and $F_i \cong F$ for some graph F, where $i = 1, 2, 3$, then the 2-Ramsey number $R_2(F,F,F)$ is the smallest order of a balanced complete bipartite graph G for which every red-blue-green coloring of G results in a monochromatic F. For example, we mentioned that it was shown in [18] that $BR(C_4,C_4,C_4) = 11$, implying that $R_2(C_4,C_4,C_4) \ge 21$. Furthermore, it was shown in [22] that $R_2(C_4,C_4,C_4) \le 21$. Consequently, $R_2(C_4,C_4,C_4) = 21$.

6. s-Bipartite Ramsey numbers

In the two preceding sections, we have seen that $BR(C_4,C_4) = 5$ and $R_2(C_4,C_4) = 10$. From this, it follows that every red-blue coloring of $K_{5,5}$ results in a monochromatic C_4, while there exists a red-blue coloring of $K_{4,5}$ that avoids a monochromatic C_4. This brings up another question. Does every red-blue coloring of $K_{4,6}$ produce a monochromatic C_4 or is there some red-blue coloring of $K_{4,6}$ that avoids a monochromatic C_4? This can be answered by observing that (1) the graph H in Figure 4 does not contain a 4-cycle and (2) the graph $K_{4,6}$ can be decomposed into two copies of H (or $K_{4,6}$ is *H-decomposable*). Thus, by coloring one copy of H red and the other blue, a red-blue coloring of $K_{4,6}$ is produced that avoids a monochromatic C_4.

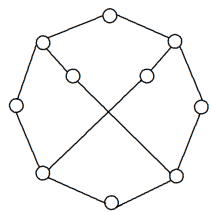


Figure 4: A graph H for which $K_{4,6}$ is H-decomposable.

This changes with the graph $K_{4,7}$, however. In fact, not only does every red-blue coloring of $K_{4,7}$ contain a monochromatic C_4, every red-blue coloring of $K_{3,7}$ contains a monochromatic C_4. To see this, let there be given a red-blue coloring of $G = K_{3,7}$ resulting in a red subgraph G_R and a blue subgraph G_B, the sizes of which are denoted by m_R and m_B, respectively. Since the size of G is 21, one of m_R and m_B is at least 11, say $m_R \ge 11$. Let U and W be the partite sets of G, where $|U| = 3$ and $|W| = 7$. If U contains vertices u_1 and u_2 such that $\deg_{G_R} u_1 + \deg_{G_R} u_2 \ge 9$, then u_1 and u_2 have two common neighbors in G_R and so G_R contains a 4-cycle. Otherwise, the degrees of the three vertices of U in G_R are either $5,3,3$, or $4,4,4$, or $4,4,3$. In any of these three cases, two vertices of U have two common neighbors in G_R, resulting in a 4-cycle in G_R.

These observations resulted in a concept introduced in [7]. For two bipartite graphs F and H and a positive integer s, the s-*bipartite Ramsey number* $BR_s(F, H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. From our discussion above, we have the following result. Here, we write $BR_s(K_{p,p}, K_{q,q})$ as $BR_s(p, q)$.

Theorem 6.1. [7] *For each integer* $s \geq 2$*,*

$$BR_s(2,2) = \begin{cases} \textit{does not exist} & \textit{if } s = 2 \\ 7 & \textit{if } s = 3,4 \\ s & \textit{if } s \geq 5. \end{cases}$$

Proof. First, let $t \geq 2$ be an integer and let $G = K_{2,t}$, where $\{u_1, u_2\}$ is one of the partite sets of G. If each edge of G incident with u_1 is colored red and each edge incident with u_2 is colored blue, then there is no monochromatic $K_{2,2}$. Thus, $BR_2(2,2)$ does not exist.

We have seen that every red-blue coloring of $K_{3,7}$ has a monochromatic $C_4 = K_{2,2}$ and there exists a red-blue coloring of $K_{4,6}$ that avoids a monochromatic $K_{2,2}$. Therefore, $BR_3(2,2) = BR_4(2,2) = 7$.

Since $BR(2,2) = 5$, it follows that $BR_s(2,2) = s$ for each integer $s \geq 5$. □

Not only has $BR_s(K_{2,2}, H)$ been determined for $H = K_{2,2}$ and $s \geq 2$, it has also been determined when $H = K_{2,3}$ or when $H = K_{3,3}$ (see [6, 7]).

Theorem 6.2. [6, 7] *For each integer* $s \geq 2$*,*

$$BR_s(K_{2,2}, K_{2,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2 \\ 10 & \textit{if } s = 3 \\ 8 & \textit{if } 4 \leq s \leq 7 \\ s & \textit{if } s \geq 8. \end{cases}$$

Theorem 6.3. [6, 7] *For each integer* $s \geq 2$*,*

$$BR_s(2,3) = BR_s(K_{2,2}, K_{3,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2,3 \\ 15 & \textit{if } s = 4 \\ 12 & \textit{if } s = 5,6 \\ 9 & \textit{if } s = 7,8 \\ s & \textit{if } s \geq 9. \end{cases}$$

While $BR_s(F, H)$ has been determined when $F = K_{2,3}$ and $H \in \{K_{2,3}, K_{3,3}\}$ for each $s \geq 2$, there are only partial results obtained when $F = H = K_{3,3}$ (see [6–9, 32]).

Theorem 6.4. [9] *For each integer* $s \geq 2$*,*

$$BR_s(K_{2,3}, K_{2,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2 \\ 13 & \textit{if } s = 3,4 \\ 11 & \textit{if } s = 5,6 \\ 9 & \textit{if } s = 7,8 \\ s & \textit{if } s \geq 9. \end{cases}$$

Theorem 6.5. [8, 32] *For each integer* $s \geq 2$*,*

$$BR_s(K_{2,3}, K_{3,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2,3 \\ 21 & \textit{if } s = 4,5 \\ 15 & \textit{if } s = 6,7 \\ 13 & \textit{if } s = 8,9 \\ 12 & \textit{if } s = 10,11 \\ s & \textit{if } s \geq 12. \end{cases}$$

Theorem 6.6. [6, 7] *For each integer* $s \geq 2$*,*

$$BR_s(3,3) = \begin{cases} \textit{does not exist} & \textit{if } s = 2,3,4 \\ 41 & \textit{if } s = 5,6 \\ 29 & \textit{if } s = 7,8. \end{cases}$$

The concept of s-bipartite Ramsey number is also related to recreational problems, an example of which is the following.

> *There are five girls at a party. What is the minimum number of boys who must be invited to the party to guarantee that there exists a group of six people, three girls and three boys, such that either* (1) *every one of the three girls is acquainted with every one of the three boys or* (2) *every one of the three girls is a stranger of every one of the three boys?*

By Theorem 6.6, the answer to this question is $BR_5(3,3) = 41$.

7. Ramsey sequences

The establishment of the existence of the classical Ramsey numbers $R(s, s)$, indirectly by Ramsey, and the classical bipartite Ramsey numbers of $BR(s, s)$ by Beineke and Schwenk for every positive integer s suggested another Ramsey concept (and another class of problems) stated in [10, p. 313] and [11].

A sequence $\{G_k\}$ of graphs is *ascending* if G_k is isomorphic to a proper subgraph of G_{k+1} for every positive integer k. Furthermore, an ascending sequence $\{G_k\}$ of graphs is a *Ramsey sequence* if for every positive integer k, there is an integer $n > k$ such that every red-blue coloring of G_n results in either a red G_k or a blue G_k, that is, a monochromatic G_k. The theorems obtained by Ramsey and by Beineke and Schwenk show, respectively, that $\{K_k\}$ and $\{K_{k,k}\}$ are both Ramsey sequences.

Even though $\{K_{k,k,k}\}$ is an ascending sequence, it is not a Ramsey sequence. To see this, let k be a given integer and let n be an integer where $n > k$. Let the partite sets of $K_{n,n,n}$ be V_1, V_2, V_3. Color each edge of $[V_1, V_2]$ red and color all remaining edges of $K_{n,n,n}$ blue. Then every monochromatic subgraph of $K_{n,n,n}$ is bipartite, while $K_{k,k,k}$ is not. Similarly, if $\{G_k\}$ is any ascending sequence for which $\{\chi(G_k)\}$ is a constant sequence of an integer 3 or more, then $\{G_k\}$ is not a Ramsey sequence. This results in the following.

Proposition 7.1. [11] *If $\{G_k\}$ is a Ramsey sequence, then either every graph G_k is bipartite or $\lim_{k\to\infty} \chi(G_k) = \infty$.*

Proof. Since the sequence $\{G_k\}$ of graphs is ascending, it follows that $\chi(G_{k+1}) \geq \chi(G_k)$ for every positive integer k. Assume that neither every graph G_k is bipartite nor $\lim_{k\to\infty} \chi(G_k) = \infty$. Therefore, there is a positive integer N such that for every integer $k \geq N$, it follows that $\chi(G_k)$ is a constant $p \geq 3$. Let n be an arbitrary integer with $n > N$. Then $\chi(G_n) = p$. Let $V_1, V_2, \ldots, V_p$ be the color classes in a proper p-coloring of the vertices of G_n. Assign the color red to all edges in $\left[V_p, \bigcup_{i=1}^{p-1} V_i\right]$ and the color blue to the remaining edges of G_n. Since the resulting red subgraph of G_n is bipartite and the chromatic number of the blue subgraph of G_n is $p-1$, it follows that there is no monochromatic subgraph of G_n that is isomorphic to G_N. Hence, $\{G_k\}$ is not a Ramsey sequence. $\square$

While $\{K_{r,r}\}$ is a Ramsey sequence of bipartite graphs, $\{K_k\}$ is a Ramsey sequence for which $\lim_{k\to\infty} \chi(K_k) = \infty$.

For a graph G_k in a Ramsey sequence $S = \{G_k\}$ of graphs, the smallest positive integer n for which every red-blue coloring of G_n results in a monochromatic G_k is referred to as the *S-Ramsey number* $R_S(G_k)$ of G_k. For example, if $S = \{K_k\}$, then $R_S(K_3) = R(3, 3) = 6$ and $R_S(K_4) = R(4, 4) = 18$; while if $S = \{K_{r,r}\}$, then $R_S(K_{2,2}) = BR(2, 2) = 5$ and $R_S(K_{3,3}) = BR(3, 3) = 17$. More generally, the following is a consequence of Theorems 2.1 and 4.1.

Corollary 7.1. [11] *Let s be a positive integer.*

- ⋆ *If $S = \{K_k\}$, then $R_S(K_s) \leq \binom{2s-2}{s-1}$.*
- ⋆ *If $S = \{K_{r,r}\}$, then $R_S(K_{s,s}) \leq \binom{2s}{s} - 1$.*

Another Ramsey sequence of graphs is that of the stars.

Proposition 7.2. [11] *If $S = \{K_{1,k}\}$, then $R_S(K_{1,t}) = 2t - 1$ for every positive integer t.*

Proof. For a positive integer t, every red-blue coloring of K_{2t-1} produces either a red $K_{1,t}$ or a blue $K_{1,t}$ and so $R_S(K_{1,t}) \leq 2t-1$. Since the red-blue coloring of $H = K_{2t-2}$ that assigns red to $t-1$ edges of H and blue the remaining $t-1$ edges of H avoids a red $K_{1,t}$ and a blue $K_{1,t}$, it follows that $R_S(K_{1,t}) \geq 2t-1$ and so $R_S(K_{1,t}) = 2t-1$. $\square$

Another simple Ramsey sequence consists of disconnected graphs. An argument similar to the one in the proof of Proposition 7.2 gives the following result.

Proposition 7.3. [11] *If $S = \{kK_2\}$, then $R_S(tK_2) = 2t - 1$ for every positive integer t.*

Theorem 7.1. [11] *The sequence $S = \{2K_k\}$ is a Ramsey sequence. Furthermore, $R_S(2K_3) = 9$.*

Proposition 7.4. [11] *If H is any connected graph of order 3 or more, then $\{kH\}$ is not a Ramsey sequence.*

Proof. The sequence $S = \{kH\}$ is clearly an ascending sequence. For an integer k, let $G = kH$ and let $H_1, H_2, \ldots, H_k$ be the k vertex-disjoint copies of H in G. Define a red-blue coloring of G by assigning red to one edge of H_i and blue to the remaining edges of H_i for $1 \leq i \leq k$. Since this coloring avoids a monochromatic H in G, it follows that $\{kH\}$ is not a Ramsey sequence. $\square$

The *n-cube* or *hypercube* Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \,\square\, K_2$ of Q_{n-1} and K_2. The n-cube can also be defined as that graph whose vertex set is the set of n-bit strings $(a_1, a_2, \ldots, a_n)$ or

$a_1 a_2 \cdots a_n$, where a_i is 0 or 1 for $1 \le i \le n$ such that two vertices are adjacent if and only if the corresponding n-bit strings differ at precisely one coordinate. Even though $\{Q_k\}$ is an ascending sequence of bipartite graphs, $\{Q_k\}$ is not a Ramsey sequence.

Theorem 7.2. [11] *The sequence $\{Q_k\}$ of hypercubes is not a Ramsey sequence.*

Proof. Let $S = \{Q_k\}$. Since Q_k is a proper subgraph of Q_{k+1} for every positive integer k, it follows that S is ascending. We show that for every integer $k \ge 3$, there is a red-blue coloring of Q_k that avoids a monochromatic $Q_2 = C_4$.

Let $v = (0, 0, \ldots, 0) \in V(Q_k)$, where $k \ge 3$. For each integer i with $0 \le i \le k$, let $V_i = \{x \in V(G) : d(v, x) = i\}$. For $0 \le i \le k$, a vertex in V_i is a k-bit string $(a_1, a_2, \ldots, a_k)$ where exactly i of the coordinates $a_1, a_2, \ldots, a_n$ are 1 and the remaining $n - i$ coordinates are 0. Thus, $V_0 = \{(0, 0, \ldots, 0)\}$ and $V_k = \{(1, 1, \ldots, 1)\}$. Furthermore, each set V_i is an independent set of $\binom{k}{i}$ vertices of Q_k. For each integer i with $0 \le i \le k-1$, let H_i denote the subgraph induced by $[V_i, V_{i+1}]$. Consequently, $E(Q_k) = \bigcup_{i=0}^{k-1} E(H_i)$. We show that H_i contains no C_4 for $0 \le i \le k-1$. This is clear for H_0 and H_{k-1}. Assume, to the contrary, that there is an integer i with $1 \le i \le k-2$ such that H_i contains a 4-cycle (w, x, y, z, w), where $w, y \in V_i$ and $x, z \in V_{i+1}$. Since $wx, wz \in E(H_i)$, there are two coordinates of the vertex w that are both 0, say coordinates p and q, where in x the coordinate p is 1 and in z the coordinate q is 1; while otherwise, the coordinates of w and x and of w and z are identical. This implies, however, that there are two coordinates p' and q' in x and z, where coordinate p' is 1 and the coordinate q' is 0 in x, while the coordinate p' is 0 and the coordinate q' is 1 in z such that changing each 1 to 0 results in the vertex y. However, the only coordinates where this can occur is when $p = p'$ and $q = q'$, which implies that $y = w$. This is impossible.

We now define a red-blue coloring Q_k by assigning red to each edge in H_i if i is even and $0 \le i \le k-1$ and blue to each edge in H_i if i is odd and $1 \le i \le k-1$. This red-blue coloring is shown in Figure 5 for Q_4, where each dashed line is a red edge and a solid line is a blue edge. Since there is no C_4 in H_i for $0 \le i \le k-1$, this red-blue coloring of Q_k avoids a monochromatic C_4. In fact, the only 4-cycles are of the form (w, x, y, z, w), where $w \in V_i$, $x, z \in V_{i+1}$, and $y \in V_{i+2}$ in which case, w has two coordinates p and q, both 0, y has coordinates p and q, both 1, and is otherwise identical to w; while x and z have exactly one of coordinates p and q to be 1 and is otherwise identical to w. This 4-cycle is not monochromatic, however. Therefore, Q_k has no monochromatic C_4; thus, S is not a Ramsey sequence. □

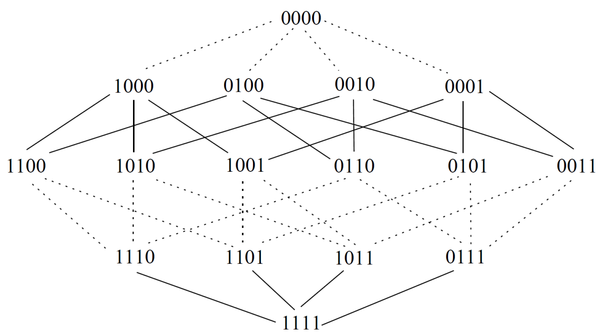


Figure 5: A red-blue coloring of Q_4 avoiding a monochromatic C_4.

We saw in Proposition 7.1 that if $\{G_k\}$ is a Ramsey sequence, then either every graph G_k is bipartite or $\lim_{k\to\infty} \chi(G_k) = \infty$. We have seen that if $\{G_k\}$ is an ascending sequence of bipartite graphs, then $\{G_k\}$ may or may not be a Ramsey sequence. We now consider ascending sequences $\{G_k\}$ for which $\lim_{k\to\infty} \chi(G_k) = \infty$. The *clique number* $\omega(G)$ of a graph G is the order of the largest clique (complete subgraph) of G. Thus, $\chi(G) \ge \omega(G)$ for every graph G. Consequently, if $\{G_k\}$ is an ascending sequence for which $\lim_{k\to\infty} \omega(G_k) = \infty$, then $\lim_{k\to\infty} \chi(G_k) = \infty$ as well.

Theorem 7.3. [11] *If $\{G_k\}$ is an ascending sequence of graphs for which $\lim_{k\to\infty} \omega(G_k) = \infty$, then $\{G_k\}$ is a Ramsey sequence.*

Proof. Let G_j be an arbitrary graph in the sequence $\{G_k\}$ and let $R(G_j, G_j) = n$. Since $\lim_{k\to\infty} \omega(G_k) = \infty$, there exists an integer p such that for every integer $k \ge p$, it follows that $\omega(G_k) \ge n$ and so $H = K_n$ is a subgraph of G_k. For every red-blue coloring of G_k, there exists a red-blue coloring of the subgraph H in G_k. Since $R(G_j, G_j) = n$, it follows that there exists a monochromatic G_j in H and so there is a monochromatic G_j in G_k. Therefore, $\{G_k\}$ is a Ramsey sequence. □

There are sequences $\{G_k\}$ of graphs for which $\lim_{k\to\infty} \chi(G_k) = \infty$ and $\lim_{k\to\infty} \omega(G_k) \ne \infty$. The question is whether there are sequences with these properties that are ascending and, if so, whether these sequences are Ramsey sequences. In [11] one such sequence was described.

Let G be a graph of order n with $V(G) = \{u_1, u_2, \ldots, u_n\}$. Applying a construction (called the *Mycielski construction*) to G, we obtain a graph, denoted by $M(G)$, of order $2n + 1$ by adding a vertex-disjoint star $K_{1,n}$ to G, where the central vertex of $K_{1,n}$ is v and the end-vertices are $v_1, v_2, \ldots, v_n$. Edges are then added between $V(G)$ and $\{v_1, v_2, \ldots, v_n\}$ so that $N(v_i) = N_G(u_i) \cup \{v\}$ for $i = 1, 2, \ldots, n$. Here, the vertex v_i is often referred to as the *shadow vertex* of u_i. This construction was introduced by Jan Mycielski [26]. If G is triangle-free, then $M(G)$ is also triangle-free and $\chi(M(G)) = \chi(G) + 1$. Thus, by successively applying the Mycielski construction, we obtain a sequence $G, M(G), M(M(G)) = M^2(G), \ldots$ of triangle-free graphs where $\lim_{k\to\infty} \chi(M^k(G)) = \infty$. For $G = K_2$, the graphs $M(G) = C_5$ and $M^2(G)$ are shown in Figure 6. The graph $M^2(G)$ is the famous *Grötzsch graph*, which is the triangle-free graph of smallest order that has chromatic number 4.

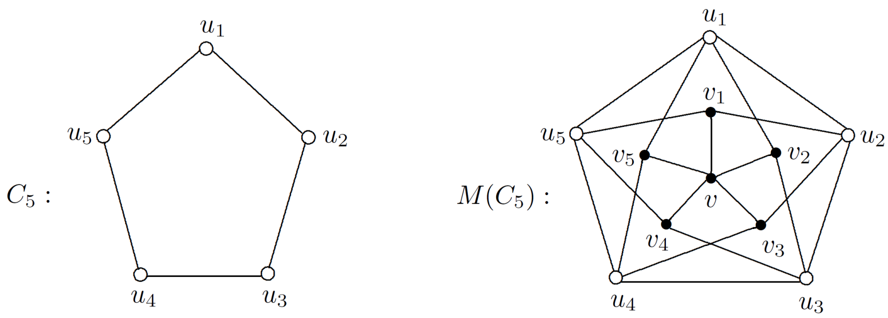


Figure 6: The graphs C_5 and Grötzsch graph.

The Mycielski construction can be applied as well to graphs that are not triangle-free. In particular, we can apply the Mycielski construction to the triangle K_3. Therefore, $G_0 = K_3$, $G_1 = M(K_3)$, $G_2 = M^2(K_3), \ldots$ is a sequence of K_4-free graphs and so $\omega(G_k) = 3$ for every nonnegative integer k with $\lim_{k\to\infty} \chi(G_k) = \infty$. Figure 7 shows red-blue colorings of K_3 and $M(K_3)$, where a red edge is denoted by a dashed line and a blue edge by a solid line. In both red-blue colorings, there is no monochromatic K_3.

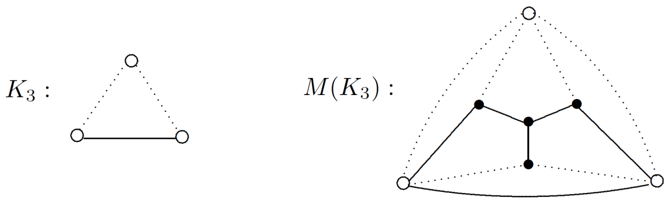


Figure 7: Red-blue colorings of K_3 and $M(K_3)$ avoiding a monochromatic K_3.

The following result was obtained in [11].

Theorem 7.4. [11] *The sequence $S = \{M^k(K_3)\}$ of graphs is ascending,*

$$\lim_{k\to\infty} \omega(M^k(K_3)) = 3, \textit{ and } \lim_{k\to\infty} \chi(M^k(K_3)) = \infty,$$

but S is not a Ramsey sequence.

Of course, we are still left with the following question:

Does there exist an ascending sequence $\{G_k\}$ of graphs with $\lim_{k\to\infty} \chi(G_k) = \infty$ and $\lim_{k\to\infty} \omega(G_k) \neq \infty$ such that $\{G_k\}$ is a Ramsey sequence?

8. Monochromatic ascending subgraph sequences

There is a problem in Ramsey theory that involves both ascending sequences of graphs and graph decompositions. A graph G of size $\binom{n+1}{2}$ for some integer $n \geq 2$ is said to have an *ascending subgraph decomposition* (ASD) if there exists an ascending sequence $\{G_k\} = \{G_1, G_2, \ldots, G_n\}$ of n subgraphs of G such that $\{G_1, G_2, \ldots, G_n\}$ is a decomposition of G where G_i has size i for $1 \leq i \leq n$. This concept and the following conjecture were introduced in [1].

The Ascending Subgraph Decomposition Conjecture. *For $n \geq 2$, every graph of size $\binom{n+1}{2}$ has an ascending subgraph decomposition.*

Let G be a graph of size m where $\binom{n+1}{2} \le m < \binom{n+2}{2}$. The *ascending Ramsey index* $AR(G)$ of G is the maximum integer k such that for every red-blue coloring of G, there exists an ascending subgraph sequence G_1, G_2, ..., G_k such that G_i is monochromatic for $1 \le i \le k$. A graph G of size $\binom{n+1}{2}$ is said to have a *monochromatic ascending subgraph decomposition* (or a *monochromatic ASD*) if for every red-blue coloring of G, there exists an ascending subgraph decomposition G_1, G_2, ..., G_n of G such that each subgraph G_i is monochromatic for $1 \le i \le n$. Consequently, if a graph G of size $\binom{n+1}{2}$ has a monochromatic ASD, then $AR(G) = n$. This concept is illustrated in the next two examples.

Example 8.1. *The graph K_4 has a monochromatic ASD and so $AR(K_4) = 3$.*

Proof. Let there be given an arbitrary red-blue coloring of K_4, resulting in the red subgraph G_R and the blue subgraph G_B of sizes m_R and m_B, respectively, where $m_R \le m_B$. We show that K_4 has a monochromatic ASD. Since $m_R \le m_B$, it follows that $0 \le m_R \le 3$. For $m_R \in \{0, 1\}$, such an ASD is clear. Suppose that $m_R = 2$. Then either $G_R = 2K_2$ or $G_R = P_3$. In either case, there is a monochromatic ASD with $G_1 = K_2$, $G_2 = G_R$, and $G_3 = P_4$. If $m_R = 3$, then $G_R \in \{K_3, K_{1,3}, P_4\}$. In each case, $G_1 = K_2$, $G_2 = K_{1,2}$, $G_3 = G_R$ is a monochromatic ASD of K_4. □

Example 8.2. *The graph $G = 3K_2 + K_{1,7}$ of size 10 has ascending Ramsey index 3.*

Proof. First, consider the red-blue coloring of G, resulting in the red subgraph $G_R \cong 4K_2$. We show that there is no monochromatic ASD of G into four graphs G_1, G_2, G_3, G_4 of G with this red-blue coloring, for suppose that there is. Then either (1) only G_4 is a red subgraph or (2) only G_1 and G_3 are red subgraphs. We consider these two cases.

Case 1. Only G_4 is a red subgraph of G. Since $G_4 = 4K_2$, it follows that $G_3 = K_{1,3}$. Because $K_{1,3}$ is not isomorphic to a subgraph of G_4, this is a contradiction.

Case 2. Only G_1 and G_3 are red subgraphs of G. Since $G_1 = K_2$ and $G_3 = 3K_2$, it follows that $G_2 = K_{1,2}$ and $G_4 = K_{1,4}$. Because G_3 is not isomorphic to a subgraph of G_4, for example, this is a contradiction.

Therefore, $AR(G) \le 3$. It remains to show that $AR(G) \ge 3$. Let there be given an arbitrary red-blue coloring of G. Let $G_1 = K_2$ be any of the three components of size 1 in G and let $G_2 = K_{1,2}$ be a monochromatic subgraph of $K_{1,7}$. The remaining subgraph $K_{1,5}$ of $K_{1,7}$ has three edges colored the same. Let $G_3 = K_{1,3}$ be such a monochromatic subgraph of $K_{1,5}$. Then G_1, G_2, G_3 is a monochromatic ascending subgraph sequence in G. Thus, $AR(G) \ge 3$ and so $AR(G) = 3$. □

If G is a star or a matching of size $\binom{n+1}{2}$, then G has a monochromatic ASD and consequently $AR(G) = n$, as we show next.

Proposition 8.1. *For each integer $n \ge 2$, every star of size $\binom{n+1}{2}$ has a monochromatic ASD.*

Proof. We proceed by induction on n. The truth of this statement is immediate for $n = 2$. Assume for an arbitrary integer $n \ge 2$ that every star of size $\binom{n+1}{2}$ has a monochromatic ASD. Let G be a star of size $\binom{n+2}{2}$ and let there be given a red-blue coloring of G. Since $n \ge 2$, it follows that $\frac{1}{2}\binom{n+2}{2} \ge n + 1$ and so there is a monochromatic substar H of G having size $n + 1$. Let U be the set of the $n + 1$ end-vertices of the substar H and let $G' = G - U$. Thus, G' is a star of size $\binom{n+1}{2}$. By the induction hypothesis, the resulting red-blue coloring of G' has a monochromatic ASD into n monochromatic subgraphs G_1, G_2, ..., G_n. Hence, G_1, G_2, ..., G_n, $G_{n+1} = H$ is a monochromatic ASD of G. □

The following result has a proof similar to that of Proposition 8.1.

Proposition 8.2. *For each integer $n \ge 2$, every matching of size $\binom{n+1}{2}$ has a monochromatic ASD.*

Among the numerous problems on this topic are the following. In addition to stars and matchings of size $\binom{n+1}{2}$, which graphs G of size $\binom{n+1}{2}$ have $AR(G) = n$? For which positive integers k, do there exist a positive integer n and a graph G of size $\binom{n+1}{2}$ such that $AR(G) = n - k$?

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