Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **6** (2021) 68–83
DOI: 10.47443/dml.2021.s109

Review Article

A survey of line digraphs and generalizations*

Jay S. Bagga[1,†], Lowell W. Beineke[2]

[1]*Department of Computer Science, Ball State University, Muncie, Indiana, USA*
[2]*Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, Indiana, USA*

(Received: 2 August 2020. Accepted: 16 August 2020. Published online: 11 March 2021.)

Abstract

The line graph transformation is one of the most widely investigated operations in graph theory. Frank Harary was the first to generalize this concept to digraphs. Since then various other generalizations have been proposed and studied. In this paper we present a survey of line digraphs and their generalizations.

Keywords: line graph; line digraph; total digraph.

2020 Mathematics Subject Classification: 05C20, 05C76.

1. Introduction and basic properties

One of the most-studied operations in graph theory – perhaps *the* most-studied – is that of the line graph. The line graph $L(G)$ of a graph G (with at least one edge) has the set of edges of G as its vertices with two adjacent if the corresponding edges have a common vertex in G. While the name line graph was introduced by Harary and Norman [15] in 1960, the concept actually originated in 1932 by Whitney [24] in his study of edge isomorphisms of graphs, where he showed that except for K_3 and $K_{1,3}$, edge-isomorphism implies isomorphism for connected graphs.

The concept of line graph has been generalized in various ways. For example, an edge in a graph can also be viewed as a path of length 1 on a set of two vertices. Then the line graph of a graph G is obtained by mapping paths of length 2 in G to paths of length 1 in $L(G)$.

The focus of this survey is line digraphs, a concept introduced in [15]. While there are several surveys in the literature on line graphs and their generalizations, including Bagga [2] and Hemminger and Beineke [18], we believe this is among the first surveys on line digraphs. We also present some generalizations. A forthcoming book by the authors [9] will cover this topic in much greater detail. Since our aim is to provide a survey of results on line digraphs and some of their generalizations, we state most results without proof which can be found in the references cited.

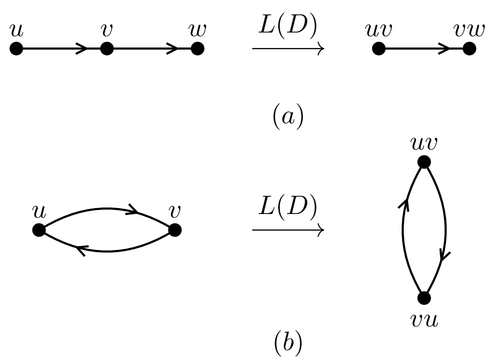

Figure 1: The line digraph operation.

In this section we provide relevant definitions and cover some basic properties of line digraphs. Section 2 discusses isomorphism properties, while in section 3 we discuss the construction of root digraphs. Section 4 presents results on characterizations of line digraphs and section 5 provides a discussion of iterated line digraphs. In sections 6–8, we present three generalizations of line digraphs: super line digraphs, total digraphs, and path digraphs.

*Dedicated to the memory of Professor Frank Harary on the occasion of his 100th birthday.
†Corresponding author (jbagga@bsu.edu).

For a digraph D, the concept corresponding to the line graph is the *line digraph* $L(D)$, defined as follows: $L(D)$ has the arcs of D as its vertices and there is an arc from a vertex a to a vertex b if the head of a is the same as the tail of b in D. Figure 1 (a) shows the line digraph transformation for two arcs uv and vw (with $u \neq w$) while (b) illustrates the case of two "symmetric" arcs uv and vu.

These indicate the basic role that paths and cycles play in line digraphs. Clearly, for $n \geq 2$, $L(\overrightarrow{P_n}) = \overrightarrow{P_{n-1}}$ and $L(\overrightarrow{C_n}) = \overrightarrow{C_n}$. Going beyond that, we have the following result.

Theorem 1.1. (a) *If W is an open trail of length k in a digraph D, then its image in $L(D)$ is a path of length $k-1$.*
(b) *If W is a circuit of length k in a digraph D, then its image in $L(D)$ is a cycle of length k.*

Much more than in the case of graphs, there is often considerable value in allowing loops. In fact, if loops are allowed (as well as symmetric pairs of arcs), then one simply has finite relations, and there are times when the properties of reflexivity and symmetry of relations on sets are appropriate to have. With the terminology of relations, Figure 2 shows the line digraph of an equivalence relation on two elements and that of a complete irreflexive symmetric relation on three elements.

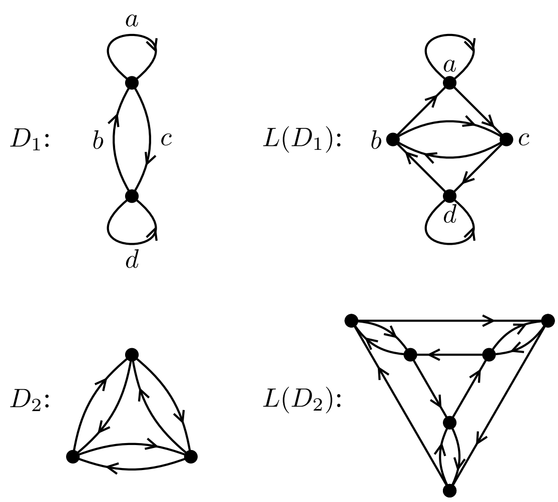

Figure 2: Examples of relations.

The next three figures show five examples of line digraphs having some very interesting features, especially when compared with line graphs.

- The line digraphs of D_1 and D_2 in Figure 3 are isomorphic; they are representative of the family of digraphs with this property. In contrast, the only two connected graphs with this property are K_3 and $K_{1,3}$.

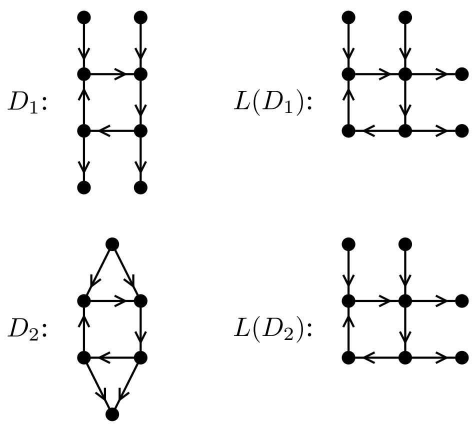

Figure 3: Two digraphs with the same line digraph.

- The line digraph of D in Figure 4 is D itself; the only connected graphs with this property are cycles.
- In Figure 5 the line digraphs of D_1 and D_2 are each other; no connected graphs have this property.

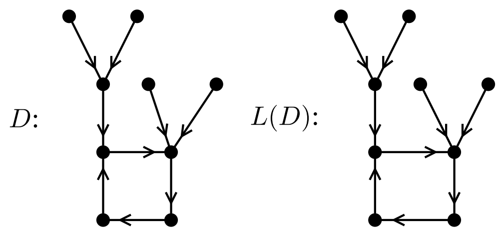

Figure 4: A digraph isomorphic to its line digraph.

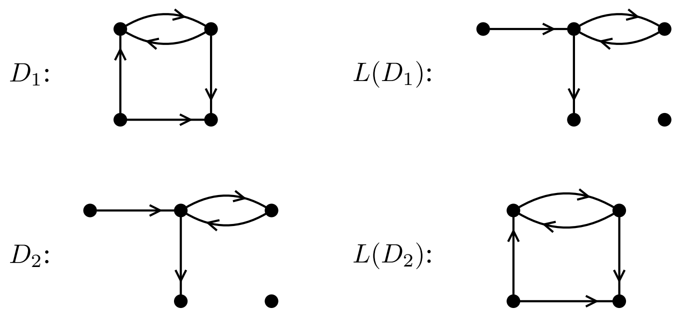

Figure 5: Two digraphs that are line digraphs of each other.

With that rather tantalizing introduction to some of the interesting features of line digraphs, we move to some simple observations. Note that $d^+(v)$ and $d^-(v)$ denote the out-degree and in-degree respectively of vertex v.

Theorem 1.2. *Let D be a digraph with n vertices and m arcs $(m > 1)$.*

(a) *If $e = vw$ is an arc of D, then its out-degree in $L(D)$ is $d^+(w)$ and its in-degree is $d^-(v)$.*

(b) *Its line digraph $L(D)$ has m vertices and $\sum_v d^+(v)d^-(v)$ arcs.*

Recall that the set of edges at a vertex v in a graph G forms a star and this yields a complete subgraph in the line graph $L(G)$. The situation is quite different for digraphs. Assume now, for purposes of illustration, that D is an oriented graph, that is, a digraph without loops or symmetric arcs (in other words, no cycles of length 1 or 2). Then, in contrast to the edges at a vertex in a graph, the arcs at a vertex v in a digraph D are of two types, the in-arcs and the out-arcs. Furthermore, if a vertex v has arcs of both types, then it gives rise to an orientation of a complete bipartite graph (if all of the arcs at v are of the same type, they give rise to an induced null subdigraph (that is, an arc-free set of vertices) of $L(D)$). We formalize this by introducing some notation and terminology connected with oriented analogues of complete bipartite graphs. We denote by $\overrightarrow{K_{r,s}}$ the digraph obtained from $K_{r,s}$ by orienting all of the edges from the set of r vertices to the set of s vertices. We call such an oriented graph a *bicomplete digraph*. It can readily be seen that it is the line digraph of the digraph $\overrightarrow{S}_{r,s}$ obtained from the star $K_{1,r+s}$ by orienting r arcs into its central vertex v and orienting the other s arcs out from v. Figure 6 shows $\overrightarrow{K_{2,3}}$ and $\overrightarrow{S_{2,3}}$.

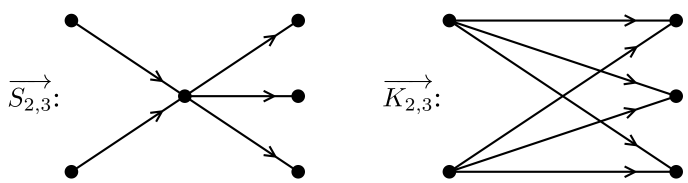

Figure 6: An oriented star and an oriented complete bipartite graph.

2. Isomorphic line digraphs

We pointed out earlier that the digraphs D_1 and D_2 in Figure 3 had isomorphic line digraphs. We observe now that the difference between D_1 and D_2 is in the vertices having either in-degree or out-degree 0. We classify vertices into four types: first, there are the *isolates*, those with no incident arcs. A vertex with in-degree 0 and positive out-degree is called a *source* while one with out-degree 0 and positive in-degree is called a *sink*. Lastly, a vertex with both in- and out-degree positive is called a *juncture*. It is essentially only in the non-juncture vertices that non-isomorphic digraphs with the same line digraph can differ. If D is a digraph with at least one juncture vertex, then the subdigraph induced by all such vertices is called its *core*. It was shown by Harary and Norman [15] that if two digraphs have isomorphic line digraphs, then either their cores are both empty or they are isomorphic.

Among the set of digraphs with a given line digraph, one can be chosen as canonical. We define a digraph to be *fundamental* if every source has out-degree 1, every sink has in-degree 1, and there are no isolated vertices. Using the result of Harary and Norman, we can deduce the following theorem that shows that the fundamental digraphs can be taken as canonical.

Theorem 2.1. *If D is a digraph with at least one arc, then there is exactly one fundamental digraph F with $L(F) \cong L(D)$.*

The theorem has some corollaries that are worth stating.

Corollary 2.1. *If D_1 and D_2 are digraphs in which every vertex has positive in- and out-degree, then $L(D_1) \cong L(D_2)$ if and only if $D_1 \cong D_2$.*

Note that this can be extended to include digraphs with at most one source and at most one sink (and of course no isolated vertices).

Corollary 2.2. *If D_1 and D_2 are strongly connected digraphs, then $L(D_1) \cong L(D_2)$ if and only if $D_1 \cong D_2$.*

3. Root digraphs

If digraphs D and F are such that $L(F) = D$, then F is called a *root* or a *root digraph* of D. In this section we give a procedure for finding a root digraph of a given line digraph D. In the preceding section, we saw that a line digraph can have many roots, but that they differ only in their sources and sinks. The procedure that we present can be used to find all of the roots, but we will focus on those that are fundamental.

Let $D = L(F)$. For each arc ab in D, there are by definition vertices u, v, and w (not necessarily all different) in F with $a = uv$ and $b = vw$. Furthermore, all of the arcs into and out of v produce a bicomplete digraph that contains ab. It follows that each juncture vertex of F gives rise to one such bicomplete digraph, and these bicomplete digraphs form a partition of the arcs of D.

To complete our analysis, it is convenient to have a directed concept corresponding to the undirected concept of the intersection graph of a collection of sets. Consider a family $\mathcal{S}$ of ordered pairs of sets $(A_1, B_1), (A_2, B_2), \ldots, (A_k, B_k)$ with each element of S in at most one A_i and in at most one B_i. The *connection digraph* of $\mathcal{S}$ has a vertex v_i for each pair (A_i, B_i), with an arc from v_i to v_j if $B_i \cap A_j \neq \emptyset$. For example, let $\mathcal{S}$ be the following four pairs of sets of elements of the set $S = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$: $\mathcal{S} = \{(\{\gamma\}, \{\alpha, \beta\}), (\{\beta, \delta\}, \{\gamma\}), (\{\alpha\}, \{\delta, \epsilon\}), (\{\epsilon, \zeta\}, \{\zeta\})\}$. With v, w, x, y four vertices corresponding respectively to the four pairs of sets, we get the connection digraph D shown in Figure 7.

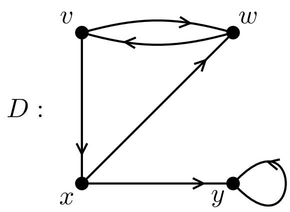

Figure 7: A connection digraph.

Before returning to line digraphs, we note a further parallel with intersection graphs: Just as every graph is the intersection graph of some family of sets, every digraph is the connection digraph of some collection of pairs of sets. This can be seen by construction: Given a digraph D without isolated vertices, for each vertex v, take the ordered pairs $(A^-(v), A^+(v))$, where $A^-(v)$ is the set of in-coming arcs at v and $A^+(v)$ is the set of out-going arcs (isolated vertices can easily be treated). It can readily be verified that the connection digraph of this family is D.

The following algorithm constructs a root digraph for an arbitrary line digraph.

Root Construction. Let $\mathcal{S} = \{(A_1, B_1), (A_2, B_2), \ldots, (A_k, B_k)\}$ be the pairs of partite sets of the decomposition of the arcs of a line digraph D into bicomplete digraphs. Let F be the connection digraph of $\mathcal{S}$. As we observed above, each vertex

v in D is in at most one first set and at most one second set of the pairs in $\mathcal{S}$, and v is a juncture if and only if it is in one of each, it is a source if and only if it is in a first set but not a second set, it is a sink if and only if it is in a second set but not a first, and an isolate if and only if it is not in any of the sets. For each arc a out of a source of D, add to F a new vertex $v(a)$ with an arc from it to the vertex corresponding to the pair in $\mathcal{S}$ in which it appears. Similarly, for each arc into a sink, add to F a new vertex with an arc to it from the pair in which it appears. Finally, to each isolated vertex of D, add to F two more new vertices and an arc from one to the other. The resulting digraph F is a root digraph for D.

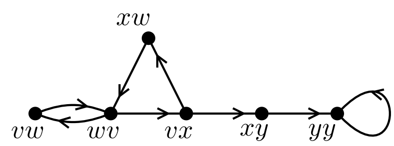

Figure 8: Illustrating root construction.

We use the digraph F in Figure 8 to illustrate this for the digraph D in Figure 7 . If we label the vertices of F as a, b, c, d, e, f, then we have the set of arcs of F partitioned into four bicomplete digraphs with these sets of two arcs each (the last a loop): $H_1 = \{ab, cb\}$, $H_2 = \{ba, bd\}$, $H_3 = \{dc, de\}$, $H_4 = \{ef, ff\}$.

4. Characterizations of line digraphs

The question that we want to answer is this: How can one tell if a given digraph D is a line digraph or not? In a way, one answer was given in the previous section, and that was to find out whether it has a root graph or not. Following that thread, we define a partition of the arcs of a digraph into bicomplete subdigraphs to be a *proper bicomplete partition* if each vertex is in at most one of the first sets and at most one of the second sets. Several other answers have been found to this question. The following was observed by Heuchenne [19]: Assume that ac, bc, and bd are arcs in a line digraph $D = L(F)$. Then the root digraph F must have arcs $a = uv$, $c = vx$, $b = xv$, and $d = vy$, and hence the arc ad must be in $L(F)$. It is in fact the case that not only does this give a necessary condition for a digraph to be a line digraph but it is also sufficient. This gives the following characterization theorem for loop-free line digraphs.

Before stating that theorem, we need the following observation. Consider the digraph D_1 with the four vertices a, b, c, d and the four arcs ab, ac, bd, cd shown in Figure 9, and suppose that it is the line digraph of some digraph F. Then F must have vertices u, v, w so that $a = uv, b = vw, d = wx$. But then it follows that c must have v as its first vertex and w as its second, and so b and c are parallel arcs, and we have assumed that all of our digraphs are simple. The same argument can be applied to the digraph D_2 in Figure 9 obtained by identifying the vertices a and d, as having three vertices a, b, and c and the two 2-cycles aba and aca.

Theorem 4.1. *Let D be a digraph that does not have two paths of length 2 from one vertex to another nor a pair of 2-cycles at a vertex. Then the following statements are equivalent*:
(1) *D is a line digraph.*
(2) *If ac, bc, and bd are arcs in D, then so is ad.*
(3) *The set of arcs of D has a proper bicomplete partition.*

We stated our theorem in terms of digraphs without loops, but it is straightforward to allow loops – some of the statements just get a bit more complicated. Take statement (2) in the theorem for instance. This is represented in the first digraph in Figure 10 by the fact that if the three solid arcs appear in a line digraph, that the dashed arc must also be present. If loops are allowed, then the other six digraphs in the figure are also needed for the argument to be complete.

A further criterion, one involving the adjacency matrix, was provided by Richards [22]. We note that when translated into matrix terms, Heuchenne's condition says that the adjacency matrix has no 2×2 submatrix with exactly one 0. Also, there is never more than one path of length 2 from one vertex to another (see Figure 9). That is, in matrix terms, there cannot be a row and a column with 1s in the same two positions. Therefore, we include those in our next theorem.

Theorem 4.2. *Let* M *be a* 0-1 *matrix with zero diagonal. Then the following statements are equivalent*:
(1) M *is the adjacency matrix of a line digraph.*
(2) M *has no 2×2 submatrix with exactly one 0.*

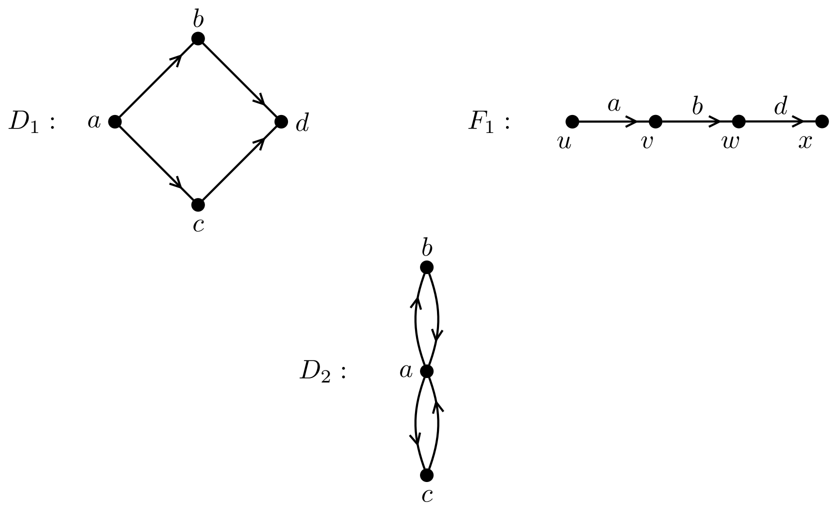

Figure 9: A forbidden digraph in a line digraph.

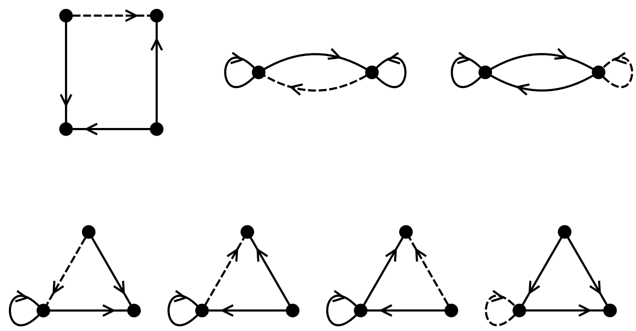

Figure 10: Seven forbidden induced subdigraphs.

(3) *Any two rows of* M *are either identical or orthogonal and the product of any row and any column is* 0 *or* 1.
(4) *Any two columns of* M *are either identical or orthogonal and the product of any row and any column is* 0 *or* 1.

5. Iterated line digraphs

5.1 Basic properties

Iterated line digraphs are defined recursively just as one would expect: Given a non-null digraph D, $L^1(D) = L(D)$, and if $L^k(D)$ is non-null, then $L^{k+1}(D) = L(L^k(D))$; if $L^k(D)$ is null, then $L^{k+1}(D)$ is not defined. We also take $L^0(D) = D$. We begin this section with some elementary examples that illustrate some of the behavior of the iterates of connected line digraphs with at most two cycles. We then concentrate on digraphs that have iterated line digraphs that repeat. This is followed by a discussion of characterizations of iterated line digraphs.

Figure 11 shows a digraph without cycles and its first three iterated line digraphs, ending up with a null digraph. Not surprisingly, this is representative of what happens with any cycle-free digraph: Eventually $L^k(D)$ has no arcs, and then of course $L^{k+1}(D)$ does not exist.

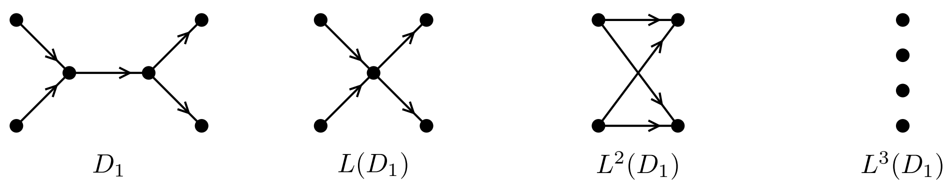

Figure 11: An example of line iterates of a digraph without cycles.

We next consider digraphs in which there is a path from one cycle to another. In our next example, shown in Figure 12, the digraph D has two arc-disjoint cycles joined by a path, and we observe that its iterated line digraphs have the same property, but with the path joining the cycles increasing in length. This suggests that the line digraph of a digraph containing such cycles increases in size, at least eventually.

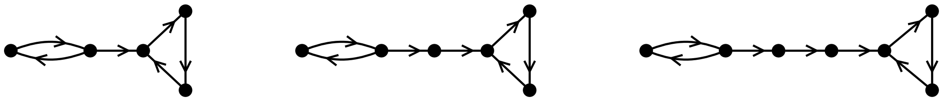

Figure 12: An example of line iterates of a digraph with a pair of joined cycles.

Another possibility in digraphs with cycles is that a pair of them have an arc in common. In the simplest case, like that shown in Figure 13, there are two vertices joined by three internally disjoint paths, two in one direction and the third in the opposite direction. We observe that here each iterated line digraphs have cycles of the same lengths but with less in common.

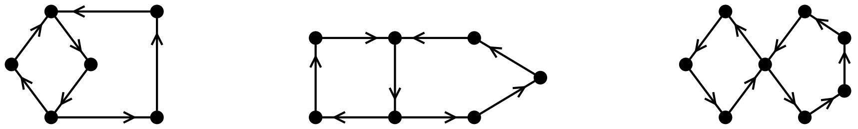

Figure 13: Another example of line iterates of a digraph with a pair of joined cycles.

The one case that we haven't covered as yet is that in which there is at least one cycle, but no two are joined by a path (remember that our cycles and paths are all directed). As the simple example in Figure 14 illustrates, eventually the iterated line digraphs of such a digraph are such that all of the cycles are in different connected components. Furthermore, although new cycle-free components can arise, their iterates become null and there are limits on their number and their size. The following theorem summarizes what can happen to the order of the iterated line digraphs of a digraph.

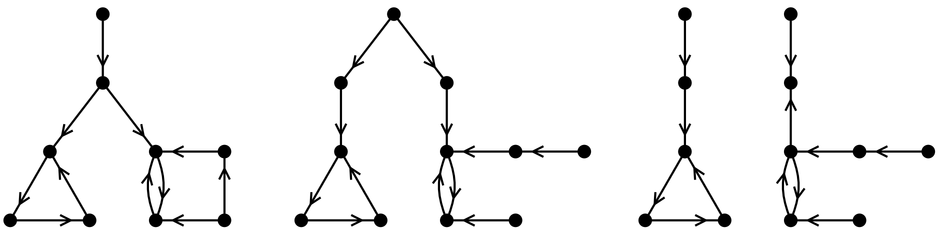

Figure 14: An example of line iterates of a digraph with no joined cycles.

Theorem 5.1. *Let D be a digraph and let n_k be the number of vertices in its kth iterated line digraph $L^k(D)$.*

(a) *For some k, $L^k(D)$ is a null graph if and only if D has no cycles. Furthermore, the value of this k equals the length of a longest path in D.*

(b) *If D has at least one cycle but there is no path between two cycles, then the sequence $\{n_k\}$ is bounded.*

(c) *If D has a pair of cycles for which there is a path from one to the other, then $\lim_{k\to\infty} n_k = \infty$.*

We note that this exhausts all of the possibilities of what happens eventually to the iterated line digraphs of any digraph.

5.2 Periodicity

It is well known that the only connected graphs that are isomorphic to their line graphs are cycles. It is also clear that cycles are among the digraphs D for which $L(D) \cong D$. It turns out that the situation is much more complicated than for graphs however. For historical reasons, we note that the first result of this type was the connected case, first proved in [15]:

A connected digraph D is isomorphic to its line digraph if and only if every vertex has out-degree 1 or every vertex has in-degree 1. This will be a corollary to a more general result in what follows.

It is useful to have some additional terminology and notation, beginning with *in-tree* and *out-tree*. First, we define *in-trees* recursively: A single vertex is an in-tree, and given an in-tree T_k^- with k vertices, the addition of a new vertex with an arc to one vertex of T_k^- results in an in-tree of order $k+1$. Less formally, we can consider in-trees to be orientations of trees with a root vertex having all paths oriented towards the root. *Out-trees* are of course defined similarly, and are the converse digraphs of in-trees. (An out-tree is sometimes called an *arborescence*, and by analogy an in-tree is a *counter-arborescence*.) An example of each is shown in Figure 15.

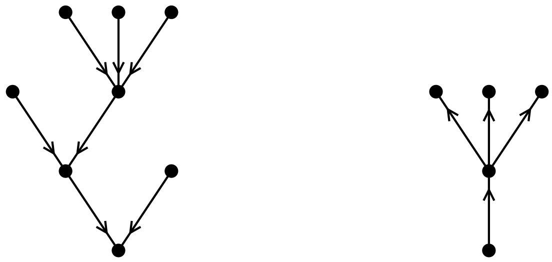

Figure 15: An in-tree and an out-tree.

A related concept is the digraph of a function on a finite set, which we call a *function digraph*; that is, a digraph in which each vertex has out-degree 1. They are characterized by each component comprising a cycle and (possibly) in-trees or out-trees (but not some of each) at some of the vertices. A simple example with three components is given in Figure 16. It is straightforward to show that every such digraph is isomorphic to its line digraph, as is that of its converse of course. The result of Harary and Norman [15] says that these are the only connected digraphs with this property.

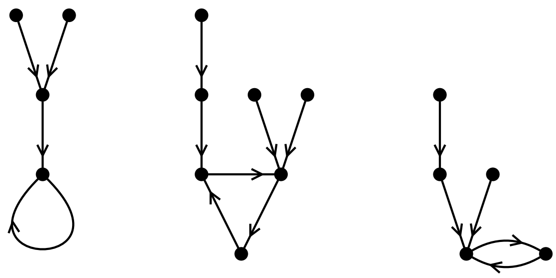

Figure 16: The digraph of a function.

However, there are other digraphs with this property. Not only are there disconnected digraphs D with a single cycle for which $L(D) \cong D$ – an example is shown in Figure 17 – but for any $p > 1$, there are digraphs D with $L^p(D) \cong D$ (and $L^k(D) \not\cong D$ for $k < p$). In fact, there are digraphs whose iterated line digraphs eventually repeat even though they do not repeat at the start. We make the following definition. A line digraph D is *periodic* if there are positive integers k and p for which $L^{k+p}(D) \cong L^k(D)$ for all k. When this occurs, the minimum such p is called the *period* of D. For example, consider the digraph D and the iterated line digraphs shown in Figure 18. Although all five of the digraphs are different, we can see that $L^5(D) \cong L^3(D)$ (since $L^4(D)$ is isomorphic to $L^2(D)$ except for its isolated vertex), and hence, for $k \geq 3$, $L^{k+2}(D) \cong L^k(D)$ (and $L^{k+1}(D) \not\cong L^k(D)$), so D is periodic with period 2.

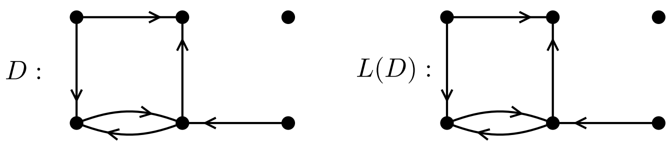

Figure 17: A disconnected digraph isomorphic to its line digraph.

The primary purpose of this section is to investigate digraphs whose line digraphs have a given period. We focus on unicyclic digraphs, those with just one cycle, since it happens, by far the nicest result is for them. It should be clear from our discussion that within a digraph D having a single cycle, the paths going towards that cycle get maintained under the line digraph operation, and therefore the unions of those paths are of interest. Of course, a similar statement holds

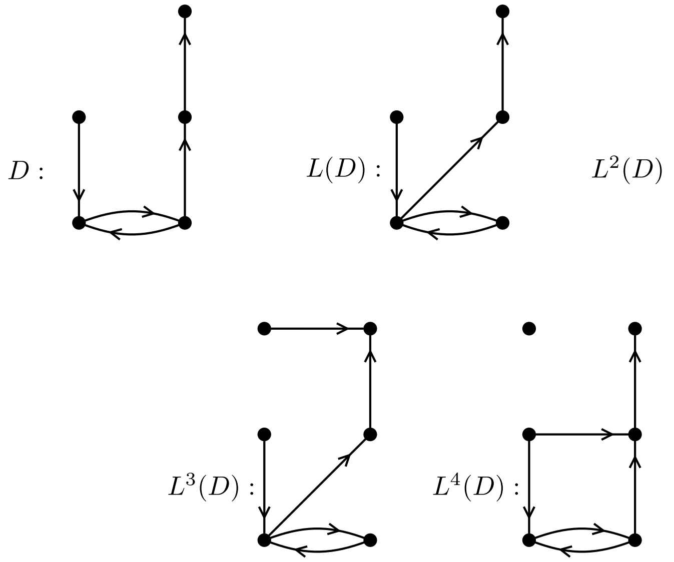

Figure 18: A digraph with period 2.

for the paths going away from the cycle. Because of their importance, we define the *foundation* $F(D)$ of digraph D to be the subdigraph formed by the cycle C and the union of all paths to and from C. An example of a unicyclic digraph and its foundation is shown in Figure 19.

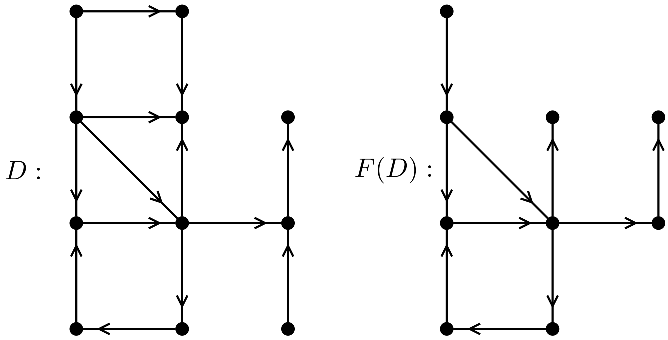

Figure 19: The foundation of a unicyclic digraph.

We now consider a special type of unicyclic digraph: an *eddy* digraph consists of a cycle $C = v_0v_1 \dots v_{k-1}v_0$ together with an in-tree A_i and an out-tree B_i (possibly trivial) at each vertex v_i of C.

Theorem 5.2. *If D is a unicyclic digraph, then for k sufficiently large the foundation of $L^k(D)$ is an eddy digraph.*

Since the line digraph of a path is another path and a path in a line digraph can only come from a path, it follows that the foundation $F(L(D))$ of a line digraph is isomorphic to the line digraph $L(F(D))$ of the foundation of D. Therefore, in illustrating this theorem, we show only the foundations of the iterated line digraphs; see Figure 20.

We now let D be an eddy digraph with cycle $C = v_0v_1 \dots v_{r-1}v_0$, and let A_i and B_i be the in-tree and out-tree respectively rooted at v_i. (Note that some of these may be trivial.) Furthermore, let $\{A_0, A_1, \dots, A_{r-1}\}$ be the cyclic (modulo k) sequence of in-trees and define the *in-tree index* of D to be the least positive integer l for which $A_{i+l} \cong A_i$ for all i. The sequence of out-trees $\{B_0, B_1, \dots, B_{r-1}\}$ and the *out-tree index* are defined similarly.

The next theorem shows how the in-trees and the out-trees in the foundation of a unicyclic digraph are related under the line digraph operation, a feature that is illustrated in Figure 21.

Theorem 5.3. *The foundation of the line digraph of an eddy digraph is isomorphic to the digraph obtained by shifting the in-tree sequence one step along the cycle relative to the out-tree sequence.*

The following result is due to Hemminger [16].

Theorem 5.4. *If D is a unicyclic digraph, then the period of D is the greatest common divisor of its out-tree and in-tree indices.*

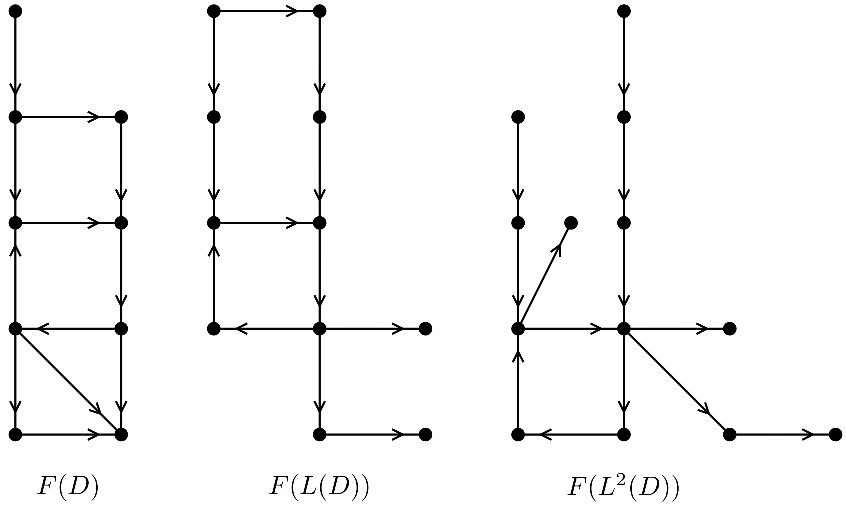

Figure 20: The foundations of iterated line digraphs.

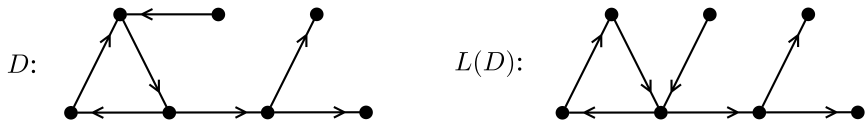

Figure 21: The foundation of the line digraph of an eddy digraph.

The period-1 case is of course of special interest:

Corollary 5.1. *A unicyclic digraph has period* 1 *if and only if its out-tree and in-tree indices are relatively prime.*

An example is shown in Figure 22, with an eddy digraph D having in-tree index 3, out-tree index 2, and a 6-cycle, and the foundation of $L(D)$.

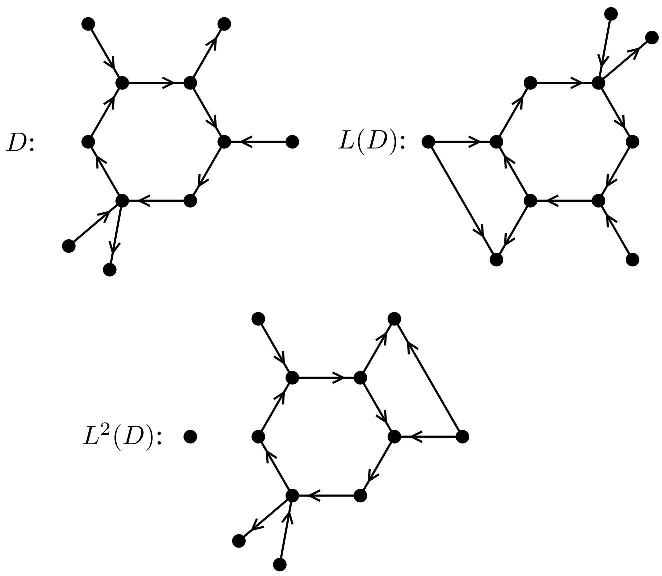

Figure 22: The line digraph of a digraph with a 6-cycle and relatively prime in-tree and out-tree indices.

5.3 A characterization of second-order line digraphs

Recall that in a characterization of line digraphs, Heuchenne [19] showed that if uv, wv, and wx are arcs in a line digraph $L(D)$, then so is ux. Here, the vertices $u, v, w,$ and x need not all be different. We call this the *first Heuchenne condition*. The following theorem shows that a similar result must hold for the second iterated line digraph $L^2(D)$.

Theorem 5.5. *If the second iterated line digraph* $L^2(D)$ *has internally disjoint* u–w, v–w, *and* v–x 2*-paths, then it also has a* u–x 2*-path internally disjoint from the other* 2*-paths.*

We call the condition given in the theorem the *second Heuchenne condition*, and it is illustrated in Figure 23 where the presence in a second order line digraph of the three paths with solid arcs require the presence of the path with dashed arcs.

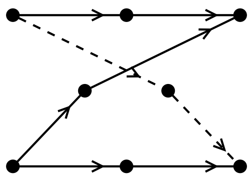

Figure 23: The second Heuchenne condition.

Our goal here is a characterization of second-order line digraphs; that is, those digraphs D for which there is a digraph F with $D \cong L^2(F)$. This result is due to Beineke and Zamfirescu [10].

We now note with two examples (shown in Figure 24(a) and (b)) that while the second Heuchenne condition is necessary for a line digraph to be a second-order line digraph, it is not sufficient even with the other observations made earlier. In both cases, D is the line digraph of E, but E itself is not a line digraph. It does not have the first Heuchenne property and consequently must have an additional arc, as in E'. It can be verified that digraph F in the figure satisfies $D' \cong L(E') \cong L^2(F)$.

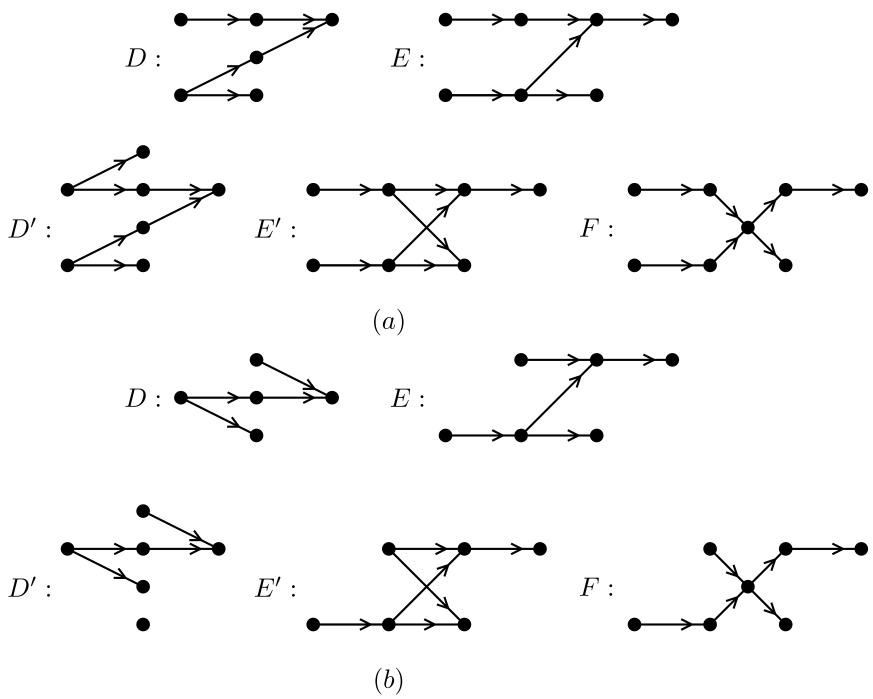

Figure 24: Illustrating Theorem 5.6.

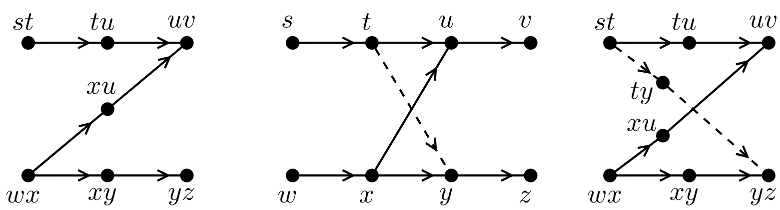

Figure 25: Levels in 2-walks.

Now consider digraph D in Figure 24(b). As in the previous example, it is the line digraph of digraph E in the figure, but E is not a line digraph. In order for E to be a line digraph, it must have the additional arc shown in E' and so D' must have an isolated vertex. Then the digraph F shown is such that $E' \cong L(F)$ and $D' \cong L^2(F)$. The necessity of these additional arcs is covered in condition (d) of the theorem.

We say that a vertex is an *end-source* if it has out-degree 1 and in-degree 0, and an *end-sink* if those values are reversed. In stating our theorem, we find it convenient to introduce a particular type of subdigraph of a digraph D. We say that two walks $\overrightarrow{P}: v_0v_1v_2$ and $\overrightarrow{Q}: w_0w_1w_2$ of length 2 are *hooked* if $v_i = w_i$ for at least value of i. In addition, we say that $\overrightarrow{P} \sim \overrightarrow{Q}$ if there is a sequence of 2-walks $\overrightarrow{P} = \overrightarrow{P_0}, \overrightarrow{P_1}, \ldots, \overrightarrow{P_r} = \overrightarrow{Q}$ with the property that $\overrightarrow{P_{i-1}}$ and $\overrightarrow{P_i}$ are hooked for each i. Clearly $\sim$ is an equivalence relation on the set of 2-walks in D. The union of the 2-walks in an equivalence class is called a *tri-level subdigraph* of D, with the first vertices in the 2-paths called *first-level*, the middle vertices *second-level*, and the last vertices *third-level*. The digraph D in Figure 25 is an example of a tri-level digraph, with the second diagram showing all of the 2-walks and how they are hooked. (The digraph can be obtained from this by identifying the vertices labeled with the same letter.)

We now have the background that we find useful for our characterization theorem.

Theorem 5.6. *A digraph D is a second-order line digraph if and only if it satisfies the following six conditions:*

(a) *It has no multiple arcs.*

(b) (*First Heuchenne condition*) *For all vertices u, v, w, and x, if uw, vw, and vx are arcs in D, then so is ux.*

(c) *For any two vertices u and v, there is at most one u–v 2-walk.*

(d) (*Second Heuchenne condition*) *For all vertices u, v, w, and x, if there is a u–w 2-walk, a v–w 2-walk, and a v–x 2-walk, then there is also a u–x 2-walk.*

(e) *Within any tri-level subdigraph, there is the same number of sinks at all first-level vertices and the same number of sources at all third-level vertices.*

(f) *For each tri-level subdigraph, D has pq isolated vertices, where p is the number of sinks at its first-level vertices and q is the number of sources at its third-level vertices.*

In closing we note that Hemminger [17] extended this result to iterated line digraphs of the kth order.

6. Super line digraphs

Bagga, Beineke, and Varma [3] defined and studied *super line graphs* which turns out to be a generalization of $L(G)$ that has yielded many interesting properties. For a graph G with $m \geq 1$ edges and no isolated vertices, and for an integer r with $1 \leq r \leq m$, the *super line graph* $\mathcal{L}_r(G)$ of *index* r has the sets of r edges in G as its vertices, and two of these are joined by an edge if some edge in one set is adjacent to some edge in the other. Thus the order of $\mathcal{L}_r(G)$ is $\binom{m}{r}$. There is extensive literature on super line graphs (see Bagga, Beineke and Varma [4–7]). However, the generalization to super line digraphs is relatively unexplored.

We consider digraphs without loops. Given a digraph D with $m \geq 1$ arcs and an integer r with $1 \leq r \leq m$ the *super line digraph $\mathcal{L}_r(D)$ of index* r is the digraph whose vertices are r-subsets of $A(D)$. For two distinct vertices S and T of $\mathcal{L}_r(D)$ there is an arc from S to a T if and only if there exist $s \in S$ and $t \in T$ such that s and t form a 2-path in D, i.e. $head(s) = tail(t)$. Clearly, $\mathcal{L}_1(D) = L(D)$. We observe that our definition does not allow loops. Figure 26 illustrates a digraph D and its super line digraphs.

We list below some basic properties. We first describe some notation. For an arc a in D, we denote by $N^+(a)$ the set of arcs that have the same tail as the head of a. The notation $N^-(a)$ has a similar meaning.

For a vertex ab of $\mathcal{L}_2(D)$, we define $\partial^+_{ab} = |N^+(a) \cup N^+(b)|$, $\partial^-_{ab} = |N^-(a) \cup N^-(b)|$ and $\mu_{ab} = 1$ if a, b induce a 2-path or a 2-cycle in D, and $\mu_{ab} = 0$, otherwise. More generally, for $r \geq 2$, if S is a vertex of $\mathcal{L}_r(D)$, we use the notation ∂^+_S and ∂^-_S.

Theorem 6.1. *If D is a digraph with m arcs and H is a subdigraph of D, then*

(a) *$\mathcal{L}_r(H)$ is an induced subdigraph of $\mathcal{L}_r(D)$;*

(b) *for $1 \leq r < m/2, \mathcal{L}_r(D)$ is a subdigraph of $\mathcal{L}_{r+1}(D)$.*

Theorem 6.2. *For a vertex ab of $\mathcal{L}_2(D)$*

(a) $d^+(ab) = \binom{\partial^+_{ab}}{2} + \partial^+_{ab}(q - \partial^+_{ab}) - \mu_{ab}$;

(b) $d^-(ab) = \binom{\partial^-_{ab}}{2} + \partial^-_{ab}(q - \partial^-_{ab}) - \mu_{ab}$.

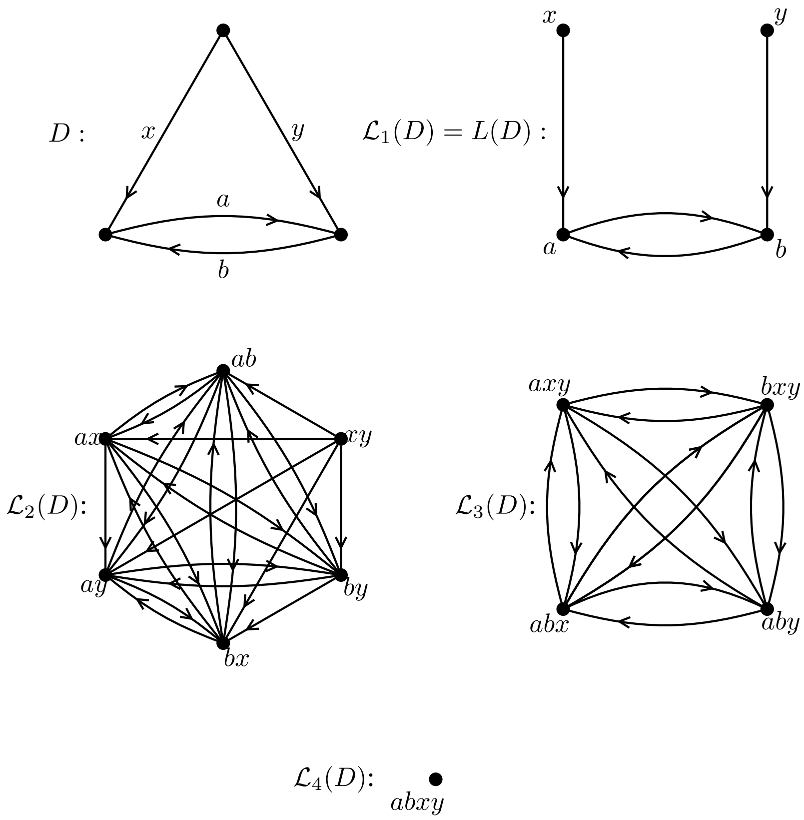

Figure 26: A digraph and its super line digraphs.

Aigner [1] showed that if D is a digraph with at least three vertices (none of which is isolated), then $L(D)$ is strongly connected if and only if D is strongly connected. However, for $r > 1$, the situation for $\mathcal{L}_r(D)$ is quite different, as the next result of Ferrero [14] shows.

Theorem 6.3. *If $r \geq 2$ and if D is a digraph with the property that for any vertex S of $\mathcal{L}_r(D)$, $\partial_S^+ > 0$ and $\partial_S^- > 0$, then $\mathcal{L}_r(D)$ is strongly connected and the distance between any two vertices is at most 2.*

We now consider the special case when the distance between any two vertices in $\mathcal{L}_r(D)$ is at most 1, that is, $\mathcal{L}_r(D)$ is complete. It is easily seen that if $\mathcal{L}_r(D)$ is complete, so is $\mathcal{L}_{r+1}(D)$. If D has m arcs, them $\mathcal{L}_m(D) = K_1$. We define the *line-completion number* of D as the least r for which $\mathcal{L}_r(D)$ is complete. Clearly, the line completion number of D is 1 if and only if D is P_2 or C_2.

Theorem 6.4. *If D is a digraph with $m \geq 3$ arcs, then $\mathcal{L}_2(D)$ is complete if and only if D does not have disjoint C_2s and the subdigraph of D generated by any three arcs contains $\overrightarrow{C_2}$, $\overrightarrow{C_3}$, or $\overrightarrow{P_4}$.*

7. Total digraphs

Behzad [8] generalized the concept of line graph in a different way. He considered vertices and edges of a graph G as its elements, which then make up the vertices of the *total graph* $T(G)$ of G, with adjacency if the corresponding vertex-vertex, edge-edge or vertex-edge pairs are adjacent or incident in G.

Chartrand and Stewart [13] extended the concept of total graphs to total digraphs. In this section, we again assume that digraphs have no loops or multiple arcs. The *total digraph* $T(D)$ of a digraph D has the vertices and arcs of D as its vertices. For a pair of vertices x and y in $T(D)$, there are four ways to have an arc from x to y: (i) if both x and y are vertices in D and xy is an arc; (ii) both x and y are arcs in D and the head of x is the tail of y; (iii) x is a vertex and y is an arc in D and x is the tail of y; or (iv) x is an arc and y is a vertex in D and y is the head of x. It follows easily from the definition that D and $L(D)$ are disjoint subdigraphs of $T(D)$. Furthermore, each arc in D contributes to two arcs in $T(D)$ that have a head and a tail in each of the two subdigraphs. These observations give our first basic result.

Theorem 7.1. *If D is a digraph with n vertices and m arcs $(m, n > 1)$, then $T(D)$ has $m+n$ vertices and $3m + \sum_{v \in V(D)} d^+(v)d^-(v)$ arcs. If v is a vertex of D, then its out-degree in $T(D)$ is $2d^+(v)$ and its in-degree is $2d^-(v)$. If $e = vw$ is an arc of D, then its out-degree in $T(D)$ is $1 + d^+(w)$ and its in-degree is $1 + d^-(v)$.*

We observe that each arc in D gives a transitive triple in $T(D)$, each transitive triple gives a transitive triple and a 3-cycle gives two 3-cycles. It is also not hard to see that each 2-cycle in D gives two 2-cycles, two 3-cycles and two transitive triples in $T(D)$.

Theorem 7.2. *Let D be a digraph with n vertices and m arcs $(m, n > 1)$. If D has a transitive triples, b 2-cycles, and c 3-cycles, then $T(D)$ has $2b$ 2-cycles, $a + 2b + m$ transitive triples, and $2b + 2c$ 3-cycles.*

Total digraphs can be characterized in terms of the subdivision digraph of a digraph. We discuss this next. The *subdivision* of an arc uv in a digraph is the replacement of this arc by two new arcs uw and wv, where w is a new vertex. The *subdivision digraph* $S(D)$ is obtained from D by subdividing each arc of D. The *square* D^2 of a digraph D has the same vertices as those of D and uw is an arc in D^2 if the distance from u to w in D is 1 or 2. The next result follows easily from the definitions.

Theorem 7.3. *For a digraph D, $T(D) = S(D)^2$.*

Figure 27 (see Skowrońska, Sysło and Zamfirescu [23]) illustrates the concepts above for a digraph D.

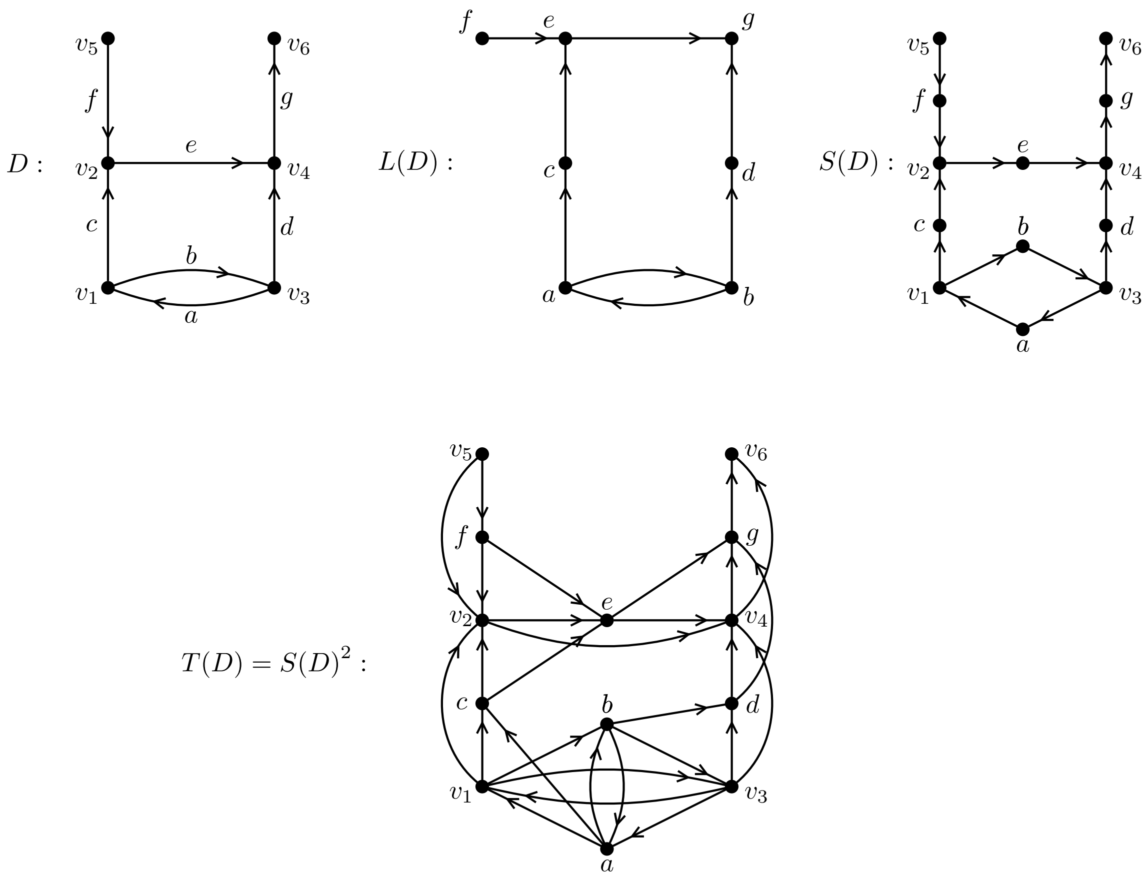

Figure 27: A digraph and its friends.

We turn to connectedness properties of total digraphs. Assume first that D is strongly connected. If u and v are vertices in $T(D) = S(D)^2$ that are also vertices in D, there is u–v path. If u and v are both arcs in D, we get a u–v path in $L(D)$ that begins with the arc from u to $head(u)$, followed by a path in D from $head(u)$ to $tail(v)$, and ends with the arc from $tail(v)$ to v. The cases when exactly one of u and v is a vertex and the other an arc in D are similar. Conversely, it follows directly from the definitions that if $T(D)$ is strongly connected, then so is D.

Theorem 7.4. *For a digraph D,*
(a) *D is strongly connected if and only if $T(D)$ is strongly connected.*
(b) *D is (weakly) connected if and only if $T(D)$ is weakly connected.*
(c) *D is acyclic if and only if $T(D)$ is acyclic.*

Liu and Meng [21] studied connectivity properties of total digraphs. For a digraph D, its minimum degree $\delta(D)$ is defined to be the smaller of the minimum in-degree and the minimum out-degree of D. An *arc-cut* of a strongly connected digraph D is a set of arcs whose removal leaves a digraph that is not strongly connected. The *arc-connectivity* $\lambda(D)$ of a strongly connected digraph D is the minimum cardinality of an arc-cut. Clearly, $\lambda(D) \leq \delta(D)$. We say that D is *maximally arc-connected* if $\lambda(D) = \delta(D)$. A strongly connected digraph D is said to be *super-arc-connected* if every minimum arc cut is a set of out-arcs or in-arcs at some vertex v. The (vertex) connectivity $\kappa(D)$, maximal (vertex) connectivity and super (vertex) connectivity are similarly defined. We can now state the results.

Theorem 7.5. *If a digraph D is strongly connected and has order at least 2, then*
(a) *$T(D)$ is maximally arc-connected. Additionally, if D has order at least 3, then $T(D)$ is super-arc-connected.*
(b) $\kappa(T(D) \geq min(\delta(T(D), \kappa(D) + \lambda(D))$.
(c) *If $\kappa(D) + \lambda(D) \geq \delta(D) + 1$, then $T(D)$ has maximum connectivity.*
(d) *If $\kappa(D) + \lambda(D)) > \delta(D) + 1$, then $T(D)$ is super (vertex) connected.*

In section 4 we provided some characterizations of line digraphs. We conclude this section with a similar characterization result for total digraphs [23]. We first define some terminology and notation. If three vertices u, v, w in a digraph D induce a transitive triple, that is, (without loss of generality) uv, uw, and vw are arcs in D, then, following [23], we call the arcs uv, and vw the *legs* and the arc uw the *hypotenuse* of the transitive triple. For a subset of arcs B of $A(D)$, the subdigraph of D generated by B is the digraph such that every vertex is on an arc of B, and its set of arcs is B. We denote by D^* the digraph generated by arcs that are not hypotenuses in D, and by D^{**} the subdigraph of D obtained by removing the arcs that are not hypotenuses. A vertex v of D is a *carrier* if $d^+(v) = d^-(v) = 1$.

Theorem 7.6. *A digraph D is a total digraph of a digraph if and only if the following conditions are satisfied:*
(a) *Every arc belongs to a transitive triple.*
(b) *Every hypotenuse is the hypotenuse of exactly one transitive triple whose legs are in D^*.*
(c)*] If arcs uv and vw are not hypotenuses, then uw is an arc in D.*
(d) *D^{**} has at least two (weakly connected) components.*
(e) *If u and v are not carriers in D^* and there exists a path in D^{**} from v to a vertex w, then uw and wu are not arcs in D^*.*

It is illustrative to verify the above result for the digraphs in Figure 27, for which it can be seen that $T(D)^* = S(D)$ and $T(D)^{**} = D \cup L(D)$. With the characterization above, an efficient algorithm has been developed (see [23]) for testing whether a given digraph is a total digraph.

8. Path digraphs

As we noted in section 1, the line graph of a graph G is obtained by mapping paths of length 2 in G to paths of length 1 in $L(G)$. Broersma and Hoede [11] generalized this to *path graphs*. For a graph G its path graph $\Lambda(G)$ has paths of length 2 of G as vertices, and two are adjacent if the corresponding 2-paths in G intersect in a path of length 1, or their union forms a path of length 3 or a 3-cycle.

Broersma and Li [12] generalized the concept of path graphs to path digraphs. For a given positive integer k, if D is a digraph with no loops and at least one $\overrightarrow{P_k}$, the *k-path digraph* of D is the digraph that has the k-paths of D as its vertices, with an arc from a vertex a to a vertex b if the intersection of the corresponding $\overrightarrow{P_k}$s is a $\overrightarrow{P_{k+1}}$ or their union is a $\overrightarrow{C_k}$. We denote the 3-path digraph of D by $\Lambda(D)$. Figure 28 illustrates the operations of forming $\Lambda(D)$. It follows from the definition that the k-path digraphs of D for $k = 1$ and $k = 2$ are D and $L(D)$ respectively. It is also clear that, for $2 \leq k \leq r$, the k-path digraphs of $\overrightarrow{P_r}$ and $\overrightarrow{C_r}$ are $\overrightarrow{P_{r-2}}$ and $\overrightarrow{C_r}$, respectively.

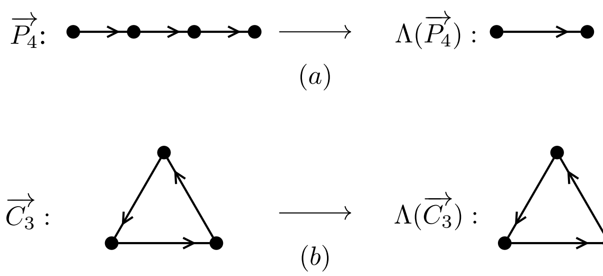

Figure 28: The operation of forming the path digraph.

We first describe an elementary result about order, in-degrees and out-degrees of path digraphs $\Lambda(D)$. We denote the set of vertices and the set of arcs in D by $V(D)$ and $A(D)$. For a vertex v in D, we set $\overline{A}_v = \{w : \{vw, wv\} \subseteq A(D)\}$, and $\overline{A}(D) = \{vw \in A(D) : wv \in A(D)\}$. The following results can be derived with a straightforward argument.

Theorem 8.1. *With notation as above, the number of vertices and arcs in $\Lambda(D)$ are given by $\sum_{v\in V(D)}(d^-(v)d^+(v) - |\overline{A}_v|)$ and $\sum_{vw\in A(D)}(d^-(v)d^+(w)) - \sum_{vw\in \overline{A}(D)}(d^-(v) + d^+(w)) + |\overline{A}(D)|$.*

Theorem 8.2. *For a vertex $a = uvw$ of $\Lambda(D)$, $d^-(a) = d^-(u) - |\{u\} \cap \overline{A}_v|$ and $d^+(a) = d^+(w) - |\{w\} \cap \overline{A}_v|$.*

The next result discusses the cycle structure in $\Lambda(D)$. We observe that it pertains to small cycles, but it is possible to derive similar results about long cycles.

Theorem 8.3. *Let D be a digraph with at least one $\overrightarrow{P_3}$. Then the following hold:*
(a) *$\Lambda(D)$ has no 2-cycles.*
(b) *If the underlying graph of $\Lambda(D)$ has a 3-cycle, then it is a (directed) 3-cycle in $\Lambda(D)$. In other words, $\Lambda(D)$ has no transitive triples.*
(c) *Each 4-cycle in the underlying graph of $\Lambda(D)$ is chordless and is a (directed) 4-cycle or has alternating arc directions in $\Lambda(D)$.*
(d) *No cycle of length 5 or more in the underlying graph of $\Lambda(D)$ is both chordless and oriented with alternating arc directions.*

In [12], the authors also investigated isomorphism questions such as when is $\Lambda(D) \cong D$, and for which digraphs does $\Lambda(D_1) \cong \Lambda(D_2)$ imply $D_1 \cong D_2$. We observe that if a digraph D is strongly connected, then D contains a spanning in-tree and a spanning out-tree. We conclude with three theorems involving isomorphisms between a path digraph and another digraph. We assume that all given path digraphs have at least one juncture vertex, that is, a vertex with both in-degee and out-degree positive (in other words, a 2-path).

Theorem 8.4. *If D is a connected digraph with no sources or sinks that has an in-tree or an out-tree, then $\Lambda(D) \cong D$ if and only if it is a cycle.*

As we observed earlier, every strongly connected digraph satisfies the hypotheses and so is not its own path digraph unless it is a cycle.

The next theorem gives a partial answer to the question of when do two digraphs have isomorphic path digraphs. We define a digraph to be *arc-extendible* if every arc is the middle arc of a path or cycle of length 3.

Theorem 8.5. *If D and F are connected arc-extendible digraphs, then every isomorphism from $\Lambda(D)$ to $\Lambda(F)$ is induced by an arc-isomorphism from D to F.*

The third isomorphism theorem combines iterated line digraphs and path digraphs.

Theorem 8.6. *If D is a digraph with a juncture vertex, then $\Lambda(D) \cong L^2(D)$.*

For a more detailed discussion, we refer the reader to [12, 20].

References

[1] M. Aigner, On the linegraph of a directed graph, *Math. Z.* **102** (1967) 56–61.
[2] J. Bagga, Old and new generalizations of line graphs, *Int. J. Math. Math. Sci.* **29** (2004) 1509–1521.
[3] K. S. Bagga, L. W. Beineke, B. N. Varma, Super line graphs, In: Y. Alavi, A. Schwenk (Eds.), *Graph Theory, Combinatorics, and Applications*, Vol. 1, Wiley, 1995, pp. 35–46.
[4] K. S. Bagga, L. W. Beineke, B. N. Varma, The line completion number of a graph, In: Y. Alavi, A. Schwenk (Eds.), *Graph Theory, Combinatorics, and Applications*, Vol. 2, Wiley, 1995, pp. 1197–1201.
[5] J. S. Bagga, L. W. Beineke, B. N. Varma, Independence and cycles in super line graphs, *Australas. J. Combin.* **19** (1999) 171–178.
[6] J. S. Bagga, L. W. Beineke, B. N. Varma, The super line graph $\mathcal{L}_2$, *Discrete Math.* **206** (1999) 51–61.
[7] J. S. Bagga, L. W. Beineke, B. N. Varma, A number-theoretic problem on super line graphs, *AKCE Int. J. Graphs Comb.* **13** (2016) 177–190.
[8] M. Behzad, *Graphs and Their Chromatic Numbers*, Doctoral thesis, Michigan State University, East Lansing, 1965.
[9] L. W. Beineke, J. S. Bagga, *Line Graphs and Line Digraphs*, Springer, To appear.
[10] L. W. Beineke, C. M. Zamfirescu, Connection digraphs and second-order line digraphs, *Discrete Math.* **39** (1982) 237–254.
[11] H. J. Broersma, C. Hoede, Path graphs, *J. Graph Theory* **13** (1989) 427–444.
[12] H. J. Broersma, X. Li, Isomorphisms and traversability of directed path graphs, *Discuss. Math. Graph Theory* **22** (2002) 215–228.
[13] G. Chartrand, M. J. Stewart, Total digraphs, *Canad. Math. Bull.* **9** (1966) 171–176.
[14] D. Ferrero, Some properties of superline digraphs, *International Conference on Information Technology Proceedings ITNG* 07, 1967, pp. 1073–1076.
[15] F. Harary, R. Z. Norman, Some properties of line digraphs, *Rend. Circ. Mat. Palermo* 2 **9** (1960) 161–168.
[16] R. L. Hemminger, Digraphs with periodic line digraphs, *Studia Sci. Math. Hungar.* **9** (1974) 27–31.
[17] R. L. Hemminger, Characterization of nth line digraphs, *Congr. Numer.* **69** (1989) 147–156.
[18] R. L. Hemminger, L. W. Beineke, Line graphs and line digraphs, In: L. W. Beineke, R. J. Wilson (Eds.), *Selected Topics in Graph Theory*, Academic Press, 1978, pp. 271–305.
[19] C. Heuchenne, Sur une certaine correspondance entre graphs, *Bull. Soc. Royal. Sci. Liège* **33** (1964) 743–753.
[20] X. Li, Y. Liu, B. Zhao, Isomorphisms of directed $\overrightarrow{P_3}$-graphs, *Ars Combin.* **101** (2011) 435–448.
[21] J. Liu, J. Meng, Super-arc-connected and super-connected total digraphs, *Ars Combin.* **87** (2008) 97–104.
[22] P. I. Richards, Precedence constraints and arrow diagrams, *SIAM Rev.* **9** (1967) 548–553.
[23] M. Skowrońska, M. Sysło, C. Zamfirescu, An algorithmic characterization of total digraphs, *J. Algorithms* **7** (1986) 120–133.
[24] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932) 150–168.