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*Research Article*

# Plane integral drawings of the platonic solid graphs with triangle faces*

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**Abstract**

The planar graphs of the platonic solids the tetrahedron, octahedron, and icosahedron can be drawn as triangulations of the plane. Such drawings are called *primitive integral plane graphs* if the edges are noncrossing straight line segments of integer lengths and if the greatest common divisor of the lengths is one. It is proved that for each of these three solids, there exist infinitely many primitive integral plane graphs. The simpler cases of the cube and dodecahedron are mentioned.

**Keywords:** platonic solids; plane integral graph drawing; pythagorean triple; diophantine equations.

**2020 Mathematics Subject Classification:** 05C10, 05C62, 11D09.

## 1. Introduction

Every planar graph can be drawn in the plane with noncrossing edges which are straight line segments (Steinitz and Rademacher, Wagner, Fáry, Stein – see a short proof in [7]). It is an open problem [2, 4] whether the edges also can be straight line segments of integral lengths. Nevertheless, one can try to construct for each planar graph $G$ such an *integral plane drawing*, denoted by $D(G)$. Moreover, it can be asked for the minimum diameter $d$ of $D(G)$, where $d$ denotes the largest edge length of a $D(G)$.

For the five platonic solids, the tetrahedron, octahedron, cube, dodecahedron, and icosahedron, the minimum diameters of their integral plane graphs have been determined in [5] to be 17, 2, 13, 2, and 159, respectively. Those three of them which are triangulations of the plane, that is, where the corresponding polyhedra have triangular faces, namely the tetrahedron, octahedron, and icosahedron, occur in connection with generalized matchstick graphs in [1].

It is asked in [1] for the minimum number $n_0(r, d)$ of vertices of an integral plane drawing (matchstick graph) of an $r$–regular planar graph ($r = 3, 4$, or 5 due to the Eulerian polyhedron formula) with given diameter $d$. Since $n_0(3, d) = 4$ for a plane tetrahedron graph, $n_0(4, d) = 6$ for a plane octahedron graph, and $n_0(5, d) = 12$ for a plane icosahedron graph are determined in [1] for many values of $d$, we will discuss here whether infinitely many values of $d$ are possible, that is, whether there exist infinitely many primitive integral plane drawings of these three planar platonic solid graphs.

## 2. Tetrahedron

The smallest integral plane drawing of the tetrahedron graph, that is, with the smallest diameter $d = 17$, can be seen in Figure 1 (see [1, 5]).

**Theorem 2.1.** *There exists an infinite number of primitive integral plane tetrahedron graphs determined by any pair of primitive pythagorean triangles.*

*Proof.* Primitive pythagorean triangles are used with legs of lengths $2mn$ and $m^2 - n^2$, and the hypotenuse of length $m^2 + n^2$ for parameters $m, n$ with $(m, n) = 1$ and $m \not\equiv n \pmod 2$ (see [11]).

Consider any two such triangles $(a, b, c)$ and $(d, e, f)$ with $a < b$ and $e < d$ for the pairs of legs $a, b$ and $d, e$, respectively. Then a multiplication of $(a, b, c)$ by $d$ and $(d, e, f)$ by $a$, and a reflection at the side of length $bd$ leads to the tetrahedron graph in Figure 2. Finally, division by $(a, d)$ results in a primitive solution. □

*Dedicated to the memory of Professor Frank Harary on the occasion of his 100th birthday with thanks for many great ideas.

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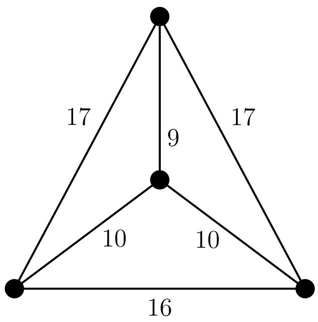


Figure 1: Tetrahedron graph with smallest diameter.

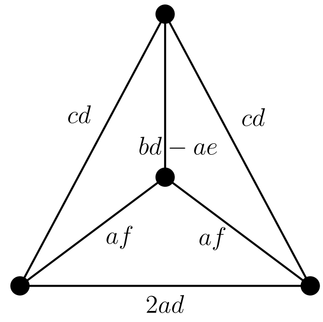


Figure 2: Tetrahedron graph from pythagorean triangles.

The smallest diameter $d$ in the case of Theorem 2.1 is $d = 17$ as in Figure 1, where pythagorean triangles $(a, b, c) = (8, 15, 17)$ and $(d, e, f) = (4, 3, 5)$ are used.

We next consider tetrahedra where edges incident to the inner vertex have equal length.

**Theorem 2.2.** *There exists an infinite number of primitive integral plane tetrahedron graphs where the three outer vertices lie on a circle with the inner vertex as its centerpoint.*

*Proof.* We use the following result of [10, pp. 7, 26]:

> "On a circle with diameter $R = \prod_{j=1}^{k} p_j^{\alpha_j}$, for any distinct prime numbers $p_j \equiv 1 \pmod 4$ there exist $n = \prod_{j=1}^{k}(\alpha_j + 1)$ pairs of opposite points with pairwise integral distances."

Then, from any $n \geq 3$ pairs of opposite points on the circle, we always can choose three points such that the center of the circle is inside of the triangle determined by the three points. Figure 3 shows an example for $R = 5^2$, which after multiplication by 2 implies the desired tetrahedron graph in Figure 4. □

By computer search (see [1]) all 499 integral plane tetrahedron graphs with diameter $d \leq 100$ have been found and it can be checked that $d = 48$ is the smallest diameter if the inner edges are of equal length as in Figure 4.

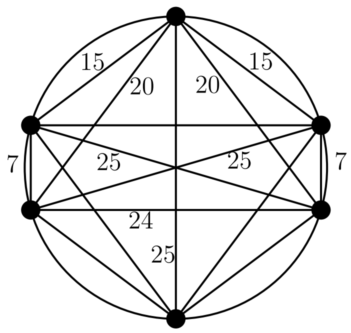


Figure 3: Pairwise integral distances for opposite pairs of points with diameter 25.

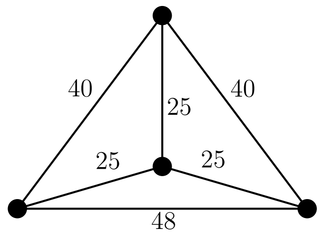


Figure 4: The smallest tetrahedron graph with equal inner edges.

What if the outer edges of the tetrahedron form an equilateral triangle?

**Theorem 2.3.** *There exists an infinite number of primitive integral plane tetrahedron graphs where the outer vertices are vertices of an equilateral triangle.*

*Proof.* We use the following result of [9, p. 48] (see also [6] and [3, D21]):

> "Any pair of rational numbers $(p, q) \neq (0, 0)$ in
>
> $$u = (p^2 + 4pq + q^2)^2 + 4(p^2 - q^2)^2$$
>
> and
>
> $$v = 2(p^2 + 4pq + q^2)(p^2 - q^2)$$
>
> determines four points with pairwise rational distances
>
> $$s,\ x = \left|\frac{s(u+v)}{u}\right|,\ y = \left|\frac{s(u-v)}{u}\right|,$$

$$\text{and } z = \left| \frac{8s}{u}(p^2 - q^2)(p^2 + pq + q^2) \right|$$

where any of these four distances can serve as the side lengths of an equilateral triangle, and then the fourth point has the remaining distances to the vertices of this triangle."

Choosing $s = u$ we obtain

$$\begin{aligned} s = u &= 5p^4 + 8p^3q + 10p^2q^2 + 8pq^3 + 5q^4, \\ x = u + v &= 7p^4 + 16p^3q + 10p^2q^2 + 3q^4, \\ y = u - v &= 3p^4 + 10p^2q^2 + 16pq^3 + 7q^4, \\ z &= 8p^4 + 8p^3q - 8pq^3 - 8q^4. \end{aligned}$$

Furthermore, we may choose $p = 4t$ and $q = 2t + 1$ for $t \geq 1$ to get

$$\begin{aligned} u &= 3280t^4 + 1696t^3 + 472t^2 + 72t + 5, \\ x &= 4528t^4 + 1760t^3 + 232t^2 + 24t + 3, \\ y &= 2032t^4 + 1632t^3 + 712t^2 + 120t + 7, \\ z &= 2688t^4 - 128t^3 - 384t^2 - 96t - 8. \end{aligned}$$

If now $x$ serves as the side length of the equilateral triangle, then it can be checked that for $t \geq 1$, the lengths $u, y$, and $z$ are less than $\frac{x}{2}\sqrt{3}$, so that the fourth point lies inside the triangle. For example, for $t = 1$, the tetrahedron graph in Figure 5 is determined. □

It was checked by computer in [9] that $d = 112$ is the smallest diameter for an equilateral triangle as shown in Figure 6.

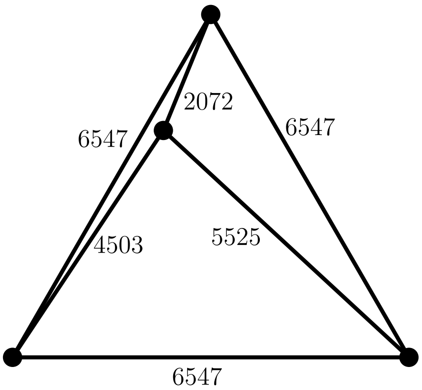


Figure 5: Equilateral triangle for $t = 1$.

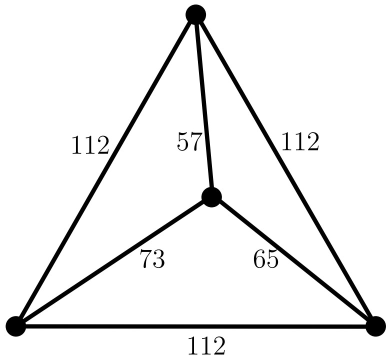


Figure 6: Minimum $d = 112$ for an equilateral triangle.

It may be remarked that due to a computer search it was conjectured in [1] that a primitive integral plane tetrahedron graph exists for any $d \geq 68$.

**Question 2.1.** *Is $d = 67$ the largest diameter $d$ such that a primitive integral plane tetrahedron graph does not exist?*

## 3. Octahedron

For the octahedron graph the smallest diameter was determined in [1,5] to be $d = 13$ (see Figure 8).

**Theorem 3.1.** *There exists an infinite number of primitive integral plane octahedron graphs.*

*Proof.* We use the primitive integral $120°$-triangles where the largest side is of length $m^2 + mn + n^2$ and where the legs of the angle of $120°$ are of lengths $2mn + n^2$ and $m^2 - n^2$ with parameters $m, n$ for $m \not\equiv n \pmod 3$ and $(m, n) = 1$ (see [8]).

For such a triangle $(a, b, c)$, where $a$ and $b$, with $a > b$, are the legs of the angle of $120°$, we consider an equilateral triangle of side length $2a + b$. Incident to its vertices we insert three pairs of triangles $(a, b, c)$, as in Figure 7. Then $x = a + b - 2b = a - b$ and after the deletion of the inner six edges of length $b$ the desired integral octahedron graph is complete. □

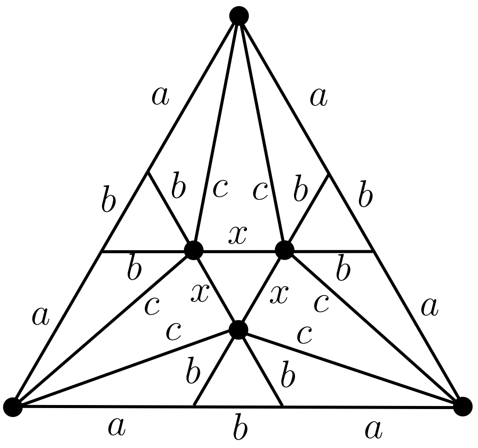


Figure 7: Construction of an integral octahedron graph.

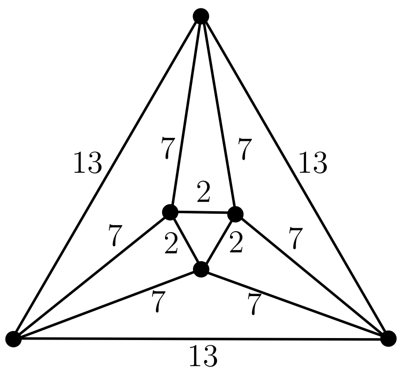


Figure 8: Octahedron minimum $d = 13$ $(m = 2,\ n = 1)$.

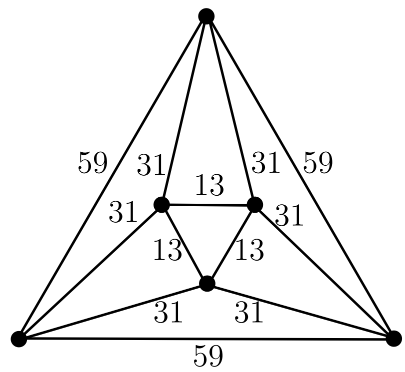


Figure 9: Octahedron graph $(m = 5,\ n = 1)$.

As examples, for $m = 2, n = 1$, the minimum case of Figure 8 (see [1, 5]) is given, and for $m = 5, n = 1$, the graph in Figure 9 is obtained.

In [1] for diameters up to 100 all 22 primitive integral plane octahedron graphs are listed.

**Question 3.1.** *Is there a largest distance $d$ such that a primitive integral plane octahedron graph does not exist?*

## 4. Icosahedron

The smallest diameter for the icosahedron graph is $d = 159$ (see [1, 5] and Figure 10).

**Theorem 4.1.** *There exists an infinite number of primitive integral plane icosahedron graphs.*

*Proof.* We will generalize the minimum drawing of Figure 10 which is symmetrical by rotation and which is determined by the three integral $120^o$-triangles

$$(b, c, d) = (40, 24, 56),$$
$$(u, v, w) = (16, 39, 49), \text{ and}$$
$$(p, q, r) = (95, 24, 109)$$

together with the three conditions (i) $b > c$, (ii) $u = b - c$, and (iii) $q = c$, $p = b + u + v$. Due to (i) the angle at $b$ in $(b, c, d)$ is less than $30^o$ so that the isosceles triangle with side lengths $d$ and $u$ does exist. Both, $b + c$ and $u + 2c$ are side lengths of the same equilateral triangle which determines (ii). Condition (iii) follows from the symmetry of Figure 10.

With parameters $i, j$ and $s, t$ we choose the two rational $120^o$-degree triangles

$$(b, c, d) = (2ij + j^2, i^2 - j^2, i^2 + j^2 + ij) \text{ and}$$

$$(u, v, w) = (k(s^2 - t^2), k(2st + t^2), k(s^2 + t^2 + st))$$

where $k = (2ij + 2j^2 - i^2)/(s^2 - t^2)$ is determined by (ii). Then all edges of the generalized Figure 10 are of rational lengths, except for those edges of length $r$. If $i = 2j + 1$ is chosen for $j \geq 2$, $j \not\equiv 2 \pmod 3$, then $(b, c, d)$ is primitive and $(i)$ is fulfilled.

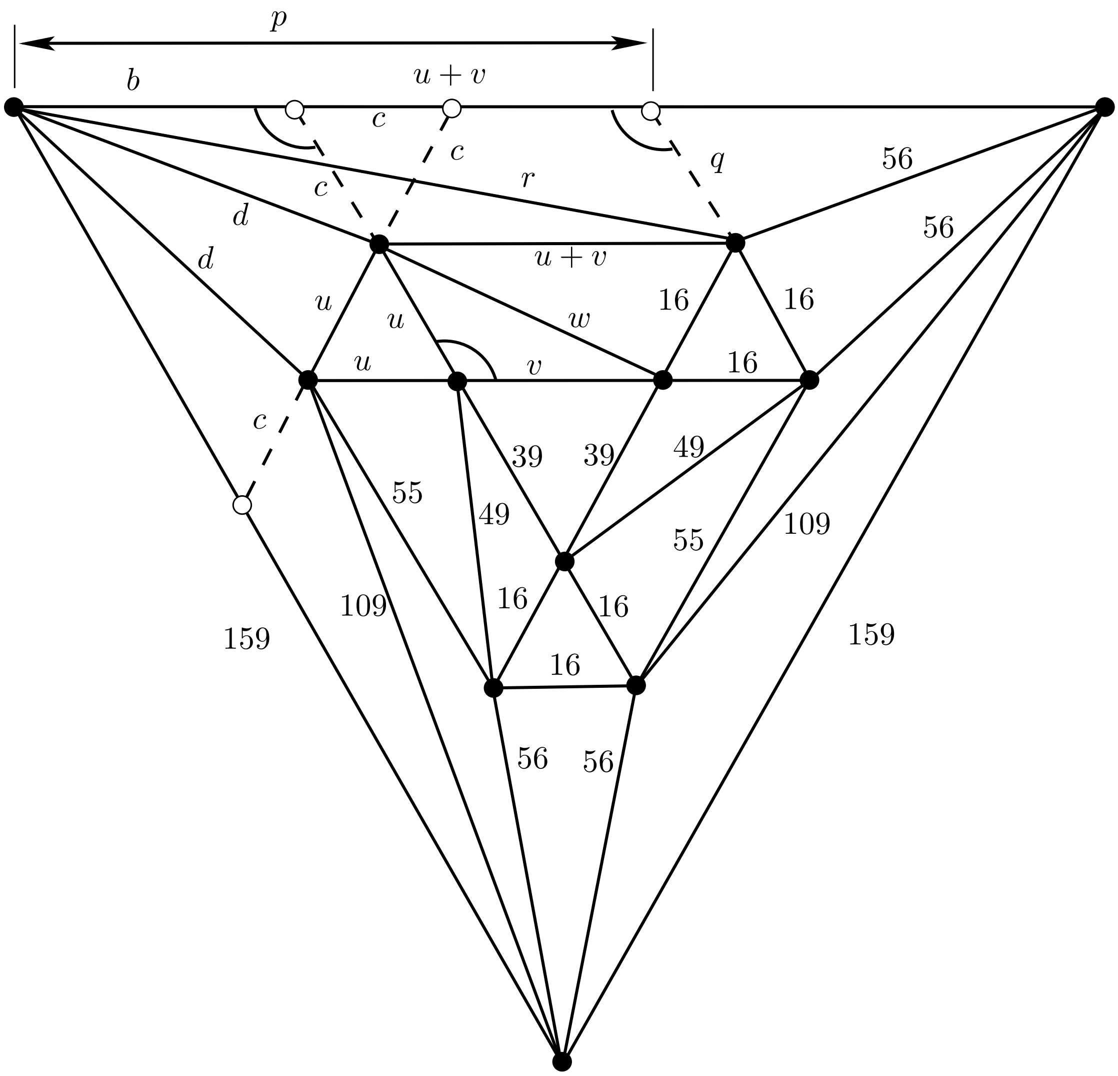


Figure 10: Icosahedron, minimum $d = 159$, and its generalization.

Moreover, we may choose $t = 1$. For $r$ the triangle $(p, q, r)$ yields $r^2 = p^2 + q^2 + pq$ and (iii) determines

$$(s^2 - 1)p = (s^2 - 1)(7j^2 - 1) + (2s + 1)(2j^2 - 2j - 1) \text{ and}$$
$$(s^2 - 1)q = (s^2 - 1)(3j^2 + 4j + 1).$$

By insertion of $p$ and $q$ into $r^2 = p^2 + q^2 + pq$ we obtain

$$(s^2 - 1)^2 r^2 = R^2 = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

with

$$a_4 = 79j^4 + 52j^3 + 12j^2 + 4j + 1,$$
$$a_3 = 68j^4 - 52j^3 - 54j^2 - 4j + 2,$$
$$a_2 = -108j^4 - 162j^3 - 51j^2 + 6j + 3,$$
$$a_1 = -52j^4 + 20j^3 + 54j^2 + 20j + 2,$$
$$a_0 = (7j^2 + 5j + 1)^2 = T^2.$$

Since $a_0 = T^2$ is a square number we can assume $R^2 = ms^2 + ns + T$ to get

$$R^2 = (ms^2 + ns + T)^2 = m^2 s^4 + 2mns^3 + (n^2 + 2mT)s^2 + 2nTs + T^2.$$

By comparison of the coefficients we obtain $a_4 s^4 + a_3 s^3 = m^2 s^4 + 2mns^3$, $a_2 = n^2 + 2mT$, and $a_1 = 2nT$. Insertion of the values for $a_\gamma$, $0 \le \gamma \le 4$, yields

$$n = \frac{a_1}{2T} = -\frac{26}{7} j^2 + \mathcal{O}(j),$$

$$m = \frac{a_2 - n^2}{2T} = -\frac{2984}{343} j^2 + \mathcal{O}(j), \text{ and}$$

$$s = s_0 = \frac{a_3 - 2mn}{m^2 - a_4} = \frac{68j^4 - 2\frac{2984}{343}\frac{26}{7}j^4 + \mathcal{O}(j^3)}{\frac{2984^2}{343^2}j^4 - 79j^4 + \mathcal{O}(j^3)} = -1{,}017\ldots + \mathcal{O}(j^{-1}).$$

This gives a rational solution $(s_0, R_0)$ yielding a rational $r$. However, since $s_0$ is negative a geometrical realization is impossible. If we then replace $s$ by $x + s_0$ in the equation for $R^2$ we obtain

$$R^2 = a_4' x^4 + a_3' x^3 + a_2' x^2 + a_1' x + a_0'$$

where

$$\begin{aligned}
a_4' &= a_4 = 79j^4 + \mathcal{O}(j^3),\\
a_3' &= 4a_4 s_0 + a_3 = -253{,}578\ldots j^4 + \mathcal{O}(j^3),\\
a_2' &= 6a_4 s_0^2 + 3a_3 s_0 + a_2 = 175{,}281\ldots j^4 + \mathcal{O}(j^3),\\
a_1' &= 4a_4 s_0^3 + 3a_3 s_0^2 + 2a_2 s_0 + a_1 = 46{,}046\ldots j^4 + \mathcal{O}(j^3),\\
a_0' &= a_4 s_0^4 + a_3 s_0^3 + a_2 s_0^2 + a_1 s_0 + a_0 = R_0^2 = \{1{,}770\ldots + \mathcal{O}(j^3)\}^2.
\end{aligned}$$

Since $a_0' = R^2$ is a square number again we can use the same procedure as above to obtain

$$\begin{aligned}
n' &= \frac{a_1'}{2R_0} = 13{,}005\ldots j^2 + \mathcal{O}(j),\\
m' &= \frac{a_2' - n'^2}{2R_0} = 1{,}733\ldots j^2 + \mathcal{O}(j), \text{ and}\\
x = x_0 &= \frac{a_3' - 2m'n'}{m'^2 - a_4} = 3{,}930\ldots + \mathcal{O}(j^{-1}).
\end{aligned}$$

Then $s_1 = x_0 + s_0 = 2{,}912\ldots + \mathcal{O}(j^{-1})$ gives a desired rational solution $(s_1, R_1)$ yielding a positive rational $r$ for any sufficiently large $j \not\equiv 2 \pmod 3$. After multiplication with the least common denominator a proof is complete. □

Of course, these constructed examples have extremely large integers. However, there are many other types than the minimum one and corresponding generalizations. For $d \leq 250$ we know 20 icosahedron graphs with diameters $d = 159, 160, 168, 205, 209, 218,$ and 247 with 2 and 13 different drawings for 247 and 205, respectively. With the denotations of Figure 11 these drawings are presented in Table 1.

**Question 4.1.** *Is there an arbitrarily large diameter for which a primitive integral plane icosahedron graph does not exist?*

| $(a_1a_2a_3)(a_{11}a_{12}a_{13}a_{21}a_{22}a_{23}a_{31}a_{32}a_{33})(b_{11}b_{12}b_{21}b_{22}b_{31}b_{32})(c_{11}c_{12}c_{13}c_{21}c_{22}c_{23}c_{31}c_{32}c_{33})(c_1c_2c_3)$ |
|---|
| (159,159,159)(56,56,109,56,56,109,56,56,109)(16,55,16,55,16,55)(49,16,16,49,16,16,49,16,16)(39,39,39) |
| (160,160,160)(77,77,93,77,77,93,77,77,93)(22,17,22,17,22,17)(21,13,13,21,13,13,21,13,13)(16,16,16) |
| (168,168,168)(79,79,103,79,79,103,79,79,103)(11,26,11,26,11,26)(14,14,9,14,14,9,14,14,9)(15,15,15) |
| (205,205,168)(117,110,117,92,75,85,85,75,92)(29,29,29,40,40,29)(25,6,25,25,6,25,25,25,25,)(40,48,40) |
| (205,205,168)(117,110,117,92,75,85,85,75,92)(29,29,29,40,40,29)(25,6,25,25,6,25,25,39,25,)(30,48,30) |
| (205,205,168)(117,110,117,92,75,85,85,75,92)(29,29,29,40,40,29)(25,6,25,25,6,25,30,50,30,)(25,48,25) |
| (205,205,168)(117,110,117,92,75,85,85,75,92)(29,29,29,40,40,29)(20,21,20,20,21,20,24,32,24,)(20,24,20) |
| (205,205,168)(117,110,117,92,75,85,85,75,92)(29,29,29,40,40,29)(20,21,20,25,36,25,24,32,24,)(20,15,7) |
| (205,205,168)(117,110,117,92,75,85,85,75,92)(29,29,29,40,40,29)(25,6,25,25,36,25,25,25,25,)(40,30,14) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(25,6,25,25,6,25,25,3,25)(40,48,40) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(25,6,25,25,6,25,25,17,25)(30,48,30) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(25,6,25,25,6,25,30,28,30)(25,48,25) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(20,21,20,20,21,20,24,10,24)(20,24,20) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(20,21,20,25,36,25,24,10,24)(20,15,7) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(20,21,20,20,21,20,25,17,25)(15,24,15) |
| (205,205,168)(117,110,117,92,75,91,91,75,92)(29,29,29,26,26,29)(25,36,25,25,6,25,25,3,25)(14,30,40) |
| (209,209,209)(91,91,129,91,91,129,91,91,129)(26,40,26,40,26,40)(39,19,19,39,19,19,39,19,19)(29,29,29) |
| (218,218,218)(112,112,110,112,112,110,112,112,110)(32,42,32,42,32,42)(13,35,13,13,35,13,13,35,13)(13,13,13) |
| (247,247,247)(120,91,133,120,91,133,120,91,133)(49,49,49,49,49,49)(26,21,35,35,21,56,56,21,35)(49,35,56) |
| (247,247,247)(120,91,133,120,91,133,120,91,133)(49,49,49,49,49,49)(56,21,35,56,21,35,56,21,35)(49,49,49) |

Table 1: Known primitive integral icosahedron graphs with diameter $d = a_1 \leq 250$ and denotation of Figure 11.

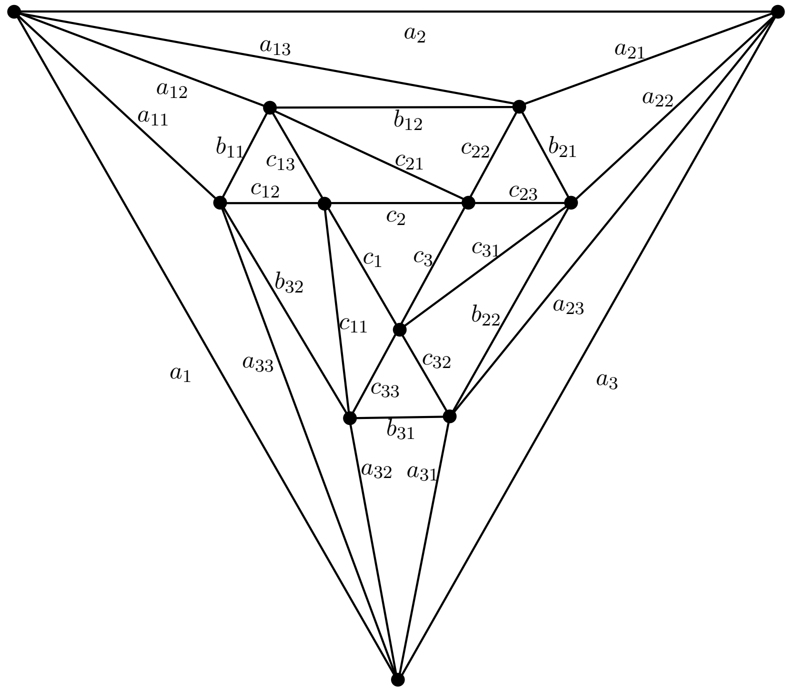


Figure 11: Denotations: $(a_i)(a_{ij})(b_{ik})(c_{ij})(c_i)$, $i = 1, 2, 3$; $j = 1, 2, 3$; $k = 1, 2$.

## 5. Concluding remarks

For the two remaining platonic solids, the cube and the dodecahedron, infinitely many primitive integral plane drawings can be constructed as follows (see Figures 12 and 13).

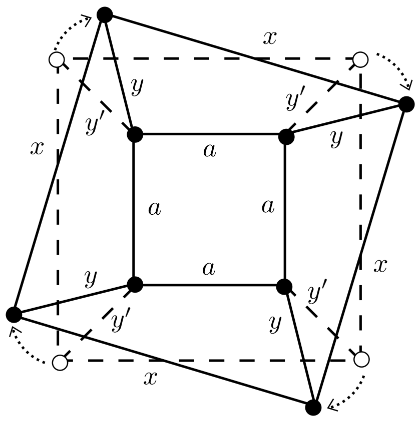


Figure 12: Integral cube graph.

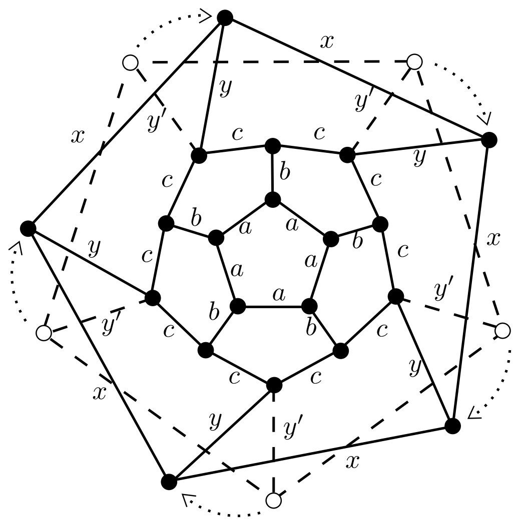


Figure 13: Integral dodecahedron graph.

We start with a square or a regular pentagon of side length $a$ together with edges of integer lengths $b$ and $c$ as in Figures 12 and 13, respectively. Now consider the dashed square or pentagon of side length $x, (x, a) = 1$, being concentric and with sides parallel to the first square or pentagon, respectively. Then lift it up parallel to the plane as far as the distances $y'$ become an integer $y$, then turn it about the common centerpoint to screw it back into the plane preserving the rigid edges of length $y$, and the desired drawing is constructed. Altogether, we have the following result.

**Theorem 5.1.** *There exist infinitely many primitive integral plane drawings for each of the five platonic solid graphs.*

In general, we may expand the open problem [2, 4] mentioned in the Introduction in the following way.

**Conjecture 5.1.** *Every planar graph can have infinitely many primitive integral plane drawings and there is a maximum diameter $d = d_0$ for which such a drawing does not exist.*

For the cube and dodecahedron $d_0 = 1$ follows from the above constructions.

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