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Research Article

On 4-colorable robust critical graphs*

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Abstract

Given a proper k-coloring of a graph G, a vertex v is locally recolorable if there is a proper k-coloring of the graph that changes the color of v and limits any other color changes to the neighbors of v. The coloring is robust if every vertex is locally recolorable. The robust chromatic number of G, $\chi_R(G)$, is the smallest number k for which G has a robust k-coloring. If $\chi_R(G) = \chi(G)$, the graph is χ-robust and if deleting any vertex of a χ-robust graph decreases $\chi_R(G)$, the graph is χ-robust-critical. We conjecture that only complete graphs are χ-robust-critical. This paper investigates this conjecture for $\chi = 4$ and supports the conjecture for a large class of such graphs. Furthermore, conditions that must be satisfied for such graphs are determined.

Keywords: robust coloring; chromatic number; χ-robust-critical.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

As introduced by Anderson, Brigham, Dutton and Vitray in 2014 [1], if c is a proper k-coloring of a graph G, a vertex v is **locally recolorable with respect to c** if there is a proper k-coloring c^v of G such that $c^v(v) \neq c(v)$ and $c^v(x) = c(x)$ for all $x \in V(G) - N[v]$ ($N(v)$ and $N[v]$ are the open and closed neighborhoods, respectively, of a vertex v). The coloring c^v is called a **local recoloring of v with respect to c**. A proper coloring c is **robust** if every vertex of G is locally recolorable with respect to c, and G is k-robust if it has a robust k-coloring. The smallest k such that G has a robust k-coloring is the **robust chromatic number of G**, denoted $\chi_R(G)$.

Any proper k-coloring of a graph is also a robust $(k+1)$-coloring of that graph, since the extra color can be used to locally recolor any vertex. Therefore,

$$\chi(G) \leq \chi_R(G) \leq \chi(G) + 1$$

where $\chi(G)$ is the chromatic number of G. There are a number of results throughout graph theory where one parameter is known to be one of two consecutive numbers. Because of these inequalities, all graphs G fall into one of two classes. A graph G is **χ-robust** if $\chi_R(G) = \chi(G)$, and G is **χ-robust-critical** if G is both χ-robust and $\chi_R(G - v) < \chi_R(G)$ for all $v \in V(G)$. Note that, if G is χ-robust-critical, then it is vertex χ-critical, that is, $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

The complete graph K_n is χ-robust-critical for $n \geq 3$, since $\chi(K_n) = \chi_R(K_n) = n$ and $\chi_R(K_n - v) = n - 1$ for any vertex v. A natural and useful relation between G and its induced subgraphs is expressed in the following proposition.

Proposition 1.1. *If H is an induced subgraph of G, then $\chi_R(H) \leq \chi_R(G)$.*

Proof. If H is an induced subgraph of G and c is a proper k-coloring of G, then the restriction $c|_H$ of c to vertices in H is a proper coloring of H. Furthermore, for any vertex v of H, if c^v is a local recoloring of v with respect to c, then $c^v|_H$ is a local recoloring of v with respect to c_H. □

The previous proposition is not true without the word "induced." For the graph $K_2 \times K_3$ in Figure 1, removing an edge can increase the robust chromatic number from three to four. This graph is vertex transitive, and, up to symmetry, has only one proper 3-coloring, illustrated as shown in Figure 1. Figure 1 also shows a local recoloring of the white vertex. So $\chi_R(K_2 \times K_3) = 3$. The non-induced subgraph shown in Figure 2 has two proper 3-colorings, up to symmetry, both shown. Neither is robust; the white vertex in each graph is not locally recolorable.

*This paper is dedicated to the memory of Frank Harary.
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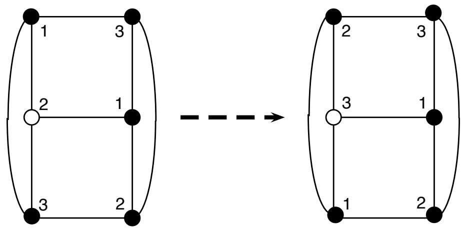


Figure 1: Every proper 3-coloring of $K_2 \times K_3$ is robust.

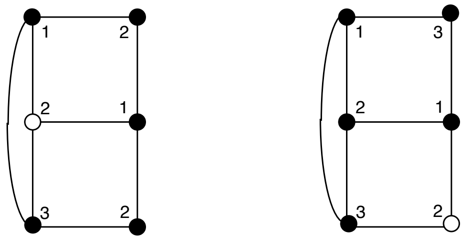


Figure 2: The two proper 3-colorings of a subgraph of $K_2 \times K_3$. In each, the white vertex is not locally recolorable.

We use K_n^- to denote the graph K_n minus an edge. The robust chromatic number of K_n^- is n, implying the following result.

Proposition 1.2. *If a graph G is $(n-1)$-robust, then G does not contain the graph K_n^- as a subgraph.*

Proof. If G contains K_n^- as a subgraph, then either K_n^- or K_n is an induced subgraph of G and so, by Proposition 1.1, $\chi_R(G) \geq n$. □

This leads to a result about χ-robust-critical graphs.

Corollary 1.1. *If a χ-robust-critical graph G with chromatic number n contains K_n^- as a subgraph, then $G = K_n$.*

If a bipartite graph G has a vertex v of degree two or larger, then for any 2-coloring of G, no neighbor of v is locally-recolorable. Therefore, the only connected graphs which have robust chromatic number 2 are K_1 and K_2. This implies there are no χ-robust-critical graphs with chromatic number 2. It also implies that deleting a vertex from a χ-robust-critical graph with chromatic number 3 will result in a graph with maximum degree less than or equal to 1. Since K_3 is the only connected, 3-chromatic graph with this property, it is the only χ-robust-critical graph with chromatic number 3. Failure to find any χ-robust-critical graphs other than complete graphs leads us to propose the following conjecture.

Conjecture 1.1. *A graph is χ-robust-critical if and only if it is a complete graph on 3 or more vertices.*

This paper supports Conjecture 1.1 for a large collection of graphs having chromatic number 4. If $\chi(G) = 4$, G is a χ-robust-critical graph, and $v \in V(G)$, then $G - v$ is 3-robust. Accordingly, the next section establishes results for 3-robust graphs.

2. Graphs with robust chromatic number 3

Throughout this section, we assume that H is a graph with $\chi_R(H) = 3$.

Definition 2.1. *Let v be a vertex in a triangle T and z an adjacent vertex not in T. Vertex z is a* **sidekick of T at v** *if v is the only vertex in T adjacent to z. A set of sidekicks of T is* **free** *if no two are adjacent to the same vertex in T.*

In the graph in Figure 3, vertices t, x, y, and z are all sidekicks of triangle T at u, u, w, and v, respectively. The sets $\{x, y, z\}$, and $\{t, y, z\}$ are free, whereas $\{x, t\}$ is not. More informally, we say y and z are free sidekicks of T, whereas x and t are not.

Proposition 2.1. *If c is a robust 3-coloring of a triangle with two free sidekicks, then the two sidekicks must have different colors.*

Proof. Let c be a robust 3-coloring of a triangle T with two free sidekicks, x and y, as shown in Figure 4. Assume, by way of contradiction, that $c(x) = c(y)$. Since $c(v) \neq c(u)$ and $c(v) \neq c(w)$, $c(v) = c(x) = c(y)$. It follows that, v is not locally recolorable, contradicting the robustness of c. □

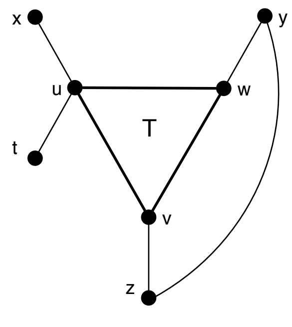


Figure 3: A triangle with four sidekicks.

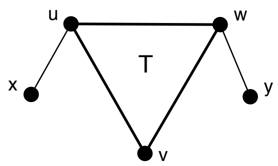


Figure 4: A triangle with two free sidekicks.

Proposition 2.2. *If c is a robust 3-coloring of H and $\delta(H) \geq 3$, then*
(i) any three free sidekicks of a triangle are assigned three different colors,
(ii) any set of sidekicks of a triangle at the same vertex have the same color,
(iii) no two triangles share a vertex, and
(iv) the set of sidekicks of a triangle at a single vertex forms an independent set.

Proof. (i) This is an immediate consequence of Proposition 2.1.
(ii) Let T be a triangle with vertices u, v, and w, and suppose t and x are sidekicks of T at vertex u, as in Figure 3. Since $\delta(H) \geq 3$, there is at least one sidekick y of T at w and one sidekick z at v, and by Proposition 1.2, $y \neq z$. By (i), t, y and z have different colors and x, y and z have different colors. Therefore, since c is proper 3-coloring, $c(t) = c(x)$.
(iii) Suppose T_1 and T_2 are two different triangles containing u. By Proposition 1.2, no subgraph of H is isomorphic to K_4^-, and therefore, u is the only vertex common to both triangles. However, the two vertices in $V(T_2) - \{u\}$ are both sidekicks of T_1 at u and therefore, by (ii), have the same color. This contradicts that c is a proper coloring.
(iv) This follows from (ii). □

Due to Proposition 2.2, triangles in H come in two varieties.

Definition 2.2. *Suppose c is a robust 3-coloring of H with colors 0, 1, 2. Let T be a triangle in H. If for every $u \in V(T)$, and every sidekick x of T at u, $c(x) = c(u) + 1 \mod 3$, we say T is a* **+** *triangle. Similarly, T is a* **–** *triangle when $c(x) = c(u) - 1 \mod 3$.*

Proposition 2.3. *If c is a robust 3-coloring of H and $\delta(H) \geq 3$, then*
(i) every triangle in H is either $+$ or $-$ and
(ii) any two triangles in H containing adjacent vertices must be of opposite signs.

Proof. (i) Let T be a triangle in H with vertices u_0, u_1, u_2 where $c(u_i) = i$ for $i \in \{0, 1, 2\}$. Since $\delta(H) \geq 3$, all three vertices in T are adjacent to some vertex not in T. Since H does not contain K_4^- as a subgraph, these vertices must be sidekicks. Suppose that x_0 is a sidekick of T at u_0 with $c(x_0) = 1$. By Proposition 2.1, a sidekick of T at u_2 cannot be colored 1, hence, it must be colored 0. This implies, by Proposition 2.2(i), that a sidekick of T at u_1 is colored 2. By Proposition 2.2(ii), T is $+$. Similarly, if $c(x_0) = 2$, then T is $-$.

(ii) Let u and v be adjacent vertices in distinct triangles T_u and T_v, respectively. If T_u is $+$, then $c(v) = c(u) + 1 \mod 3$, which implies $c(u) = c(v) - 1 \mod 3$. Therefore, by (i), T_v is $-$. Similarly, if T_u is $-$, then T_v is $+$. □

3. The main result

We will assume throughout this section that $\chi(G) = 4$, G is a χ-robust-critical graph not equal to K_4, and all colorings are proper.

The following lemma establishes a lower bound for the minimum degree of G. Since G is vertex χ-critical, $\delta(G) \geq \chi(G) - 1 = \chi_R(G) - 1$. Furthermore, $G \neq K_4$ implies G does not contain a subgraph isomorphic to K_4^-, by Corollary 1.1.

Lemma 3.1. *If G is a χ-robust-critical graph not equal to K_4 and $\chi(G) = 4$, then $\delta(G) \geq 4$.*

Proof. By the comment preceding the lemma, $\delta(G) \geq 3$. Suppose $v \in V(G)$ has degree 3. By definition of χ-robust-critical, there is a robust 3-coloring of $G - v$. Since $N[v]$ cannot be K_4^-, some vertex of $N(v)$ is not adjacent to either of the others. By locally recoloring that vertex, if necessary, we can obtain a 3-coloring of $G - v$ which uses no more than 2 colors for the vertices in $N(v)$. By assigning the third color to v we obtain a 3-coloring of G which contradicts $\chi(G) = 4$. □

Lemma 3.2. *If G is a χ-robust-critical graph not equal to K_4 and $\chi(G) = 4$, then $|V(G)| \geq 10$.*

Proof. The Grötzch graph [2], which has 11 vertices, is the smallest triangle-free graph with chromatic number 4; so, we may assume G contains a triangle T with vertices u_0, u_1, and u_2. By Lemma 3.1, there are two vertices x_0 and y_0 in $N(u_0) - V(T)$. Since K_4^- is not a subgraph of G, x_0 and y_0 are both sidekicks of T at u_0. Similarly, there exist sidekicks x_1 and y_1 of T at u_1 and sidekicks x_2 and y_2 at u_2. Since $G - x_1$ has a robust 3-coloring, x_0 and y_0 are not adjacent, by Lemma 2.2(iv). Likewise, x_1 and y_1 are not adjacent and x_2 and y_2 are not adjacent. There may, however, be edges between pairs of free sidekicks of T (between x_1 and x_2, for example). In any event, $c(u_i) = i$ and $c(x_i) = c(y_i) = i + 1 \mod 3$ for $i \in \{0, 1, 2\}$ is a proper 3-coloring of the subgraph H induced by $N(u_0) \cup N(u_1) \cup N(u_2)$. However, if $|V(G)| \leq 9$ then $G = H$, which contradicts $\chi(G) = 4$. □

We will use the results of Section 2 by deleting a vertex z of G to produce a subgraph with a robust 3-coloring.

Proposition 3.1. *If G is a χ-robust-critical graph not equal to K_4 and $\chi(G) = 4$, then*
(i) every vertex of G is in at most one triangle, and
(ii) the set of sidekicks of a triangle at a single vertex is independent.

Proof. (i) Suppose vertex u is on two triangles. By Lemma 3.2, there is a vertex z that is not on either triangle. Since G is χ-robust-critical, $G - z$ has a robust 3-coloring, and by Lemma 3.1, $\delta(G - z) \geq 3$. However, u is on two triangles in $G - z$, contradicting Proposition 2.2(iii).

(ii) This follows immediately from (i). □

Note that if x is a sidekick of a triangle, then a local recoloring of x cannot alter the colors assigned to the vertices of that triangle.

Definition 3.1. *A vertex is* **mono-triangular** *if it is in exactly one triangle and each of its neighbors is also in exactly one triangle.*

When a vertex v is in exactly one triangle, we use T_v to designate that triangle. If G has a mono-triangular vertex v, then by Lemma 3.1, the triangle T_v has at least two sidekicks x and y at v. The next lemma shows that there is always a vertex z, not in any of the triangles T_v, T_x, and T_y, whose deletion maintains the mono-triangularity of v.

Lemma 3.3. *Suppose G is a χ-robust-critical graph and $\chi(G) = 4$. If v is a mono-triangular vertex of G and $\{x, y\} \subseteq N(v) - V(T_v)$ then there exists a vertex $z \in V(G) - (V(T_v) \cup V(T_x) \cup V(T_y))$ such that v is a mono-triangular vertex of $G - z$.*

Proof. By Lemma 3.2, $|V(G)| \geq 10$, which implies $V(G) - (V(T_v) \cup V(T_x) \cup V(T_y)) \neq \emptyset$. Suppose $N[v] \subseteq V(T_v) \cup V(T_x) \cup V(T_y)$ and let $z \in V(G) - (V(T_v) \cup V(T_x) \cup V(T_y))$. Since T_v, T_x, and T_y are the only triangles containing a vertex in $N[v]$, v is a mono-triangular vertex of $G - z$. On the other hand, if $N[v] \not\subseteq V(T_v) \cup V(T_x) \cup V(T_y)$, there exists $z \in N(v) - (V(T_v) \cup V(T_x) \cup V(T_y))$. Since z is a sidekick of T_v at v and, by Proposition 3.1(ii), the set of all sidekicks of T_v at v is independent, $z \notin V(T_u)$, for any $u \in N[v] - \{z\}$ and so v is mono-triangular in $G - z$. □

Using Proposition 2.3, we impose a structure on the set of triangles which contain vertices in the neighborhood of a mono-triangular vertex.

Corollary 3.1. *Suppose G is a χ-robust-critical graph with $\chi(G) = 4$ and v is a mono-triangular vertex of G. If x and y are distinct sidekicks of T_v at v, then $V(T_x) \cap V(T_y) = \emptyset$ and no vertex in T_x is adjacent to a vertex in T_y.*

Proof. By Lemma 3.3, there exists $z \in V(G) - (V(T_v) \cup V(T_x) \cup V(T_y))$ such that v is a mono-triangular vertex of $G - z$. Let c be a robust 3-coloring of $G - z$. Proposition 2.2(iii), with $H = G - z$, implies $V(T_x) \cap V(T_y) = \emptyset$. Now, suppose a vertex in T_x is adjacent to a vertex in T_y. By Proposition 2.3(ii), one of T_x and T_y must be $+$ and the other must be $-$. Since x and y are in $N(v)$, Proposition 2.3(ii) also implies T_v is both $-$ and $+$, an impossibility. □

If T_v is $+$ for some 3-coloring of G, then we can obtain a local recoloring of v without changing the colors of any vertices not in $V(T_v)$ (see Figure 5) by subtracting 1 from each of the colors on the triangle. Similarly, if T_v is $-$, we can add 1 to each color on the triangle. As indicated in the figure, these recolorings change a triangle from $+$ to $-$ and vice versa.

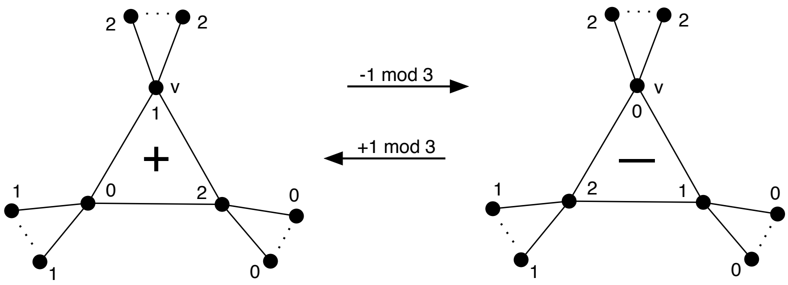


Figure 5: Recoloring signed triangles.

The colorings in Figure 5 play a key role in the proof of the next theorem.

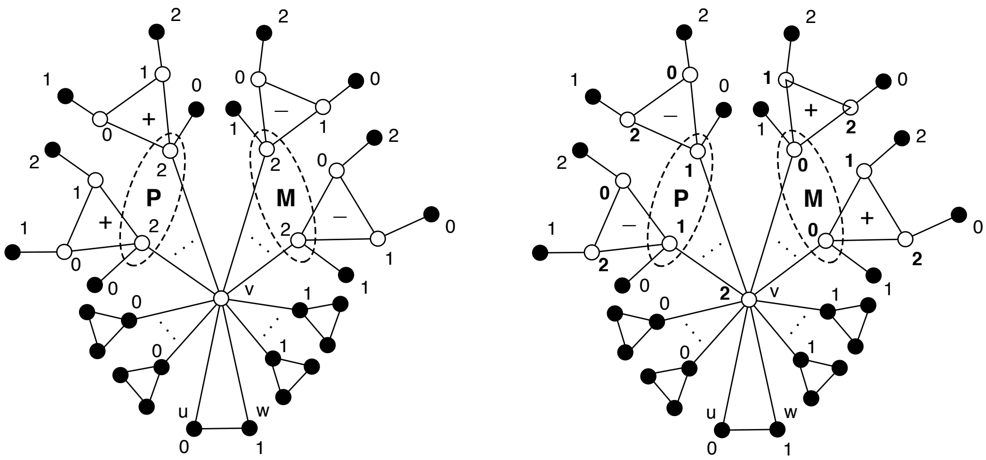
Theorem 3.1. *If G contains a mono-triangular vertex v, then G is not χ-robust-critical with chromatic number 4.*

Proof. Suppose G has a mono-triangular vertex v and is χ-robust-critical with chromatic number 4. Let c be a robust 3-coloring of $G - v$. Let u and w be the other two vertices of T_v. Without loss of generality, assume $c(u) = 0$ and $c(w) = 1$. Let $P = \{r \in N(v) - \{u, w\} : c(r) = 2 \text{ and } T_r \text{ is } +\}$ and $M = \{r \in N(v) - \{u, w\} : c(r) = 2 \text{ and } T_r \text{ is } -\}$. See Figure 6(a).

We define a coloring c' of G by,

$$c'(z) = \begin{cases} 2 & \text{if } z = v \\ c(z) - 1 \mod 3 & \text{if } z \in T_r, \text{ where } r \in P \\ c(z) + 1 \mod 3 & \text{if } z \in T_r, \text{ where } r \in M \\ c(z) & \text{otherwise.} \end{cases}$$

See Figure 6(b). Notice c' assigns a color to v and changes the colors only of vertices in the triangles containing a vertex of $N(v)$ colored 2 by c.



(a) Coloring *c* of *G-v*. (b) Coloring *c'* of *G*.

Figure 6: Constructing a 3-coloring of G.

We now show that c' is a proper 3-coloring of G, contradicting the hypothesis that $\chi(G) = 4$. Suppose x and y are adjacent vertices in G. All the arithmetic in the cases below is modulo 3.

Case 1. One of the vertices, say y, is equal to v. This implies $x \in N(v)$. If $c(x) = 2$, then $x \in P \cup M$ and $c'(x) \neq 2 = c'(v)$. On the other hand, if $c(x) \neq 2$, then $x \notin P \cup M$ and $c'(x) = c(x) \neq 2 = c'(v)$.

Case 2. Neither x nor y is v.

Subcase 2a. Neither x nor y is in $V(T_r)$ for any $r \in P \cup M$. By definition of c', $c'(x) = c(x) \neq c(y) = c'(y)$.

Subcase 2b. Both x and y are in $V(T_r)$ for some $r \in P \cup M$. If $r \in P$, then $c'(x) = c(x) - 1$ and $c'(y) = c(y) - 1$. If $r \in M$, $c'(x) = c(x) + 1$ and $c'(y) = c(y) + 1$. In either case, we have $c'(x) \neq c'(y)$, since $c(x) \neq c(y)$.

Subcase 2c. For some $r \in P \cup M$, $x \in V(T_r)$, but $y \notin V(T_r)$ (i.e., y is a sidekick of T_r at x). By Corollary 3.1, $y \notin V(T_{r'})$ for any $r' \in P \cup M$, and hence, $c'(y) = c(y)$. If $r \in P$, then $c(y) = c(x) + 1$ and $c'(x) = c(x) - 1$. Thus, $c'(x) \neq c'(y)$. Similarly, $c'(x) \neq c'(y)$ if $r \in M$.

Since $c'(x) \neq c'(y)$ in all cases, c' is a proper 3-coloring of G, which contradicts G having chromatic number 4. □

Definition 3.2. *A graph G is called an* **MT-graph** *if every vertex in G is mono-triangular.*

Theorem 3.1 shows that no MT-graph of chromatic number 4 is χ-robust-critical. The next theorem shows that all χ-robust-critical graphs with small maximum degree are MT-graphs.

Theorem 3.2. *If $G \neq K_4$ is a χ-robust-critical graph with $\chi(G) = 4$ and $\Delta(G) \leq 5$, then G is an MT-graph.*

Proof. Let v be a vertex in G. By Proposition 3.1(i), v is in at most one triangle. Thus, it suffices to show that v is in at least one triangle, that is, that $N(v)$ is not an independent set. For any proper 3-coloring of $G - v$, every color in $\{0, 1, 2\}$ is assigned to at least one vertex in $N(v)$, or else the coloring could be extended to a proper 3-coloring of G. Let c be a robust 3-coloring of $G - v$. Since $\deg(v) < 6$, there is some color i with $|c_i \cap N(v)| = 1$. Let u be the vertex in $N(v)$ with $c(u) = i$. There is a local recoloring c^u of u with respect to c and some vertex w in $N(v)$ has $c^u(w) = i$, otherwise c^u could be extended to a 3-coloring of G. Therefore, u and w are adjacent and $N(v)$ is not independent. □

Theorems 3.1 and 3.2 imply the following.

Theorem 3.3. *If G is not K_4 and G is χ-robust-critical with chromatic number 4, then $\Delta(G) \geq 6$.*

4. Open problems

1. Prove or disprove: there are no triangle-free χ-robust-critical graphs with chromatic number 4.
2. Prove or disprove: there are no χ-robust-critical graphs with $\Delta(G) \geq 6$ and chromatic number 4.
3. Find families of graphs that do or do not contain χ-robust-critical graphs.
4. Prove or disprove: there are no χ-robust-critical graphs with chromatic number 5 other than K_5.
5. Prove or disprove Conjecture 1.1.

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