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Research Article

Generalized target functions on trees*

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Abstract

A location function on a finite connected graph G takes as input any k-tuple of vertices (a profile) and outputs a single vertex. If G is a fully-gated graph, then a target location function is defined by a predetermined vertex (the target) and outputs the unique vertex belonging to the convex closure of the profile which is closest to the target. If G is a tree, which is such a fully-gated graph, then any target function on G satisfies two conditions known in the literature as Weak Pareto Efficiency and Replacement Domination. In the continuous case, where edges can be seen as segments with interior points, these two conditions fully characterize the target functions. In previous work we proved that these two conditions do not suffice to characterize the target functions on finite trees, and that a third condition is needed. In this note we study the location functions on finite trees that are characterized by precisely these two conditions.

Keywords: generalized target functions; Weak Pareto Efficiency; Replacement Domination; trees; axioms.

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1. Introduction

In this note we are concerned with location functions on a finite connected graph $G = (V, E)$, where each user/customer i has a unique most-preferred location $x_i \in V$. The preferences of customer i with respect to the other vertices are determined by the distance to x_i: the farther away a vertex is from x_i, the less it is preferred. Thus the preferences of i form a weak order on V with x_i at its peak, the vertices adjacent to x_i in the next indifference class, the vertices at distance two from x_i in the next class, etcetera. In the literature, these weak orders have been called some variation of distance-determined, symmetric, single-peaked preferences. Clearly every vertex determines a unique weak order of this type on the vertices of the graph G, so we can represent it just by its peak. A profile of length k is a k-tuple of most preferred locations of a set of k customers. We assume throughout that $k \geq 3$. The goal of a location function is to select a vertex that represents the best location given the customers individual preferences as displayed in the input profile. Simply put, a location function is a mapping $f : V^k \to V$, where $G = (V, E)$ is a finite connected graph representing an abstract geography of interest, such as a road map.

A target function is a location function that is defined by a predetermined vertex, its target: the function outputs a vertex belonging to the convex hull of the input profile that is closest to the target. The definition of fully-gated graphs guarantees that this closest vertex exists, so that in this case target functions are well-defined. We refer to Colbourn and Huybrechts [1] for more information on such graphs. Trees and n-cubes are prime examples of fully-gated graphs. In the literature [4, 5, 7, 13] various axiomatic characterizations of target functions on trees have been given. The two axioms for the location function f that are prominent in these results are

(i) Weak Pareto Efficiency (WPE): For any profile π, the output $f(\pi)$ lies in the convex hull of the vertices in π.

(ii) Replacement Domination (RD): Suppose we change the location of one customer x_i in π into y_i, thus creating a new profile ρ that differs from π only in position i. Then, all other customers of π are either closer to $f(\rho)$ or they are farther from $f(\rho)$ compared with $f(\pi)$.

Vohra [13] studied tree networks, in which edges are considered to be continuous segments, and where internal points as well as vertices are allowed as locations. The "distance-determined, symmetric, single-peaked preferences" of the customers are defined in a similar way. Vohra proved that on a tree network the target functions are the only location functions satisfying (WPE) and (RD).

*This paper is dedicated to the memory of Frank Harary, who was our graph theory mentor and friend as well as a very funny guy.

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Gordon [4], when dealing with attribute spaces, considered finite trees. But in his case the preference ranking of a customer is an arbitrary single-peaked weak order of the vertices, where each vertex is then the peak of many different weak orders. Let $\mathcal{R}_V$ be the set of all single-peaked weak orders on V. In this context a profile is a sequence of length k consisting of weak orders in $\mathcal{R}_V$. Gordon proved that, on a tree $G = (V, E)$, a location function $f : \mathcal{R}_V^k \to V$ is a target function if and only if f satisfies (WPE) and (RD).

In a previous paper [7], we studied the case of location functions on a tree with profiles in our sense, that is, the *symmetric* single-peaked weak orders, so that profiles of preference relations can be taken as sequences of vertices. Note that, in comparison with the Gordon case, we have only a very limited subclass of profiles, so that (RD) constitutes less of a restriction on the function than in the Gordon case. It turns out that (WPE) and (RD) do not suffice to characterize the target functions. An extra axiom is needed which we called the Neighborhood Condition. We refer to [7] for the necessary details.

These results give rise to the following question: what happens in the case of the "distance-determined, symmetric, single-peaked preferences" when we require only the two axioms (WPE) and (RD) for location functions on a finite tree? Our main result answers this question. We get functions that are almost target functions satisfying one extra condition on the convex hulls of profiles.

2. Definitions and main result

Throughout this note $G = (V, E)$ is a finite tree with vertex set V and edge set E. For two vertices x and y, we denote the path between x and y by $[x, y]$. Recall that a tree has the property that, for any three vertices u, v, w in G, there is a unique vertex in $[u, v] \cap [v, w] \cap [w, u]$. The **distance** $d(x, y)$ between x and y is the length of the path $[x, y]$, that is, the number of edges in $[x, y]$. A **profile** $\pi = (x_1, x_2, \ldots, x_k)$ on G is a sequence of k vertices in G, where multiple occurrences are allowed. We call $x_1, x_2, \ldots, x_k$ the **entries** in π. The set of vertices contained in π is denoted by $\{\pi\}$. The set of all profiles on G is denoted by V^k. The **convex hull** $\langle \pi \rangle$ of π is the smallest subtree of G containing all vertices in π. Note that all leaves of $\langle \pi \rangle$ are vertices in π. For a vertex $a \in V$, the **gate** of a into $\langle \pi \rangle$ is the vertex in $\langle \pi \rangle$ closest to a. Our definitions here of distance, convex hull and gate are just simplifications of the usual definitions to the tree case. If $a \in \langle \pi \rangle$, then obviously a is its own gate. A **location function** on G is a mapping $f : V^k \to V$. We assume throughout that $k \geq 3$.

A location function $f : V^k \to V$ is **Weakly Pareto Efficient**, **(WPE)**, if

$$f(\pi) \in \langle \pi \rangle, \text{ for every profile } \pi.$$

On trees it is easy to show that (WPE) is equivalent with the classical property of Pareto Efficiency (or Pareto Optimality) found in the economics literature, see [7].

For any j and for any profiles $\pi = (x_1, \ldots, x_k)$ and $\rho = (y_1, \ldots, y_k)$, we will say that π and ρ are **equal except at** j if

$$x_j \neq y_j \text{ and } x_i = y_i \text{ for all } i \neq j.$$

If π and ρ are equal except at some j we say they are **almost equal**.

A location function $f : V^k \to V$ satisfies **Replacement Domination**, **(RD)**, if, for profiles $\pi = (x_1, \ldots, x_k)$ and $\rho = (y_1, \ldots, y_k)$ that are equal except at j, either

$$d(x_i, f(\pi)) \leq d(x_i, f(\rho)) \text{ for all } i \neq j$$

or

$$d(x_i, f(\rho)) \leq d(x_i, f(\pi)) \text{ for all } i \neq j.$$

It appears that this axiom was first considered by Moulin [8], who called it Agreement. Thomson [9–11] called it the Replacement Principle, see also [12,13]. We follow Klaus [5], who seemed to have coined the term Replacement Domination, see also [2–4, 6].

Here is one way to obtain a violation of (RD), which is often useful in our proofs. There exist $x_r, x_s \in \{\pi\}$ with $r \neq j$ and $s \neq j$ such that $f(\pi)$ and $f(\rho)$ are two distinct vertices lying on the path between x_r and x_s. In this case, either

$$d(x_r, f(\pi)) < d(x_r, f(\rho)) \text{ and } d(x_s, f(\pi)) > d(x_s, f(\rho))$$

or

$$d(x_r, f(\pi)) > d(x_r, f(\rho)) \text{ and } d(x_s, f(\pi)) < d(x_s, f(\rho)).$$

We can express this situation as follows.

Remark. *If $f(\pi) \neq f(\rho)$ and $[f(\pi), f(\rho)] \subseteq [x_r, x_s]$ with $r \neq j$ and $s \neq j$, then f does not satisfy (RD).*

For the present paper, we will call the following property the **convexity condition**: If $f(\pi) \in \langle\rho\rangle$ and $f(\rho) \in \langle\pi\rangle$, then $f(\pi) = f(\rho)$, for any profiles π and ρ.

A location function on a connected fully-gated graph $G = (V, E)$ is the **target function** with target a, denoted f^a, if for any profile π

$$f^a(\pi) = \begin{cases} a & \text{if } a \in \langle\pi\rangle, \\ g & \text{if } a \notin \langle\pi\rangle \text{ and } g \text{ is the gate of } a \text{ into } \langle\pi\rangle. \end{cases}$$

In [7] we showed that the axioms (WPE) and (RD) do not suffice to characterize target functions on finite trees as they had in the continuous case, and so we had to add one additional axiom to get the target functions characterized. Now we seek to find those location functions on trees that are completely specified by the two axioms (WPE) and (RD). In order to analyze these functions, we need to extend the definition of target function. We only need this on trees, but the definition is meaningful on any fully-gated graph.

Definition. *A location function $f : V^k \to V$ on a fully-gated graph with vertex set V is called a* ***generalized target function*** *with target $a \in V$ if, for any profile π,*

- *$f(\pi) = a$ if $a \in \langle\pi\rangle$,*
- *$f(\pi)$ is the gate of a in $\langle\pi\rangle$ or a neighbor in $\langle\pi\rangle$ of this gate if $a \notin \langle\pi\rangle$.*

If f is a generalized target function, we always have $f(\pi) \in \langle\pi\rangle$. In addition either $f(\pi) = f^a(\pi)$ or $f(\pi)$ is adjacent to $f^a(\pi)$, where f^a is the corresponding target function. For the case $a \notin \langle\pi\rangle$, since $f^a(\pi)$ is the gate of a into $\langle\pi\rangle$, we have

$$d(a, f(\pi)) \leq d(a, f^a(\pi)) + 1 \leq d(a, y) + 1 \text{ for all y} \in \langle\pi\rangle.$$

This inequality is obviously still true if $a \in \langle\pi\rangle$ since $f(\pi) = a$ in this case. Clearly the target function f^a with target a is a generalized target function with target a, but the example at the end of this paper shows that the converse is not true. Also note that there can be many different generalized target functions with the same target.

We are now ready to state our main result.

Theorem 2.1. *Let $f : V^k \to V$ be a location function on a finite tree with vertex set V. Then f satisfies (WPE) and (RD) if and only if f is a generalized target function that satisfies the convexity condition.*

3. Proof of the main result

Throughout this section $G = (V, E)$ is a finite tree, and $f : V^k \to V$ is a location function.

First assume that f satisfies (WPE) and (RD). We will prove that f is a generalized target function satisfying the convexity condition. This will be done in a sequence of Lemmas 3.1 up to 3.8.

First we prove a very powerful lemma.

Lemma 3.1. *If $f(\pi) \in [x_m, x_n]$ with $\pi = (x_1, x_2, \ldots, x_k)$ and x_j with $j \neq m, n$ is replaced by $y_j \in [x_m, x_n]$ to get a profile $\pi' = (y_1, y_2, \ldots, y_k)$ which is equal to π except at j, then $f(\pi') = f(\pi)$.*

Proof. Assume $f(\pi') \neq f(\pi)$. Since x_m and x_n are in π', it follows that $f(\pi) \in [x_m, x_n] \subseteq \langle\pi'\rangle$. By (WPE) we have $f(\pi') \in <\pi'>$. So there exist $y_r, y_s \in \{\pi'\}$ such that

$$[f(\pi'), f(\pi)] \subseteq [y_r, y_s].$$

If $j \neq r$ and $j \neq s$, then $x_r = y_r$ and $x_s = y_s$ and we get our desired contradiction to (RD) from the Remark.

Now assume $j = r$ and $s \neq j$, and so $y_s = x_s$. Thus,

$$[f(\pi'), f(\pi)] \subseteq [y_j, x_s].$$

Let t be the unique vertex belonging to the intersection $[x_m, x_n] \cap [x_m, x_s] \cap [x_n, x_s]$. Since $t, y_j \in [x_m, x_n]$, we may assume without loss of generality that $y_j \in [x_m, t]$. Since $t \in [x_m, x_s]$ implies that $[x_m, t] \subseteq [x_m, x_s]$ it follows that $y_j \in [x_m, x_s]$. Similarly, $y_j \in [x_m, x_s]$ implies that $[y_j, x_s] \subseteq [x_m, x_s]$. Therefore,

$$[f(\pi'), f(\pi)] \subseteq [x_m, x_s]$$

and again we get a contradiction to (RD). □

In [7] we proved the following lemma only for two profiles π and ρ that are almost equal. Also the proof was not as complete as it should be. With Lemma 3.1 at hand, we can now prove the following stronger statement.

Lemma 3.2. *If π and ρ are two profiles such that $\langle\pi\rangle = \langle\rho\rangle$, then $f(\pi) = f(\rho)$.*

Proof. Let π and ρ be two profiles such that $\langle\pi\rangle = \langle\rho\rangle$. Assume that $f(\pi) \neq f(\rho)$. Then we can find two leaves u and v in $\langle\pi\rangle = \langle\rho\rangle$ such that $[f(\pi), f(\rho)] \subseteq [u, v]$. Note that both u and v are entries in π as well as ρ.

First we consider $f(\pi)$. Note that $u = x_m$ and $v = x_n$, for some m and n, where x_m and x_n are entries in π. By Lemma 3.1, $f(\pi) = f(\pi')$, for the profile π' obtained from π by replacing each x_i with u, for all $i \neq m, n$. Clearly π' is a profile consisting of $k-1$ entries u and one entry v in the n-th position.

If $n < k$, then, using Lemma 3.1, we can replace $x_k = u$ with v without changing the output, and then replace x_n with u without changing the output. So we may assume that $\pi' = (u, u, \dots, u, v)$, and we have shown that $f(\pi) = f(u, u, \dots, u, v)$.

In the same way we prove that $f(\rho) = f(u, u, \dots, u, v)$. Thus we have shown that $f(\pi) = f(\rho)$. □

We observe here that, by Lemma 3.2, f satisfies anonymity, which means that $f(\pi) = f(\rho)$, for any profiles π and ρ, where ρ is obtained from π by reordering the entries of π.

Next we introduce some notation that simplifies some of our proofs. Let (uv) denote an arbitrary profile of length k where $\{(uv)\} = \{u, v\}$. i.e., (uv) consists only of u's and v's. We will write $f(uv)$ instead of the correct but cumbersome $f((uv))$. Note that since $\langle(uv)\rangle = [u, v]$, it follows that $f(uv)$ is well-defined by Lemma 3.2. That is, when $f(uv) = z$ for some specific profile of type (uv), then $f(uv) = z$ for all profiles of type (uv). In addition, let (xzy) denote the profile $(x_1, \dots, x_k)$ where $x_1 = x, x_k = y$ and $x_i = z$ for all $i \notin \{1, k\}$.

The next result is a modification of Lemma 9 from [7].

Lemma 3.3. *Let $\{x, y, p\} \subset V$ with x, y, p distinct vertices and $p \in [x, y]$ with $d(x, p) \geq 2$. If $f(xp) = x$, then $f(xy) = x$.*

Proof. Since $p \in [x, y]$, Lemma 3.2 gives $f(xpy) = f(xy)$.

First assume that $d(p, y) < d(p, x)$. Suppose that $f(xp) = x$ but $f(xpy) = f(xy) = z \neq x$. By (WPE) we know that $z \in [x, y]$, and it follows that $d(p, z) < d(p, x)$. Hence we have $d(p, f(xpy)) < d(p, f(xpp))$. On the other hand we have $d(x, f(xpy)) > d(x, f(xpp)) = 0$, so we get a contradiction to (RD), considering that (xpy) and (xpp) are almost equal profiles. Therefore $f(xy) = x$.

Next assume that $d(p, x) \leq d(p, y)$. Since $d(x, p) \geq 2$, we have $d(p, y) \geq 2$. We use induction on $d(p, y)$. If $d(p, y) = 2$, then $d(x, p) = 2$ as well, and so $[x, y]$ is a 5-path $xupvy$. Consider vertex v. We have $d(p, v) < d(p, x)$. So, applying the previous argument on x, v instead of x, y gives us that $f(xp) = x$ implies that $f(xv) = x$. Now we have $v \in [x, y]$ and $d(v, y) < d(v, x)$. So, again by the previous argument, $f(xv) = x$ implies that $f(xy) = x$. This settles the base for our induction.

For the induction step we may suppose that $f(xp) = x$ implies $f(xw) = x$, for any $[x, w]$, and $p \in [x, w]$ with $d(p, w) = r \geq 2$. Consider a vertex y and a vertex $p \in [x, y]$ with $d(p, y) = r + 1$. Let w be the neighbor of y in $[p, y] \subseteq [x, y]$. Then $d(p, w) = r$. If $f(xp) = x$, the induction hypothesis gives $f(xw) = x$. Since $d(w, y) = 1 < d(w, x)$, the first part of the proof implies that $f(xy) = x$, and the proof is complete. □

Lemma 3.4. *Let $\pi = (x_1, \dots, x_k)$ be a profile and let y be any element in $\langle\pi\rangle$ such that $y \neq f(\pi)$. If $f(\pi) = p$, then $f(yp) = p$.*

Proof. By (WPE), $p = f(\pi) \in \langle\pi\rangle$. Now $\{y, p\} \subseteq \langle\pi\rangle$ implies that there exists $x_r, x_s \in \{\pi\}$ such that

$$[y, p] \subseteq [x_r, x_s].$$

We may assume that p lies on the unique path from y to x_s. So $p \in [y, x_s]$ and $y \in [x_r, p]$.

Let $j \in \{1, \dots, k\} \backslash \{r, s\}$ and assume $x_j \neq p$. Next, let π' be the profile equal to π except that the j^{th} entry is p. By Lemma 3.1, $f(\pi') = f(\pi)$. By repeating this argument a finite number of times we get a profile π'' such that $\{\pi''\} = \{x_r, x_s, p\}$, $< \pi'' >= [x_r, x_s]$, and $f(\pi'') = f(\pi) = p$. If $x_s \neq p$, then $x_s \notin [x_r, p]$. In this case, replace x_s in π'' with p to get a profile ρ such that ρ is equal to π'' except at s. By Lemma 3.1, $f(\rho) = f(\pi'') = p$. Observe that $\{\rho\} = \{x_r, p\}$ and so $f(x_r p) = p$. If $y = x_r$, then we are done. If $y \neq x_r$, then, by Lemma 3.2, $f(x_r y p) = p$. Finally, since $f(x_r y p) = p \in [y, p]$, we replace x_r with y to get $f(yp) = p$, by Lemma 3.1. □

Next we prove that f satisfies the convexity condition.

Lemma 3.5. *For any profiles π and ρ, if $f(\pi) \in \langle\rho\rangle$ and $f(\rho) \in \langle\pi\rangle$, then $f(\pi) = f(\rho)$.*

Proof. Assume that there exist profiles π and ρ such that $f(\pi) \in \langle\rho\rangle$, $f(\rho) \in \langle\pi\rangle$, and $f(\pi) \neq f(\rho)$. If $p = f(\pi)$ and $q = f(\rho)$, then, by Lemma 3.4, $f(pq) = p$ and $f(pq) = q$, contrary to the fact that $p \neq q$. □

The next Lemma provides us with a candidate for the target of the generalized target function that we get when f satisfies (WPE) and (RD). We need the following notation. For each $x \in V$,

$$A_f(x) = \{y \in V \mid f(xy) = x\}.$$

Lemma 3.6. *There exists a vertex $a \in V$ such that $A_f(a) = V$.*

Proof. Choose $a \in V$ such that $A_f(a)$ is a maximal subset of V among sets of the form $A_f(z)$ with $z \in V$. We claim that $A_f(a) = V$. Assume to the contrary that $A_f(a) \neq V$, and choose a $y \in V \setminus A_f(a)$ with $d(a, y)$ minimum amongst such vertices. Let p be the vertex in the path $[a, y]$ adjacent to y. From the choice of y, we deduce that $p \in A_f(a)$, and thus $f(ap) = a$. If $d(a, p) \geq 2$, then Lemma 3.3 implies that $f(ay) = a$, which is contrary to $y \notin A_f(a)$. Therefore $d(a, p) \leq 1$, and so $d(a, y) \leq 2$.

First we prove that $a \in A_f(y)$. Assume that $d(a, y) = 2$, so that we have the 3-path apy. Then (WPE) and $f(ay) \neq a$ give $f(ay) \in \{p, y\}$. If $f(ay) = p$, then Lemma 3.2 gives us $f(apy) = f(ay) = p$ and $f(app) = f(ap) = a$. Hence $f(apy) \neq f(app)$. Since neither p nor a is closer to both a and p, this contradicts (RD). So $f(ay) = y$. Next assume that $d(a, y) = 1$. Since $f(ay) \neq a$, (WPE) gives $f(ay) = y$ also in this case.

We now claim that $A_f(a)$ is a proper subset of $A_f(y)$, contradicting the maximality of $A_f(a)$. We know $y \in A_f(y) \setminus A_f(a)$, so we need only show that, if $z \in A_f(a)$ then $z \in A_f(y)$.

Choose any $z \in A_f(a)$, so $f(az) = a$. Consider the profile $\pi = (ayz)$. Since G is a tree, it follows that

$$\langle \pi \rangle = [a, y] \cup [a, z].$$

By (WPE), we have $f(\pi) \in \langle \pi \rangle$. So it follows that

$$f(\pi) \in [a, y] \cup [a, z].$$

For profiles (ay) and (az), we have $< (ay) >= [a, y]$ and $< (az) >= [a, z]$. Therefore, either

$$f(\pi) \in < (ay) > \text{ and } f(ay) = y \in \langle \pi \rangle$$

or

$$f(\pi) \in < (az) > \text{ and } f(az) = a \in \langle \pi \rangle.$$

By Lemma 3.5, either $f(\pi) = f(ay) = y$ or $f(\pi) = f(az) = a$. If $f(\pi) = a$, then $f(\pi) \in [a, y] =< (ay) >$ and so, by Lemma 3.5 again, $f(\pi) = f(ay) = y$. Since $a \neq y$, we must have $f(\pi) = y$. By Lemma 3.5, $f(\pi) = y \in [y, z] =< (yz) >$ and $f(yz) \in \langle \pi \rangle$ implies that $f(yz) = y$. Thus, $z \in A_f(y)$. This contradicts the maximality of the set $A_f(a)$, from which we deduce that indeed $A_f(a) = V$. □

Observe that the vertex a with $A_f(a) = V$ is unique, because if there were a vertex $b \neq a$ with $A_f(b) = V$, then we would have $a = f(ab) = f(ba) = b$.

Lemma 3.7. *Let a be the vertex such that $A_f(a) = V$, and let π be a profile with $a \in \langle \pi \rangle$. Then $f(\pi) = a$.*

Proof. Assume $f(\pi) \neq a$ and let $p = f(\pi)$. Since $a \in \langle \pi \rangle$ it follows from Lemma **??** that $f(ap) = p$, contrary to $A_f(a) = V$. Hence, $f(\pi) = a$. □

Lemma 3.8. *Let a be the vertex such that $A_f(a) = V$, and π a profile with $a \notin \langle \pi \rangle$, and let g be the gate of a in $\langle \pi \rangle$. Then $f(\pi) \in \langle \pi \rangle$ with $d(f(\pi), g) \leq 1$.*

Proof. Assume that there exists a profile π such that $a \notin \langle \pi \rangle$ with $d(f(\pi), g) \geq 2$. Since $g \in \langle \pi \rangle$ it follows from Lemma **??** that $f(gp) = p$ where $p = f(\pi)$. Observe that $g \in [a, p]$, $d(p, g) \geq 2$, and $f(pg) = p$. Therefore, by Lemma 3.3, $f(pa) = p$ contrary to $A_f(a) = V$. Hence $d(f(\pi), g) \leq 1$. □

At this stage of the proof we have shown that if f is a location function on a tree $G = (V, E)$ satisfying (WPE) and (RD), then f is a generalized target function, where the target is the vertex a such that $A_f(a) = V$. Moreover, f satisfies the convexity condition. This is one implication in Theorem 2.1. The following lemma gives the converse of Theorem 2.1.

Lemma 3.9. *Let $f : V^k \to V$ be a generalized target function on a finite tree with vertex set V that satisfies the convexity condition. Then f satisfies (WPE) and (RD).*

Proof. From the definition of a generalized target function it follows that f satisfies (WPE). Our final goal is to prove that f satisfies (RD). So, let $\pi = (x_1, x_2, \ldots, x_k)$ and $\rho = (y_1, y_2, \ldots, y_k)$ be two profiles that are equal except at j. We have to prove that

$$d(x_i, f(\pi)) \leq d(x_i, f(\rho)), \text{ for all } i \neq j, \text{ or}$$

$$d(x_i, f(\rho)) \leq d(x_i, f(\pi)), \text{ for all } i \neq j.$$

If $f(\pi) \in \langle\rho\rangle$, and $f(\rho) \in \langle\pi\rangle$, then $f(\pi) = f(\rho)$ by the convexity condition, and we are done.

Now consider the case where $f^a(\pi) = f^a(\rho)$ and $f(\pi) \notin \langle\rho\rangle$. Let $u = f^a(\pi) = f^a(\rho)$ and note that $f(\pi) \notin \langle\rho\rangle$ implies that $f(\pi) \notin [u, x_i]$ for $i \neq j$. By the definition of a generalized target function, $f(\pi)$ is adjacent to u. Therefore, since $f(\rho) = u$ or $f(\rho)$ is adjacent to u,

$$d(x_i, f(\rho)) \leq d(x_i, u) + 1 = d(x_i, f(\pi)) \text{ for all } i \neq j.$$

The case where $f^a(\pi) = f^a(\rho)$ and $f(\rho) \notin \langle\pi\rangle$ is proved in a similar way. So at this point we may assume $f^a(\pi) \neq f^a(\rho)$. We do allow for the possibility that $f^a(\pi) = a$. Let $i \neq j$ belong to the set $\{1, \ldots, k\}$. Since $f^a(\pi)$ is the gate for a in $\langle\pi\rangle$ and $x_i \in \langle\pi\rangle$ it follows that

$$f^a(\pi) \in [a, x_i].$$

Similarly, $f^a(\rho)$ being the gate for a in $\langle\rho\rangle$ along with $x_i = y_i \in \langle\rho\rangle$ implies that

$$f^a(\rho) \in [a, x_i].$$

So given that $f^a(\pi) \neq f^a(\rho)$ and $\{f^a(\pi), f^a(\rho)\} \subseteq [a, x_i]$, we may assume without loss of generality that $d(a, f^a(\pi)) < d(a, f^a(\rho))$. Since $f(\pi) = f^a(\pi)$ or $f(\pi)$ is adjacent to $f^a(\pi)$ it follows that $f^a(\rho) \in [f(\pi), x_i]$. If $f^a(\rho) \neq f(\pi)$, then

$$d(f(\rho), x_i) \leq d(f^a(\rho), x_i) + 1 \leq d(f^a(\rho), x_i) + d(f^a(\rho), f(\pi)) = d(f(\pi), x_i).$$

Since i was arbitrary we have shown that

$$d(x_i, f(\rho)) \leq d(x_i, f(\pi)), \text{ for all } i \neq j.$$

Hence f satisfies (RD).

The final case is when $f^a(\rho) = f(\pi)$. Note that thus we have $f(\pi) \in \langle\rho\rangle$. Since $d(f(\pi), f^a(\pi)) \leq 1$ and $f^a(\pi) \neq f^a(\rho)$ it follows that $f^a(\pi)$ and $f^a(\rho)$ are adjacent. If $f(\rho) \in \langle\pi\rangle$, then, by the convexity condition, $f(\pi) = f(\rho)$ and we're done. If $f(\rho) \notin \langle\pi\rangle$, then $f(\rho) \notin [f^a(\rho), x_i]$ for any $i \neq j$. Since $f(\rho)$ is adjacent to $f^a(\rho)$ it follows that

$$d(x_i, f(\rho)) = d(x_i, f^a(\rho)) + 1 > d(x_i, f^a(\rho)) = d(x_i, f(\pi)) \text{ for all } i \neq j.$$

In this final case, we have shown that f satisfies (RD) and we're done. □

To give a concrete illustration of what we have been working with, we close with an example of a generalized target function satisfying the convexity condition.

Example. For $n \geq 2$, let $K_{1,n}$ be the star with center x_0 and leaves $x_1, x_2, \ldots, x_n$. Define $f : V^k \to V$ as follows: for any profile π, $f(\pi) = x_m$ where $m = \max\{i : x_i \in \{\pi\}\}$. The location function f is a generalized target function with x_n the target. Also, it is easy to check that for any profiles π and ρ, $f(\pi) \in \langle\rho\rangle$ along with $f(\rho) \in \langle\pi\rangle$ implies that $f(\pi) = f(\rho)$. Therefore, by Theorem 2.1, f satisfies (WPE) and (RD).

4. Concluding remarks

On finite trees we have characterized those location functions which satisfy two classical axioms that were used to characterize target location functions on tree networks. In the finite case we obtain the generalized target function satisfying an additional condition. We have an example, not given in this paper, showing Theorem 2.1 does not hold even for the 3-cube. So future research will include a study of location functions on general fully-gated graphs that satisfy the two conditions of Replacement Domination and Weakly Pareto Efficient. In addition, the new convexity condition merits further study.

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