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Research Article

Decomposition of $4k$-regular graphs into k 4-regular K_5-free and $(K_5 - e)$-free subgraphs*

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Abstract

Let G be a $4k$-regular graph with $k \geq 2$. We show that G can be decomposed into k 4-regular spanning subgraphs $G_1, G_2, \dots, G_k$, each of which does not contain an induced subgraph that is isomorphic to K_5 or $K_5 - e$. We then use a result of Heinrich et al. [*J. Graph Theory* **31** (1999) 135–143] which provides a triangle-free Euler tour in each of $G_1, G_2, \dots, G_k$ to show that G has a triangle-free Euler tour. In the case when m is even, our results imply a result by Oksimets [Ph.D thesis, Umeå University, Umeå, 2003] which states that every connected $2m$-regular graph G with $m \geq 2$ and $|E(G)|$ divisible by 3 can be decomposed into paths of length 3.

Keywords: decomposition; Eulerian; forbidden subgraph.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the degree of a vertex $v \in V(G)$ by deg(v, G). For $S \subseteq V(G)$ we denote by $\langle S \rangle$ the subgraph of G induced by S. A *k-decomposition* of G is a partition of its edge set into edge-disjoint subgraphs $H_1, H_2, \dots, H_k$ of G; if each $H_i, i = 1, 2, \dots, k$ is isomorphic to H then we have an *H-decomposition* of G and we say that H *decomposes* G. It is well known that every connected graph G, each of whose vertices has even degree, has an Euler tour; we call such a graph *Eulerian*. A *triangle-free* Euler tour in G is an Euler tour in which no three consecutive edges form a triangle in G. For graphs G and H we say that G is *H-free* if G does not contain an induced subgraph that is isomorphic to H. We refer the reader to [3] and [8] for all terminology and notation that is not defined in this paper.

In this paper we prove the following two theorems. Theorem 1.1 is a decomposition theorem for $4k$-regular graphs into k 4-regular spanning subgraphs that do not contain dense subgraphs. Bertram and Horak [1] showed that the problem of determining whether a 4-regular graph can be decomposed into two triangle-free 2-regular graphs can be solved in polynomial time. A natural extension of this is to ask when an 8-regular graph can be decomposed into K_5-free 4-regular graphs. Theorem 1.1 shows that this is always possible, and in fact we can say much more.

Theorem 1.1. *Every $4k$-regular graph with $k \geq 2$ can be decomposed into k 4-regular spanning subgraphs, each of which is K_5-free and $(K_5 - e)$-free.*

Theorem 1.2 allows us to concatenate two triangle-free Euler tours to obtain a larger triangle-free Euler tour.

Theorem 1.2. *Let G be a graph with a decomposition into subgraphs G_1 and G_2, each having a triangle-free Euler tour. If there exists $v \in V(G)$ with deg$(v, G_1) \geq 4$ and deg$(v, G_2) \geq 4$ then G has a triangle-free Euler tour.*

Heinrich et al. [4] proved the following theorem giving necessary and sufficient conditions for the existence of a triangle-free Euler tour in a 4-regular graph.

Theorem 1.3. [4] *A connected 4-regular graph G has a triangle-free Euler tour if and only if G is K_5-free and $(K_5 - e)$-free.*

Let P_4 denote the path on 4 vertices. We note that our results in Theorems 1.1 and 1.2 together with Theorem 1.3 yield the following corollary.

Corollary 1.1. *Let G be a connected $4k$-regular graph with $k \geq 2$. Then G has a P_4-decomposition if and only if $|E(G)|$ is divisible by 3.*

*Dedicated to Frank Harary, who introduced me (Michael J. Plantholt) and many others to the beauty of Graph Theory. He was a truly unique and unforgettable character: dance instructor, weather forecaster, and of course, “Mr. Graph Theory”.

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We note that the above corollary is a special case of the following result of Oksimets [5] when m is even. Oksimets' proof of Theorem 1.4 is long and available only in her PhD thesis. Theorems 1.1 and 1.2, besides being of independent interest, also yield a streamlined proof of Oksimets' result in the case when m is even.

Theorem 1.4. [5] *Let G be a connected $2m$-regular graph with $m \geq 2$. Then G has a P_4-decomposition if and only if $|E(G)|$ is divisible by 3.*

2. Proofs of Theorems 1.1 and 1.2

A *2-factor* of a graph is a spanning subgraph with each vertex having degree two. We will use the following classic theorem of Petersen [6].

Theorem 2.1. *[6] Every $2k$-regular graph can be decomposed into k 2-factors.*

A *halving* of a graph $G = (V, E)$ is a decomposition of G into spanning subgraphs G_1 and G_2 (called halves) with deg(v, G_1) = deg(v, G_2) = $\frac{1}{2}$deg(v, G) for each $v \in V(G)$. Given a graph H, we say that a halving of G is H-free if each half of the halving is H-free. Placing alternate edges of an Eulerian graph G into two halves gives the following lemma.

Lemma 2.1. *Let G be an Eulerian multigraph. Then G has a halving if and only if $|E(G)|$ is even.*

We now prove Theorem 1.1 from the Introduction.

Proof of Theorem 1.1. We first note that it suffices to prove the theorem for $k = 2$. If $k > 2$ then Theorem 2.1 gives a decomposition of G into an 8-regular spanning subgraph G_0 of G and a $4(k-2)$-regular spanning subgraph H of G. Applying the theorem for $k = 2$ to G_0 gives a spanning 4-regular subgraph G_1 of G that is K_5-free and $(K_5 - e)$-free. Now, $G \setminus E(G_1)$ is $4(k-1)$-regular and the result follows inductively. We prove the following stronger statement of Theorem 1.1 for $k = 2$.

Lemma 2.2. *Let G be a graph with the degree of each of its vertices being 0, 4 or 8. Then G has a K_5-free and $(K_5 - e)$-free halving.*

The proof is by induction on the number of vertices of degree 8. Clearly the lemma is true if there are no vertices of degree 8, because then in every halving of G, each vertex has degree at most 2. Now, let G be a graph with $j > 0$ vertices of degree 8 that satisfies the conditions of the lemma, and assume inductively that the lemma is true for any such graph with less than j vertices of degree 8. Each component of G is Eulerian with its number of edges being even. Clearly, it suffices to prove the lemma for each component of G, so assume from here on that G is connected.
Lemma 2.1 implies that G has a halving. If both parts in such a decomposition are K_5-free and K_5e-free, we are done; so assume every halving of G contains either K_5 or K_5e in at least one of its halves. Thus G itself must contain a set S of five vertices that induce K_5 or $K_5 - e$; also all vertices of S have degree 8 in G. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$.

Case 1. There is a set S such that $\langle S \rangle$ is isomorphic to K_5.

Let $G_0 = G - E(K_5)$ be the graph obtained by deleting all 10 edges of $\langle S \rangle$. All vertices in S have their degrees reduced by 4, so the induction hypothesis holds for G_0. In order to avoid a problem condition, we perform an additional reduction in one special case. Suppose that there exist disjoint edges xy and uv in G_0, where x, y, u, v are not in S, but such that $x, y, u,$and v are all adjacent to the same set of three vertices $S_3 \subset S$. Note that these 14 edges (the 12 edges out of S_3 and the edges xy and uv) induce a 4-regular subgraph, call it W. In this case we remove these edges and call the resulting graph $G_0 - E(W)$. If there is no such additional reduction, we consider any (K_5 and $K_5 - e$)-free halving of G_0 into graphs A and B. If there was such a reduction, we first take a (K_5 and $K_5 - e$)-free halving of $G_0 - E(W)$ into graphs A and B. It is easy to see that the 4-regular graph W has a decomposition into Hamilton cycles; we place the edges from one cycle into A, the other into B, to get once again a halving of G_0 into A, B. In any such decomposition, edges xy and uv get placed into different graphs. We claim that adding these 14 edges keep A and B free of any K_5 or $K_5 - e$ subgraph. To see this, observe that vertices in S_3 cannot be in a forbidden K_5 or $K_5 - e$ subgraph in A or B, because these vertices have degree 4 in G_0, and, adding for example, xy to A will not form a copy of K_5 or $K_5 - e$ because x and y are each incident to one of the vertices of S_3 in A.
To complete our desired splitting of G, we will partition the edges of $\langle S \rangle \simeq K_5$ into two 5-cycles C_1 and C_2, and add those edges to A and B respectively. This could form a forbidden graph K_5 or $K_5 - e$ in A and/or B, but we claim we can make adjustments to avoid such subgraphs.
Suppose wlog that in A after adding these 5-cycles we have a forbidden subgraph Z. Subgraph Z has 9 or 10 edges, and some of those edges must be in C_1 because A was previously (K_5 and $K_5 - e$)-free. It is easy to see that it is impossible for

exactly one of the edges of C_1, say v_1v_2, to be in Z, for then v_1 and v_2 each have degree at most three in Z, which implies that Z contains at most 8 edges. Arguing in a similar fashion, it is straightforward to check that the only way subgraph Z can have more than eight edges in A is if Z contains exactly three vertices (call this set of vertices V_1) from S; the two vertices x', y' not in V_1 must be adjacent to each other in A, and to each vertex of V_1 in A. Observe that there can be only one such structure in A, because a second such structure would need to contain one of the three vertices v of V_1, and thus its two neighbors in A, etc. Similarly, at most one such structure can exist in B, and such a structure contains a set V_2 of three vertices from S and two vertices u', v' not in S_3 that are distinct from x', y'. We refer to these subgraphs before adding the 5-cycles as A-critical and B-critical subgraphs.

If A and B previously had no critical subgraph, then the final graphs after adding the 5-cycles give us the desired halving. If only one of A and B previously had a critical subgraph, and we get a subgraph isomorphic to $K_5 - e$, we reverse the roles of C_1 and C_2 to get the desired halving. Finally, suppose the splitting of G_0 into A and B has both A-critical and B-critical subgraphs. The sets V_1 and V_2 must overlap; we consider the possible size of that overlap in turn.

If $|V_1 \bigcap V_2| = 1$, suppose wlog $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_1, v_4, v_5\}$. Letting $C_1 = v_1v_5v_2v_3v_4v_1$ adds only one edge between the vertices of V_1 in A, and similarly the remaining edges in C_2 add only one edge between the vertices of V_2, so the desired halving is formed.

If $|V_1 \bigcap V_2| = 2$, wlog let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_1, v_2, v_4\}$. Then letting $C_1 = v_1v_2v_4v_3v_5v_1$, we get the desired halving by similar reasoning.

Lastly, suppose that $|V_1 \bigcap V_2| = 3$, so V_1 and V_2 are identical. In this case, the critical structures in A and B together form a subgraph W^* that is identical to W, so the additional reduction of W at the start must have previously occurred using a set V of 3 vertices in S. Clearly S_3 must have at least one vertex, call it v^* in common with $V_1 = V_2$. We claim that in fact, this cannot happen. In both W and W^*, v^* has degree 4, and is incident with $\{x, y, u, v\}$ and $\{x', y', u', v'\}$ respectively, so $\{x, y, u, v\} = \{x', y', u', v'\}$. But in A, x' must be adjacent to exactly the four vertices consisting of y' and the three vertices of V_1. But then either $x'y' = xy$ or $x'y' = uv$, because the edges xy and uv are in different graphs A, B; wlog say $x'y' = xy$. We then get a contradiction because the vertices x, y, v^* form a triangle in A, but the edges of W form 7-cycles in each of A and B.

Case 2. G contains no K_5 subgraph, but does contain a subgraph isomorphic to $K_5 - e$.

Let S be a set of vertices in G that induce $K_5 - e$. Let $V(\langle S\rangle) = \{v_1, v_2, v_3, v_4, v_5\}$, and let v_1 and v_2 be the two vertices that are non-adjacent in $\langle S\rangle$. Let $G_1 = G - E(\langle S\rangle)$. Let M be the graph with 7 edges and with vertices $\{x_1, x_2, y_1, y_2, y_3\}$ in which each of x_1 and x_2 is adjacent with each of y_1, y_2, y_3, and x_1 is also adjacent to x_2. Then, let $G^* = G_1 \bigcup M$ with the six additional edges from $\{v_1, v_2\}$ to $\{y_1, y_2, y_3\}$.

Now all vertices of G^* have degree 4 or 8, so that by the induction hypothesis, there is a halving of G^* into graphs A and B that are both (K_5 and $K_5 - e$)-free. In any such splitting, the seven edges of M are split in a 3-4 fashion between the two graphs because x_1 and x_2 must have degree 2 in both A and B; wlog we assume 3 edges are in A, an 4 are in B. Thus it follows that wlog either of the following two subcases must occur.

> Subcase 1. Exactly two edges from v_1 to $\{y_1, y_2, y_3\}$ are in A, and exactly two of the edges from v_2 to $\{y_1, y_2, y_3\}$ are in A.
>
> Subcase 2. The 3 edges from v_1 to $\{y_1, y_2, y_3\}$ are in A, and exactly one of the edges from v_2 to $\{y_1, y_2, y_3\}$ is in A.

We consider these two subcases separately.

Subcase 1. Exactly two edges from v_1 to $\{y_1, y_2, y_3\}$ are in A, and exactly two of the edges from v_2 to $\{y_1, y_2, y_3\}$ are in A.

We delete the vertices of M and all incident edges, and add back the 9 edges deleted from G. To complete the halving of G, we need to split these edges between A and B such that at each of v_1 and v_2 exactly two of the three added incident edges are in A, and similarly for v_3, v_4 and v_5 exactly two of the four added incident edges are in A.

Let Z be a copy of $K_5 - e$ (isomorphic to $\langle S\rangle$) with non-adjacent vertices labeled v_1, v_2 , and the remaining vertices labeled a, b, c. To do so, we need to show a 1-1 correspondence between the vertices a, b, c and vertices v_3, v_4, v_5 that completes our splitting into (K_5 and $K_5 - e$)-free halves. Decompose the edges of Z into graphs A and B so that there are four edges $v_1c, v_2b, ac, ab,$ in B ; the remaining 5 edges are placed in A. Note that v_1 and v_2 will now have the desired degree split between A and B, as will the remaining vertices. When we replace $\langle S\rangle$ by Z, we need to avoid forming a copy of $K_5 - e$ in A or in B. Clearly, no subset of 5 vertices from G that contains 4 or 5 vertices of Z will induce such a forbidden subgraph,

because that set of vertices will induce edges in both A and B.
Now suppose a set of 5 vertices induces the forbidden $K_5 - e$ in A or B and contains exactly three vertices of Z. Every triangle in Z has edges in both A and B and incident edges from both A and B, so the only possibility here is that the three vertices from Z are v_1, v_2 and a (because the edges to a are both in A). If such a situation occurs, there must be vertices p, q in G but not in Z such that a, v_1, v_2 are all adjacent to p and q in A. Clearly, only one of the vertices v_3, v_4, v_5, (assume wlog v_5) can match this description of vertex a ; therefore, we assign vertex a the label v_3 and avoid this forbidden graph.
Next suppose a set of 5 vertices induces the forbidden $K_5 - e$ in A or B and contains exactly two vertices of Z. The two vertices must be adjacent in Z, and if that edge is in A (respectively B), there can be at most one other edge from A (respectively B) incident with it in Z. Thus the only possible singleton edges of this type are the edges v_1c and v_2b, which are both in B. Then, if the other 3 vertices in this forbidden graph are r, s, t, we must have v_1 adjacent to each of those vertices in B, and c adjacent to two of the vertices r, s, t in B. Clearly at most one vertex from v_4, v_5, call it c^*, can have the property of this c vertex. Similar reasoning shows that at most one vertex from v_4, v_5 can have the property required to form a forbidden subgraph that uses that vertex and v_2; call it b^*. Finally we note that c^* must be different from b^*, since the vertices are incident with only 4 vertices in B. Thus we can designate $\{v_4, v_5\}$ to correspond to $\{b, c\}$ in such a way so as to avoid the forbidden subgraph, and the result follows.

Subcase 2. The 3 edges from v_1 to $\{y_1, y_2, y_3\}$ are in A, and exactly one of the edges from v_2 to $\{y_1, y_2, y_3\}$ is in A.

Delete the vertices of M and all incident edges, and add back the 9 edges deleted from G. To complete the halving of G, we need to split these edges between A and B such that at v_1 all three of the added incident edges are in A, at v_2 exactly one of the added edges is in A, and at v_3, v_4 and v_5 exactly two of the four added incident edges are in A. As before, let Z be a copy of $K_5 - e$ (isomorphic to $\langle S\rangle$) with non-adjacent vertices labeled v_1, v_2 , and the remaining vertices labeled a, b, c. We need to show a 1-1 correspondence between the vertices a, b, c and vertices v_3, v_4, v_5 that completes our splitting into (K_5 and $K_5 - e$)-free halves. Decompose the edges of Z into graphs A and B so that the four edges v_2a, v_2b, ac, bc are in B ; the remaining 5 edges are placed in A. Note that v_1, v_2 will now have the desired degree split between A and B, as will the remaining vertices. When we replace $\langle S\rangle$ by Z, we need to avoid forming a copy of $K_5 - e$ in A or in B. Clearly no subset of 5 vertices from G that contain 4 or 5 vertices of Z will induce such a forbidden subgraph, because that set of vertices will induce edges in both A and B.
Now suppose a set of 5 vertices induces the forbidden $K_5 - e$ in A or in B and contains exactly three vertices of Z. Unlike Subcase 1, v_1 and v_2 cannot be two of these vertices, since too many edges from A would be adjacent to the set. However, it is possible that Z is formed using the vertices $\{v_1, a, b\}$ because those three vertices form a triangle in A and have only one other A edge incident in Z. If such a situation occurs, there must be vertices p, q in G but not in Z such that v_1p is an edge in A, as are all edges induced by $\{a, b, p, q\}$. Clearly, only one pair of the vertices from $\{v_3, v_4, v_5\}$ (wlog v_3 and v_4) can match this description of vertices $\{a, b\}$; therefore, we must avoid assigning the set of two vertices $\{a, b\}$ to $\{v_3, v_4\}$.
Finally suppose a set of 5 vertices induces the forbidden $K_5 - e$ in A or in B and contains exactly two vertices of Z. Arguing as in Subcase 1, the only possible singleton edge of this type is the edge v_2c, which is in A. Then if the other 3 vertices in this forbidden graph are r, s, t we must have that v_2 is adjacent to each of r, s, t in A, and that c is adjacent to two of the vertices r, s, t in A. Clearly at most one vertex from $\{v_3, v_4, v_5\}$, call it c^*, can have the property of this vertex c. We then match vertex c with a vertex of $\{v_3, v_4\}$ that is different from c^*, and match a and b with the remaining two vertices of $\{v_3, v_4, v_5\}$. This gives the desired splitting. □

We now prove Theorem 1.2 from the Introduction.

Proof of Theorem 1.2. First suppose that either $\deg(v, G_1) \geq 6$ or $\deg(v, G_2) \geq 6$; without loss of generality assume that $\deg(v, G_1) \geq 6$. Let $E_1 = a_1b_1c_1d_1 \ldots a_2b_2c_2d_2 \ldots a_3b_3c_3d_3 \ldots,$ be the edges of a triangle-free Euler tour in G_1 with b_i, c_i being incident with v, $i = 1, 2, 3$. Similarly let $E_2 = w_1x_1y_1z_1 \ldots w_2x_2y_2z_2 \ldots$ be a triangle-free Euler tour in G_2 with x_i, y_i being incident with v, $i = 1, 2$. Let E_2' denote the triangle-free Euler tour in G_2 obtained by traversing E_2 in reverse order. We claim that we can get a triangle-free Euler tour in G by inserting E_2 or E_2' into E_1 in an appropriate way. Specifically, we claim that one of the following Euler tours in G must be triangle-free.

(1) Insert E_2 beginning with edge y_1 into E_1 after edge b_1.

(2) Insert E_2 beginning with edge y_2 into E_1 after edge b_1.

(3) Insert E_2 beginning with edge y_1 into E_1 after edge b_2.

(4) Insert E_2 beginning with edge y_2 into E_1 after edge b_2.

(5) Insert E_2 beginning with edge y_1 into E_1 after edge b_3.

(6) Insert E_2 beginning with edge y_2 into E_1 after edge b_3.

(7) Insert E_2' beginning with edge x_1 into E_1 after edge b_1.

(8) Insert E_2' beginning with edge x_2 into E_1 after edge b_1.

(9) Insert E_2' beginning with edge x_1 into E_1 after edge b_2.

(10) Insert E_2' beginning with edge x_2 into E_1 after edge b_2.

(11) Insert E_2' beginning with edge x_1 into E_1 after edge b_3.

(12) Insert E_2' beginning with edge x_2 into E_1 after edge b_3.

Since E_1 and E_2 are triangle-free Euler tours, the only possible triangles in any of these 12 Euler tours must include vertex v and two consecutive edges from either E_1 or E_2 and one from the other. So, for example, the only possible triangles in Euler tour 1 above consist of the following *connection triples* of edges: $a_1b_1y_1, b_1y_1z_1, w_1x_1c_1,$ and $x_1c_1d_1$. Thus, one of the 12 Euler tours listed above must be triangle-free unless at least one of the 4 corresponding connection triples forms a triangle for each Euler tour. Note that no connection triple appears in the list for more than one Euler tour. Moreover, each of the 10 pairs of consecutive edges $a_ib_i, c_id_i, i = 1,2,3$ and $w_jx_j, y_jz_j, j = 1,2$ can appear in only one triangle. It follows that at most 10 out of the 12 Euler tours above can have a triangle. Thus, one (actually at least two) of the 12 Euler tours must be triangle-free as desired.

Now suppose that deg(v, G_1) = deg(v, G_2) = 4. Let $E_1 = a_1b_1c_1d_1 \ldots a_2b_2c_2d_2 \ldots$ be a triangle-free Euler tour in G_1 with b_i, c_i being incident with v, $i = 1,2$, and let $E_2 = w_1x_1y_1z_1 \ldots w_2x_2y_2z_2 \ldots$ be a triangle-free Euler tour in G_2 with x_i, y_i being incident with v, $i = 1,2$. Then, the 8 tours above numbered 1-4 and 7-10 are Euler tours in G; moreover, at least one of these 8 Euler tours is triangle-free unless at least one of the following forms a triangle in each of the corresponding tours.

(1) $a_1b_1y_1$ or $b_1y_1z_1$ or $w_1x_1c_1$ or $x_1c_1d_1$

(2) $a_1b_1y_2$ or $b_1y_2z_2$ or $w_2x_2c_1$ or $x_2c_1d_1$

(3) $a_2b_2y_1$ or $b_2y_1z_1$ or $w_1x_1c_2$ or $x_1c_2d_2$

(4) $a_2b_2y_2$ or $b_2y_2z_2$ or $w_2x_2c_2$ or $x_2c_2d_2$

and

(7) $a_1b_1x_1$ or $b_1x_1w_1$ or $z_1y_1c_1$ or $z_1c_1d_1$

(8) $a_1b_1x_2$ or $b_1x_2w_2$ or $z_2y_2c_1$ or $z_2c_1d_1$

(9) $a_2b_2x_1$ or $b_2x_1w_1$ or $z_1y_1c_2$ or $z_1c_2d_2$

(10) $a_2b_2x_2$ or $b_2x_2w_2$ or $z_2y_2c_2$ or $z_2c_2d_2$

No triple appears more than once in the list above, and each of the 8 consecutive pairs of edges $a_ib_i, c_id_i, w_ix_i, y_iz_i, i = 1,2$ can appear in only one triangle. Thus, if none of the 8 Euler tours listed above are triangle-free, we can assume that each of the 8 Euler tours has exactly one triangle and each of the consecutive pairs of edges listed above is in one of these triangles. We show that under these assumptions, the two $v - v$ "half-tours" of E_1 given by $c_1d_1 \ldots a_2b_2$ and $c_2d_2 \ldots a_1b_1$, and the two corresponding half-tours of E_2 can be combined (possibly with a reverse traversal) to get a triangle-free Euler tour of G. To see this, we construct a graph Z on 8 vertices that represent the 8 consecutive pairs of edges mentioned above; thus, the vertices of Z are $\{a_ib_i\}, \{c_id_i\}, \{w_ix_i\}, \{y_iz_i\}, i = 1,2$. Join two vertices in Z by an edge if their corresponding half-tours do not form a triangle when combined as indicated by the two vertices of Z (for example, a_1b_1 and c_1d_1) or if the edges are in the same half-tour (for example, c_1d_1 and a_2b_2). Since E_1 and E_2 are triangle-free, H contains two 4-cycles: $a_1b_1 - c_1d_1 - a_2d_2 - c_2d_2 - a_1b_1$ and $w_1x_1 - y_1z_1 - w_2x_2 - y_2x_2 - w_1x_1$.

For simplicity we rename the vertices in these 4-cycles and represent the 4-cycles by $ABCDA$ and $EFGHE$. It is straightforward to check that G has a desired triangle-free Euler tour if Z has a Hamilton cycle containing the edges of the matching $\{M_1, M_2, M_3, M_4\}$, where $M_1 = BC, M_2 = DA, M_3 = FG, M_4 = HE$. By our assumptions, the complement Z^* of Z has 8 edges. Let $U = \{A, B, C, D\}$, and $W = \{E, F, G, H\}$.

Now suppose that no such Hamilton cycle exists in Z. Then for each edge of Z that joins vertices in U with W, there is a corresponding edge between U and W that must be in the complement Z^*. For example, if AE is in Z, then DH must be in Z^*, else Z has the desired Hamilton cycle $AEFGHDCBA$. Therefore all eight edges of Z^* join U and W, so the vertices of U and W both induce complete subgraphs in Z. It follows that the desired Hamilton cycle in H will exist if there are two edges between U and W that together are incident with all four of the edges M_1, M_2, M_3,and M_4. But because there are exactly 8 edges between U and W in Z, and because each vertex has degree at least 1 in Z^*, no single edge $M_i, i = 1, 2, 3, 4$ can be incident with all 8 of these edges. It follows that two edges with the desired property exist, and therefore the desired Hamilton cycle exists. The result now follows. □

We thank a referee for noting that if a $4k$-regular graph has a decomposition into Hamilton cycles, then pairing these cycles gives a decomposition into k 4-regular graphs, each of which has at most 8 edges in any subgraph with five vertices. Robinson and Wormald [7] showed that for each even fixed $r \geq 4$, almost all r-regular graphs have a decomposition into Hamilton cycles, and Csaba, Kühn, Lo, Osthus and Treglown [2] showed that for any r-regular graph G of order n sufficiently large, if r is even and $r \geq \frac{n}{2}$, G has a decomposition into Hamilton cycles. These two results immediately yield the following.

Theorem 2.2. *For fixed $r = 4k$ $(k > 1)$, almost all r-regular graphs have a decomposition into k 4-regular graphs, each having the property that any five vertices induce at most 8 edges.*

Theorem 2.3. *For n sufficiently large, and $r = 4k \geq \frac{n}{2}$, every r-regular graph has a decomposition into k 4-regular subgraphs, each having the property that any five vertices induce at most 8 edges.*

Our Theorem 1.1 extends these results to all $4k$-regular graphs. We conjecture this extends naturally to larger subgraphs, and provide the statement for the next case.

Conjecture. *Every $6k$-regular $(k > 1)$ graph has a decomposition into k 6-regular subgraphs, each having the property that any seven vertices induce at most 18 edges.*

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