Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **5** (2021) 56–62
DOI: 10.47443/dml.2020.0066

Research Article

Symmetric polynomials associated with numerical semigroups

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(Received: 24 November 2020. Received in revised form: 18 March 2021. Accepted: 19 March 2021. Published online: 22 March 2021.)

Abstract

We study a new kind of symmetric polynomials $P_n(x_1,\ldots,x_m)$ of degree n in m real variables, which have arisen in the theory of numerical semigroups. We establish their basic properties and find their representation through the power sums $E_k=\sum_{j=1}^m x_j^k$. We observe a visual similarity between normalized polynomials $P_n(x_1,\ldots,x_m)/\chi_m$, where $\chi_m=\prod_{j=1}^m x_j$, and a polynomial part of a partition function $W(s,\{d_1,\ldots,d_m\})$, which gives the number of partitions of $s\geq 0$ into m positive integers d_j, and we put forward a conjecture about their relationship.

Keywords: symmetric polynomials; numerical semigroups; theory of partition.

2020 Mathematics Subject Classification: 20M14, 11P81, 05A17.

1. Symmetric polynomials $P_n(\mathbf{x}^m)$ and their factorization

In 2017, while studying the polynomial identities of arbitrary degree for syzygies degrees of the numerical semigroups $\langle d_1,\ldots,d_m\rangle$, we introduced a new kind of symmetric polynomials $P_n(x_1,\ldots,x_m)$ of degree n in m real variables x_j (see [1], Section 5.1):

$$P_n(\mathbf{x}^m)=\sum_{j=1}^m x_j^n-\sum_{1\leq j<r}^m (x_j+x_r)^n+\sum_{1\leq j<r<i}^m (x_j+x_r+x_i)^n-\ldots-(-1)^m\left(\sum_{j=1}^m x_j\right)^n, \tag{1}$$

where $\mathbf{x}^m=\{x_1,\ldots,x_m\}$ and $P_n(\mathbf{x}^m)$ is invariant under the action of the symmetric group S_m on a set of variables $\{x_1,\ldots,x_m\}$ by their permutations. Such polynomials arise in the rational representation of the Hilbert series for the complete intersection semigroup ring associated with a symmetric semigroup $\langle d_1,\ldots,d_m\rangle$. According to [1], the polynomials in (1) satisfy

$$P_n(\mathbf{x}^m)=0,\quad 1\leq n\leq m-1 \qquad \text{and} \qquad P_m(\mathbf{x}^m)=(-1)^{m+1}m!\prod_{j=1}^m x_j. \tag{2}$$

In this paper, we study a factorization of $P_n(\mathbf{x}^m)$ for $n>m$ and by making use of this property, we find a representation of $P_n(\mathbf{x}^m)$ through the power sums $E_k=\sum_{j=1}^m x_j^k$, i.e., $P_n(\mathbf{x}^m)=P_n(E_1,\ldots,E_m)$.

Lemma 1.1. *The polynomial $P_n(\mathbf{x}^m)$ vanishes if at least one of the variables x_j vanishes.*

Proof. Since $P_n(\mathbf{x}^m)$ is invariant under all permutations of variables $\{x_1,\ldots,x_m\}$, we have to prove

$$P_n(0,x_2,\ldots,x_m)=0.$$

Denote $P_n(0,x_2,\ldots,x_m)=P_n(0,\mathbf{x}^{m-1})$ and substitute $x_1=0$ into (1),

$$\begin{aligned}P_n(0,\mathbf{x}^{m-1})=&\sum_{j=2}^m x_j^n-\left[\sum_{j=2}^m x_j^n+\sum_{2\leq r<j}^m (x_j+x_r)^n\right]+\left[\sum_{2\leq r<j}^m (x_j+x_r)^n+\sum_{2\leq i<r<j}^m (x_j+x_r+x_i)^n\right]\\&-\left[\sum_{2\leq i<r<j}^m (x_j+x_r+x_i)^n+\sum_{2\leq i<r<j<t}^m (x_t+x_j+x_r+x_i)^n\right]+\ldots\\&+(-1)^m\left[\sum_{j=2}^m\left(\sum_{r=2}^m x_j+x_r\right)^n+\left(\sum_{j=2}^m x_j\right)^n\right]-(-1)^m\left(\sum_{j=2}^m x_j\right)^n.\end{aligned}$$

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Recasting the terms in the last sum in m pairs, we obtain

$$P_n(0,\mathbf{x}^{m-1}) = \left[\sum_{j=2}^m x_j^n - \sum_{j=2}^m x_j^n\right] - \left[\sum_{2\le r<j}^m (x_j+x_r)^n - \sum_{2\le r<j}^m (x_j+x_r)^n\right] + \left[\sum_{2\le i<r<j}^m (x_j+x_r+x_i)^n \right.$$
$$\left. - \sum_{2\le i<r<j}^m (x_j+x_r+x_i)^n\right] - \ldots + (-1)^m \left[\left(\sum_{j=2}^m x_j\right)^n - \left(\sum_{j=2}^m x_j\right)^n\right] = 0,$$

and Lemma 1.1 is proven. □

Corollary 1.1. *The polynomial $P_n(\mathbf{x}^m)$ is divisible by the product $\chi_m = \prod_{j=1}^m x_j$.*

Proof. Since $P_n(\mathbf{x}^m)$ is invariant under all permutations of variables $\{x_1,\ldots,x_m\}$, by Lemma 1.1 it holds that

$$P_n(0,x_2,\ldots,x_m) \quad = \quad P_n(x_1,0,\ldots,x_m) = \ldots = P_n(x_1,x_2,\ldots,0) = 0.$$

Thus, the equation $P_n(x_1,\ldots,x_m) = 0$ has at least m independent roots $x_1 = x_2 = \ldots = x_m = 0$. Then, by the polynomial factor theorem, $P_n(\mathbf{x}^m)$ is divisible by the product χ_m. □

In full agreement with (2), by Corollary 1.1, it follows that $P_n(\mathbf{x}^m) = 0$ if $n < m$ and $P_m(\mathbf{x}^m)/\chi_m$ does not depend on x_j.

Lemma 1.2. *The polynomial $P_n(\mathbf{x}^m)$ is divisible by the sum $E_1 = \sum_{j=1}^m x_j$ if $n - m \equiv 1 \pmod 2$.*

Proof. Rewrite $P_n(\mathbf{x}^m)$ as follows

$$\begin{aligned} P_n(\mathbf{x}^m) \quad = \quad & \sum_{j=1}^m x_j^n - \sum_{1\le j_2<j_1}^m \left(\sum_{k=1}^2 x_{j_k}\right)^n + \sum_{1\le j_3<j_2<j_1}^m \left(\sum_{k=1}^3 x_{j_k}\right)^n - \ldots \\ & -(-1)^m \sum_{1\le j_2<j_1}^m \left(E_1 - \sum_{k=1}^2 x_{j_k}\right)^n + (-1)^m \sum_{j=1}^m (E_1 - x_j)^n - (-1)^m E_1^n, \end{aligned}$$

and substitute there $E_1 = 0$,

$$\begin{aligned} P_n(\mathbf{x}^m) \quad = \quad & \sum_{j=1}^m x_j^n - \sum_{1\le j_2<j_1}^m \left(\sum_{k=1}^2 x_{j_k}\right)^n + \sum_{1\le j_3<j_2<j_1}^m \left(\sum_{k=1}^3 x_{j_k}\right)^n - \ldots \\ & +(-1)^{m+n} \sum_{1\le j_3<j_2<j_1}^m \left(\sum_{k=1}^3 x_{j_k}\right)^n - (-1)^{m+n} \sum_{1\le j_2<j_1}^m \left(\sum_{k=1}^2 x_{j_k}\right)^n + (-1)^{m+n} \sum_{j=1}^m x_j^n. \end{aligned}$$

Recast the terms in the last sum as follows,

$$P_n(\mathbf{x}^m) \quad = \quad \left[1 + (-1)^{m+n}\right] R_{1.n}(\mathbf{x}^m) + \frac{(-1)^\mu}{2}\left[1 + (-1)^m\right] R_{2,n}(\mathbf{x}^m), \tag{3}$$

$$R_{1.n}(\mathbf{x}^m) \quad = \quad \sum_{j=1}^m x_j^n - \sum_{1\le j_2<j_1}^m \left(\sum_{k=1}^2 x_{j_k}\right)^n + \ldots - (-1)^\mu \sum_{1\le j_\mu<\ldots<j_2<j_1}^m \left(\sum_{k=1}^\mu x_{j_k}\right)^n,$$

$$R_{2,n}(\mathbf{x}^m) \quad = \quad \sum_{1\le j_{\mu+1}<\ldots<j_2<j_1}^m \left(\sum_{k=1}^{\mu+1} x_{j_k}\right)^n, \qquad \mu = \left\lfloor \frac{m-1}{2} \right\rfloor, \tag{4}$$

where $\lfloor a \rfloor$ denotes the integer part of a.

According to (3), if $m + n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$, then $P_n(\mathbf{x}^m) = 0$. Consider another case when $m + n \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 2$. Put $m = 2q$ and $n = 2l + 1$ in (3) and (4), and obtain

$$P_{2l+1}\left(\mathbf{x}^{2q}\right) = (-1)^{q-1} \sum_{1\le j_q<\ldots<j_2<j_1}^{2q} \left(\sum_{k=1}^q x_{j_k}\right)^{2l+1}. \tag{5}$$

In (5), a summation in the external sum $\sum_{j_1>j_2>\ldots>j_q=1}^{2q}$ runs over all $(2q)!/(q!)^2$ permutations of $2q$ variables x_j in terms $\left(\sum_{k=1}^q x_{j_k}\right)^{2l+1}$. That is why every such term has in (5) its counterpart,

$$\left(x_{j_1} + x_{j_2} + \ldots + x_{j_q}\right)^{2l+1} \quad \longleftrightarrow \quad \left(x_{i_1} + x_{i_2} + \ldots + x_{i_q}\right)^{2l+1},$$

$$\begin{aligned}&\{x_{j_1},\dots,x_{j_q}\}\cap\{x_{i_1},\dots,x_{i_q}\}=\emptyset,\\&\#\{x_{j_1},\dots,x_{j_q}\}=\#\{x_{i_1},\dots,x_{i_q}\}=q,\\&\sum_{k=1}^{q}x_{j_k}+\sum_{k=1}^{q}x_{i_k}=E_1.\end{aligned}\tag{6}$$

Recomposing the external sum in (5) as a sum over pairs, described in (6),

$$\left(\sum_{k=1}^{q}x_{j_k}\right)^{2l+1}+\left(\sum_{k=1}^{q}x_{i_k}\right)^{2l+1}$$

and making use of the last equality in (6), where $E_1=0$, we arrive at $P_{2l+1}\left(\mathbf{x}^{2q}\right)=0$.

Thus, the polynomial $P_n\left(\mathbf{x}^m\right)$ is divisible by E_1 if $n+m\equiv 1 \pmod 2$. That finishes the proof of Lemma 1.2 since the two equalities, $n+m\equiv 1 \pmod 2$ and $n-m\equiv 1 \pmod 2$, are equivalent.

□

Lemma 1.3. *If $x_i>0$ for all i, then $P_n\left(\mathbf{x}^m\right)$, $n\geq m$, satisfies the following inequalities,*

$$P_n\left(\mathbf{x}^m\right)>0,\quad m\equiv 1 \pmod 2;\qquad P_n\left(\mathbf{x}^m\right)<0,\quad m\equiv 0 \pmod 2.\tag{7}$$

Proof. We prove (7) by induction. First, start with three simple inequalities,

$$\begin{aligned}P_n\left(\mathbf{x}^2\right)&=x_1^n+x_2^n-(x_1+x_2)^n<0,\qquad n\geq 2,\quad x_1,x_2>0,\end{aligned}\tag{8}$$

$$\begin{aligned}P_n\left(\mathbf{x}^3\right)&=-\sum_{k=1}^{n-1}\binom{n}{k}x_3^{n-k}x_1^k-\sum_{k=1}^{n-1}\binom{n}{k}x_3^{n-k}x_2^k+\sum_{k=1}^{n-1}\binom{n}{k}x_3^{n-k}(x_1+x_2)^k\\&=-\sum_{k=1}^{n-1}\binom{n}{k}x_3^{n-k}P_k\left(\mathbf{x}^2\right)>0,\qquad n\geq 3,\quad x_1,x_2,x_3>0,\\P_n\left(\mathbf{x}^4\right)&=-\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}x_1^k-\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}x_2^k-\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}x_3^k+\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}(x_1+x_2)^k+\\&\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}(x_2+x_3)^k+\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}(x_3+x_1)^k-\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}(x_1+x_2+x_3)^k\\&=-\sum_{k=1}^{n-1}\binom{n}{k}x_4^{n-k}P_k\left(\mathbf{x}^3\right)<0,\qquad n\geq 4,\quad x_1,x_2,x_3,x_4>0.\end{aligned}$$

Next, establish an identity for $P_n\left(\mathbf{x}^m\right)$ relating the last one with symmetric polynomials $P_k\left(\mathbf{x}^{m-1}\right)$ of a smaller tuple $\mathbf{x}^{m-1}=\{x_1,\dots,x_{m-1}\}$,

$$P_n\left(\mathbf{x}^m\right)=-\sum_{k=1}^{n-1}\binom{n}{k}x_m^{n-k}\sum_{j=1}^{m-1}x_j^k+\sum_{k=1}^{n-1}\binom{n}{k}x_m^{n-k}\sum_{1\leq j<r}^{m-1}(x_j+x_r)^k-\ldots\pm\sum_{k=1}^{n-1}\binom{n}{k}x_m^{n-k}\left(\sum_{j=1}^{m-1}x_j\right)^k$$

or

$$P_n\left(\mathbf{x}^m\right)=-\sum_{k=1}^{n-1}\binom{n}{k}x_m^{n-k}P_k\left(\mathbf{x}^{m-1}\right),\quad n\geq m,\quad x_1,\dots,x_m>0,\tag{9}$$

which follows by careful recasting the terms in (1) and further simplification.

According to (9), if $P_k\left(\mathbf{x}^{m-1}\right)>0$ irrespectively to k, then $P_k\left(\mathbf{x}^m\right)<0$, and vice versa, if $P_k\left(\mathbf{x}^{m-1}\right)<0$, then $P_k\left(\mathbf{x}^m\right)>0$. On the other hand, the first terms (8) of the alternating sequence $P_n(\mathbf{x}^m)$ with growing m satisfy (7). Then, by induction, inequalities (7) hold for every m. □

2. Representation of the polynomial $P_n(\mathbf{x}^m)$

In this section, we emphasize a hidden relationship between the polynomial $P_n\left(\mathbf{x}^m\right)$ and the polynomial part $W_1(s,\mathbf{d}^m)$ of a restricted partition function. To provide $P_n(\mathbf{x}^m)$ with properties (2) and satisfy Corollary 1.1, we choose the following representation for the polynomial,

$$P_n(\mathbf{x}^m)=\frac{(-1)^{m+1}n!}{(n-m)!}\,\chi_m T_{n-m}\left(\mathbf{x}^m\right),\tag{10}$$

where $T_r(\mathbf{x}^m)$ is a polynomial of degree r in m variables x_j. Combining (10) and Lemma 1.3, we obtain

$$T_r(\mathbf{x}^m) > 0, \qquad x_i > 0 \quad \text{for all} \quad i. \tag{11}$$

A straightforward calculation (with help of Mathematica software) of the first eight polynomials $T_r(\mathbf{x}^m)$ results in the following expressions,

$$\begin{aligned}
T_0(\mathbf{x}^m) &= 1, \\
T_1(\mathbf{x}^m) &= \frac{1}{2}\,E_1, \\
T_2(\mathbf{x}^m) &= \frac{1}{3}\frac{3E_1^2+E_2}{4}, \\
T_3(\mathbf{x}^m) &= \frac{1}{4}\frac{E_1^2+E_2}{2}\,E_1, \\
T_4(\mathbf{x}^m) &= \frac{1}{5}\frac{15E_1^4+30E_1^2E_2+5E_2^2-2E_4}{48}, \\
T_5(\mathbf{x}^m) &= \frac{1}{6}\frac{3E_1^4+10E_1^2E_2+5E_2^2-2E_4}{16}\,E_1, \\
T_6(\mathbf{x}^m) &= \frac{1}{7}\frac{63E_1^6+315E_1^4E_2+315E_1^2E_2^2-126E_1^2E_4+35E_2^3-42E_2E_4+16E_6}{576}, \\
T_7(\mathbf{x}^m) &= \frac{1}{8}\frac{9E_1^6+63E_1^4E_2+105E_1^2E_2^2-42E_1^2E_4+35E_2^3-42E_2E_4+16E_6}{144}\,E_1.
\end{aligned} \tag{12}$$

Formulas (12) for $T_r(\mathbf{x}^m)$ are valid irrespective to the ratio r/m, or, in other words, to the fact how many power sums E_k are algebraically independent. In fact, if $r > m$ then expressions may be compactified by supplementary relations $E_k = E_k(E_1, \ldots, E_m)$, $k > m$.

Unlike to elementary symmetric polynomials $\sum_{i_1<i_2<\ldots<i_r}^m x_{i_1}x_{i_2}\ldots x_{i_r}$ and power sums $E_r(\mathbf{x}^m)$, the symmetric polynomials $T_r(\mathbf{x}^m)$, $0 \le r \le 7$, are algebraically dependent. Indeed, by (12) we get

$$\begin{aligned}
\frac{T_3(\mathbf{x}^m)}{T_1^3(\mathbf{x}^m)} &= 3\frac{T_2(\mathbf{x}^m)}{T_1^2(\mathbf{x}^m)} - 2, \\
\frac{T_5(\mathbf{x}^m)}{T_1^5(\mathbf{x}^m)} &= 5\frac{T_4(\mathbf{x}^m)}{T_1^4(\mathbf{x}^m)} - 20\frac{T_2(\mathbf{x}^m)}{T_1^2(\mathbf{x}^m)} + 16, \\
\frac{T_7(\mathbf{x}^m)}{T_1^7(\mathbf{x}^m)} &= 7\frac{T_6(\mathbf{x}^m)}{T_1^6(\mathbf{x}^m)} - 70\frac{T_4(\mathbf{x}^m)}{T_1^4(\mathbf{x}^m)} + 336\frac{T_2(\mathbf{x}^m)}{T_1^2(\mathbf{x}^m)} - 272.
\end{aligned} \tag{13}$$

It is unlikely to arrive at a general formula for $T_r(\mathbf{x}^m)$ with arbitrary r by observation of the fractions in (12). However, one can recognize a visual similarity between (12) and the other known expressions of special polynomials arisen in the theory of partition [5].

Recall formulas for a polynomial part $W_1(s,\mathbf{d}^m)$ of a restricted partition function $W(s,\mathbf{d}^m)$, where $\mathbf{d}^m = \{d_1,\ldots,d_m\}$, which gives the number of partitions of $s \ge 0$ into m positive integers $(d_1,\ldots,d_m)$ and vanishes, if such partition does not exist. Following formulas (3.16), (7.1) in [1], we obtain

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\,\pi_m}\sum_{r=0}^{m-1}\binom{m-1}{r} f_r(\mathbf{d}^m)s^{m-1-r}, \qquad f_r(\mathbf{d}^m) = \left(\sigma_1 + \sum_{i=1}^m \mathcal{B}\,d_i\right)^r, \tag{14}$$

where $\pi_m = \prod_{j=1}^m d_j$ and $\sigma_1 = \sum_{j=1}^m d_j$. In (14) the formula for $f_r(\mathbf{d}^m)$ presumes a symbolic exponentiation [4]: after binomial expansion the powers $(\mathcal{B}\,d_i)^r$ are converted into the powers of d_i multiplied by Bernoulli's numbers $\mathcal{B}_r$, i.e., $d_i^r\mathcal{B}_r$. A straightforward calculation of first eight polynomials $f_r(\mathbf{d}^m) = f_r(\sigma_1,\ldots,\sigma_r)$ in terms of power sums $\sigma_k = \sum_{j=1}^m d_j^k$ were performed in [1], formulas (7.2),

$$\begin{aligned}
f_0(\mathbf{d}^m) &= 1, \\
f_1(\mathbf{d}^m) &= \frac{1}{2}\,\sigma_1, \\
f_2(\mathbf{d}^m) &= \frac{1}{3}\frac{3\sigma_1^2-\sigma_2}{4},
\end{aligned} \tag{15}$$

$$\begin{aligned}
f_3(\mathbf{d}^m) &= \frac{1}{4}\frac{\sigma_1^2-\sigma_2}{2}\,\sigma_1,\\
f_4(\mathbf{d}^m) &= \frac{1}{5}\frac{15\sigma_1^4-30\sigma_1^2\sigma_2+5\sigma_2^2+2\sigma_4}{48},\\
f_5(\mathbf{d}^m) &= \frac{1}{6}\frac{3\sigma_1^4-10\sigma_1^2\sigma_2+5\sigma_2^2+2\sigma_4}{16}\,\sigma_1,\\
f_6(\mathbf{d}^m) &= \frac{1}{7}\frac{63\sigma_1^6-315\sigma_1^4\sigma_2+315\sigma_1^2\sigma_2^2+126\sigma_1^2\sigma_4-35\sigma_2^3-42\sigma_2\sigma_4-16\sigma_6}{576},\\
f_7(\mathbf{d}^m) &= \frac{1}{8}\frac{9\sigma_1^6-63\sigma_1^4\sigma_2+105\sigma_1^2\sigma_2^2+42\sigma_1^2\sigma_4-35\sigma_2^3-42\sigma_2\sigma_4-16\sigma_6}{144}\,\sigma_1.
\end{aligned}$$

An absence of power sums σ_k with odd indices k are strongly related to the presence of Bernoulli's numbers $\mathcal{B}_r$ in formula (14). A simple comparison of formulas (12) and (15) manifests a visual similarity between polynomials $T_r(\mathbf{x}^m)$ and $f_r(\mathbf{d}^m)$, which we resume in the next conjecture.

Conjecture 2.1. *Let $T_r(\mathbf{x}^m)$ and $f_r(\mathbf{x}^m)$ be symmetric polynomials, defined in (10) and (14), respectively. Then, the following relation holds*

$$T_r(E_1,E_2,\dots,E_r)=f_r(E_1,-E_2,\dots,-E_r),\qquad r\ge 2, \tag{16}$$

where signs of arguments E_j are changed only at $E_2,\dots,E_r$.

3. Parity properties of $W_1(s,\mathbf{d}^m)$ and generalization of identities for $T_r(\mathbf{x}^m)$

The polynomials $T_r(\mathbf{x}^m)$ and $f_r(\mathbf{d}^m)$ possess one more kind of similarity besides of formulas in (12) and (15). It is easy to verify that identities (13) hold for functions $f_r(\mathbf{d}^m)$ by replacing $T_r(\mathbf{x}^m)\to f_r(\mathbf{d}^m)$. Keeping in mind such similarity, let us find a general form of identities for $f_r(\mathbf{d}^m)$. Making use of a recursive relation in [5], formula (12), for their generating function $W_1(s,\mathbf{d}^m)$,

$$W_1(s,\mathbf{d}^m)=W_1(s-d_m,\mathbf{d}^m)+W_1\left(s,\mathbf{d}^{m-1}\right),\qquad \mathbf{d}^{m-1}=\{d_1,\dots,d_{m-1}\}, \tag{17}$$

prove the parity properties

$$W_1\left(s-\frac{\sigma_1}{2},\mathbf{d}^{2m}\right)=-W_1\left(-s-\frac{\sigma_1}{2},\mathbf{d}^{2m}\right),\quad W_1\left(s-\frac{\sigma_1}{2},\mathbf{d}^{2m+1}\right)=W_1\left(-s-\frac{\sigma_1}{2},\mathbf{d}^{2m+1}\right), \tag{18}$$

following a similar proof for the whole partition function $W(s,\mathbf{d}^m)$ in [3], Lemma 4.1. Indeed, the recursive relation (17) may be rewritten for $V_1(s,\mathbf{d}^m)=W_1(s-\sigma_1/2,\mathbf{d}^m)$, where $\sigma_1/2=f_1(\mathbf{d}^m)$,

$$V_1(s,\mathbf{d}^m)=V_1(s-d_m,\mathbf{d}^m)+V_1\left(s-\frac{d_m}{2},\mathbf{d}^{m-1}\right).$$

Making use of a new variable $q=s-d_m/2$, the last relation reads

$$\begin{aligned}
V_1\left(q,\mathbf{d}^{m-1}\right) &= V_1\left(q+\frac{d_m}{2},\mathbf{d}^m\right)-V_1\left(q-\frac{d_m}{2},\mathbf{d}^m\right),\\
-V_1\left(-q,\mathbf{d}^{m-1}\right) &= V_1\left(-q-\frac{d_m}{2},\mathbf{d}^m\right)-V_1\left(-q+\frac{d_m}{2},\mathbf{d}^m\right).
\end{aligned}$$

Hence, if $V_1(q,\mathbf{d}^m)$ is an even function of q, then $V_1\left(q,\mathbf{d}^{m-1}\right)$ is an odd one, and vice versa. But, according to (14), for $m=1$ we have $V_1\left(q,\mathbf{d}^1\right)=W_1\left(q-d_1/2,\mathbf{d}^1\right)=1/d_1$, where $\mathbf{d}^1=\{d_1\}$, or in other words, the function $V_1\left(q,\mathbf{d}^1\right)$ is even in q. Therefore we obtain

$$V_1\left(s,\mathbf{d}^{2m}\right)=-V_1\left(-s,\mathbf{d}^{2m}\right),\qquad V_1\left(s,\mathbf{d}^{2m+1}\right)=V_1\left(-s,\mathbf{d}^{2m+1}\right),$$

that finally leads to (18).

Identities (18) impose a set of relations on $f_r(\mathbf{d}^m)$. To find them, we have to cancel in a series expansion (14) for $W_1\left(s-f_1(\mathbf{d}^{2m}),\mathbf{d}^{2m}\right)$ all terms with even degrees of s

$$s^{2m-1-r}\sum_{k=0}^{r}(-1)^k\binom{2m-1}{r-k}\binom{2m-1-r+k}{k}f_1^k\left(\mathbf{d}^{2m}\right)f_{r-k}\left(\mathbf{d}^{2m}\right), \tag{19}$$

and for $W_1\left(s-f_1(\mathbf{d}^{2m+1}),\mathbf{d}^{2m+1}\right)$ all terms with odd degrees of s

$$s^{2m-r}\sum_{k=0}^{r}(-1)^k\binom{2m}{r-k}\binom{2m-r+k}{k}f_1^k\left(\mathbf{d}^{2m+1}\right)f_{r-k}\left(\mathbf{d}^{2m+1}\right). \tag{20}$$

Making use of identity for binomial coefficients

$$\binom{A-1}{B-1-C}\binom{A-B+C}{C}=\binom{A-1}{B-1}\binom{B-1}{C},\qquad A>B>C\geq 0,$$

and substituting $r=2n-1$ into (19) and (20), and equating them to zero, we obtain, respectively,

$$s^{2(m-n)}\binom{2m-1}{2n-1}\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}f_1^k\left(\mathbf{d}^{2m}\right)f_{2n-1-k}\left(\mathbf{d}^{2m}\right)=0, \tag{21}$$

$$s^{2(m-n)+1}\binom{2m}{2n-1}\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}f_1^k\left(\mathbf{d}^{2m+1}\right)f_{2n-1-k}\left(\mathbf{d}^{2m+1}\right)=0. \tag{22}$$

By comparison (21) and (22) and keeping in mind $f_1\left(\mathbf{d}^m\right)\neq 0$, we arrive at universal relation irrespectively to the parity of m,

$$\frac{f_{2n-1}\left(\mathbf{d}^m\right)}{f_1^{2n-1}\left(\mathbf{d}^m\right)}=\sum_{k=1}^{2n-1}(-1)^{k+1}\binom{2n-1}{k}\frac{f_{2n-1-k}\left(\mathbf{d}^m\right)}{f_1^{2n-1-k}\left(\mathbf{d}^m\right)}. \tag{23}$$

Note, that for $n=1$ equality (23) holds identically. Applying a recursive procedure to formula (23), the last expression may be represented as follows,

$$\begin{aligned}\frac{f_{2n-1}\left(\mathbf{d}^m\right)}{f_1^{2n-1}\left(\mathbf{d}^m\right)} &= \sum_{k_1=1}^{n}\binom{2n-1}{2k_1-1}\frac{f_{2(n-k_1)}\left(\mathbf{d}^m\right)}{f_1^{2(n-k_1)}\left(\mathbf{d}^m\right)}- \\ &\sum_{k_1,k_2=1}^{n}\binom{2n-1}{2k_1}\binom{2(n-k_1)-1}{2k_2-1}\frac{f_{2(n-k_1-k_2)}\left(\mathbf{d}^m\right)}{f_1^{2(n-k_1-k_2)}\left(\mathbf{d}^m\right)}+ \\ &\sum_{k_1,k_2,k_3=1}^{n}\binom{2n-1}{2k_1}\binom{2(n-k_1)-1}{2k_2}\binom{2(n-k_1-k_2)-1}{2k_3-1}\frac{f_{2(n-k_1-k_2-k_3)}\left(\mathbf{d}^m\right)}{f_1^{2(n-k_1-k_2-k_3)}\left(\mathbf{d}^m\right)}-\ldots\end{aligned} \tag{24}$$

where the number of summation is equal n. Finally, formula (24) may be presented in a simpler way

$$\frac{f_{2n-1}\left(\mathbf{d}^m\right)}{f_1^{2n-1}\left(\mathbf{d}^m\right)}=\sum_{r=1}^{n}(-1)^{r+1}C_{n,r}\frac{f_{2(n-r)}(\mathbf{d}^m)}{f_1^{2(n-r)}(\mathbf{d}^m)},\qquad C_{n,r}\in\mathbb{Z}_>, \tag{25}$$

where coefficients $C_{n,r}$ with $r=1,2,3,4$ are calculated below

$$\begin{aligned}C_{n,1} &= \binom{2n-1}{1}, \\ C_{n,2} &= \binom{2n-1}{2}\binom{2n-3}{1}-\binom{2n-1}{3}, \\ C_{n,3} &= \binom{2n-1}{2}\binom{2n-3}{2}\binom{2n-5}{1}-\binom{2n-1}{2}\binom{2n-3}{3}-\binom{2n-1}{4}\binom{2n-5}{1}+\binom{2n-1}{5}, \\ C_{n,4} &= \binom{2n-1}{2}\binom{2n-3}{2}\binom{2n-5}{2}\binom{2n-7}{1}-\binom{2n-1}{2}\binom{2n-3}{4}\binom{2n-7}{1}- \\ &\binom{2n-1}{4}\binom{2n-5}{2}\binom{2n-7}{1}-\binom{2n-1}{2}\binom{2n-3}{2}\binom{2n-5}{3}+ \\ &\binom{2n-1}{2}\binom{2n-3}{5}+\binom{2n-1}{4}\binom{2n-5}{3}+\binom{2n-1}{6}\binom{2n-7}{1}-\binom{2n-1}{7},\end{aligned} \tag{26}$$

and the higher $C_{n,r}$ have to be determined recursively by (24). The total number of terms (products of binomial coefficients) contributing to formula (26) for $C_{n,r}$ is given by 2^{r-1}.

It is easy to verify that formulas (25) do nicely provide the integer coefficients in (13) for $n=2,3,4$ successively. That observation leads us to the conjecture.

Conjecture 3.1. *Let $T_r(\mathbf{x}^m)$ be symmetric polynomials, defined in (10), then $T_r(\mathbf{x}^m)$ satisfy the identities,*

$$\frac{T_{2n-1}(\mathbf{x}^m)}{T_1^{2n-1}(\mathbf{x}^m)} = \sum_{k=1}^{2n-1}(-1)^{k+1}\binom{2n-1}{k}\frac{T_{2n-1-k}(\mathbf{x}^m)}{T_1^{2n-1-k}(\mathbf{x}^m)},$$

$$\frac{T_{2n-1}(\mathbf{d}^m)}{T_1^{2n-1}(\mathbf{d}^m)} = \sum_{r=1}^{n}(-1)^{r+1}C_{n,r}\frac{T_{2(n-r)}(\mathbf{d}^m)}{T_1^{2(n-r)}(\mathbf{d}^m)}, \qquad \sum_{r=1}^{n}(-1)^{r+1}C_{n,r}=1.$$

If Conjecture 3.1 is true, then we can continue the list (13) of identities for polynomials $T_r(\mathbf{x}^m)$, e.g.,

$$\frac{T_9(\mathbf{x}^m)}{T_1^9(\mathbf{x}^m)} = 9\frac{T_8(\mathbf{x}^m)}{T_1^8(\mathbf{x}^m)} - 168\frac{T_6(\mathbf{x}^m)}{T_1^6(\mathbf{x}^m)} + 2016\frac{T_4(\mathbf{x}^m)}{T_1^4(\mathbf{x}^m)} - 9792\frac{T_2(\mathbf{x}^m)}{T_1^2(\mathbf{x}^m)} + 7936, \tag{27}$$

$$\frac{T_{11}(\mathbf{x}^m)}{T_1^{11}(\mathbf{x}^m)} = 11\frac{T_{10}(\mathbf{x}^m)}{T_1^{10}(\mathbf{x}^m)} - 330\frac{T_8(\mathbf{x}^m)}{T_1^8(\mathbf{x}^m)} + 7392\frac{T_6(\mathbf{x}^m)}{T_1^6(\mathbf{x}^m)} - 89760\frac{T_4(\mathbf{x}^m)}{T_1^4(\mathbf{x}^m)} + 436480\frac{T_2(\mathbf{x}^m)}{T_1^2(\mathbf{x}^m)} - 353792,$$

even in the absence of explicit formulas for $T_r(\mathbf{x}^m)$, $r = 8, 9, 10, 11$, in (12).

By observation of formulas (13) and (27), a sequence $C_{n,r}$, $1 \le r \le n$, is unimodal and log-concave for $2 \le n \le 6$. We left open a question whether these properties are preserved for any n.

4. Concluding remarks

The present paper is devoted to the study of polynomials $P_n(\mathbf{x}^m)$ possessing nice algebraic properties and a hidden relationship with restricted partition functions. We put forward two conjectures about such relationship and left open a problem to continue it in different aspects.

There is another reason to study polynomials $P_n(\mathbf{x}^m)$ and their associates, symmetric polynomials $T_{n-m}(\mathbf{x}^m)$, that put them in some wider context. Recent studies [2] of algebraic relations between higher genera† of numerical semigroups with arbitrary embedding dimension, multiplicity and inner symmetries (non-Gorenstein's, Gorenstein's and complete intersection) has shown an important role of polynomials $T_r(\mathbf{x}^m)$, which are involved in these relations (see formulas (22) and (27) in [2]). This makes them interesting objects in commutative algebra.

Acknowledgement

The author is grateful to the referees for their valuable suggestions, which improved the article.

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†Higher genera G_r of numerical semigroup $S_m = \langle d_1, \ldots, d_m\rangle$ are defined as $G_r = \sum_{s\in\Delta_m} s^r$, where $\Delta_m = \mathbb{Z}_> \setminus S_m$ and $\{d_1, \ldots, d_m\}$ denote a set of semigroup gaps and generators of semigroup, respectively. They were studied in the preprint [2], where a finite number of algebraic relations for G_r were found.