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*Research Article*

# Binomial sums about Bernoulli, Euler and Hermite polynomials

Xiaoyuan Wang[1], Wenchang Chu[2,*]

[1]*School of Science, Dalian Jiaotong University, Dalian 116028, P. R. China*
[2]*Department of Mathematics and Physics, University of Salento, Lecce 73100, Italy*

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**Abstract**

Binomial sums about Bernoulli, Euler and Hermite polynomials are examined by making use of the symmetric summation theorem on polynomial differences, which is due to Chu and Magli [*European J. Combin.* **28** (2007) 921–930]. Several summation formulae are also obtained, including Barbero's recent one on Bernoulli polynomials reported in [*Comptes Rendus Math.* **358** (2020) 41–44].

**Keywords:** binomial coefficient; Bernoulli polynomial; Euler polynomial; Hermite polynomial; recurrence relation.

**2020 Mathematics Subject Classification:** 11B68, 05A10.

## 1. Introduction and motivation

In classical analysis and combinatorics, the Bernoulli and Euler numbers play an important role, that are defined respectively by

$$\frac{\tau}{e^\tau-1}=\sum_{n\geq 0}B_n\frac{\tau^n}{n!} \quad\text{and}\quad \frac{2e^\tau}{e^{2\tau}+1}=\sum_{n\geq 0}E_n\frac{\tau^n}{n!}.$$

The corresponding polynomials have the following generating functions:

$$\frac{\tau e^{x\tau}}{e^\tau-1}=\sum_{n\geq 0}B_n(x)\frac{\tau^n}{n!} \quad\text{and}\quad \frac{2e^{x\tau}}{e^{\tau}+1}=\sum_{n\geq 0}E_n(x)\frac{\tau^n}{n!}.$$

Both Bernoulli and Euler polynomials can be expressed by the corresponding numbers through the binomial relations

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} \quad\text{and}\quad E_n(x)=\sum_{k=0}^{n}\binom{n}{k}E_k(0)x^{n-k}.$$

They can be characterized by the following general polynomials associated to an arbitrary sequence $\{a_n\}$ by the binomial sums

$$A_n(x)=\sum_{k=0}^{n}a_k\binom{n}{k}x^{n-k} \quad\text{for}\quad n=0,1,2,\cdots. \tag{1}$$

Chu and Magli [5] found that these polynomials satisfy the following general algebraic identity, which has interesting applications to classical combinatorial numbers and polynomials, such as Bernoulli and Euler polynomials (cf. [9]).

**Lemma 1.1** (Symmetric Difference)**.** *For two variables $x, y$ and three integer parameters $m, n, \ell$ with $m, n$ being nonnegative, the following algebraic identity holds:*

$$\sum_{k=0}^{m}\binom{m}{k}\frac{A_{n+k+\ell}(x)}{(n+k+1)_\ell}(y-x)^{m-k}-\sum_{k=0}^{n}\binom{n}{k}\frac{A_{m+k+\ell}(y)}{(m+k+1)_\ell}(x-y)^{n-k}$$
$$=\frac{m!n!\chi(\ell>0)}{(m+n+\ell)!}\sum_{k=1}^{\ell}\binom{m+n+\ell}{\ell-k}\binom{-k}{m}A_{\ell-k}(y)(x-y)^{m+n+k}.$$

Here and forth, $\chi$ denotes, for brevity, the logical function with $\chi(\text{true})=1$ and $\chi(\text{false})=0$, otherwise. For two integers $i, j$ and a natural number $m$, the notation "$i\equiv_m j$" stands for that "$i$ is congruent to $j$ modulo $m$".

There exist numerous summation formulae and identities about the Bernoulli and Euler numbers and polynomials (cf. [1, 2, 4, 6, 7]). Recently, Barbero [3] discovered a new identity about Bernoulli polynomials. We find that Barbero's

*Corresponding author (chu.wenchang@unisalento.it)

identity is an implication of Lemma 1.1 when $\ell = 1$, $m = n$ and $A_n(x)$ is specified to Bernoulli polynomial. This suggests us to examine further applications of Lemma 1.1. In the next section, we shall prove a general theorem about Bernoulli polynomials, which contains Barbero's identity as the special case $\ell = 1$. Then in Section 3, an analogous theorem for Euler polynomials will be shown, where three interesting formulae corresponding to $\ell < 1, \ell = 1$ and $\ell = 2$ will be highlighted. Finally, we illustrate an application to Hermite polynomials in Section 4, where some unusual identities are deduced.

## 2. Bernoulli polynomials

In Lemma 1.1, performing first the replacements $n \to m$, $y \to n - x$ and then specifying $A_n(x)$ to Bernoulli polynomial, we have the equality (cf. [9])

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k}-\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+\ell}(n-x)}{(m+k+1)_\ell}(2x-n)^{m-k}\\=\frac{m!^2\chi(\ell>0)}{(2m+\ell)!}\sum_{k=1}^{\ell}\binom{2m+\ell}{\ell-k}\binom{-k}{m}B_{\ell-k}(n-x)(2x-n)^{2m+k}.\end{aligned}\tag{2}$$

By iterating the recurrence relation

$$B_m(1+x)=B_m(x)+mx^{m-1},$$

we can reformulate the polynomial

$$\begin{aligned}B_m(n-x)&=B_m(n-1-x)+m(n-1-x)^{m-1}\\&=B_m(n-2-x)+m(n-1-x)^{m-1}+m(n-2-x)^{m-1}\\&=B_m(1-x)+m\sum_{i=1}^{n-1}(n-x-i)^{m-1}.\end{aligned}$$

According to the reciprocal relation

$$B_m(1-x)=(-1)^mB_m(x),$$

we deduce further the expression

$$B_m(n-x)=(-1)^mB_m(x)+m\sum_{i=1}^{n-1}(i-x)^{m-1}.$$

Substituting this into (2) and then simplifying the resultant equation, we get the identity

$$\begin{aligned}\Phi_\ell(m,n)=2\chi(\ell\equiv_2 1)\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k}\\-\frac{m!^2\chi(\ell>0)}{(2m+\ell)!}\sum_{k=1}^{\ell}\binom{2m+\ell}{\ell-k}\binom{-k}{m}B_{\ell-k}(n-x)(2x-n)^{2m+k},\end{aligned}\tag{3}$$

where $\Phi_\ell(m,n)$ is a double sum defined by

$$\Phi_\ell(m,n)=\sum_{k=0}^{m}\binom{m}{k}(2x-n)^{m-k}\sum_{i=1}^{n-1}\frac{(i-x)^{m+k+\ell-1}}{(m+k+1)_{\ell-1}}.\tag{4}$$

The rightmost fraction can be expressed as a multiple integral with the integration domain

$$\Big\{x\le y_{\ell-1}\le y_{\ell-2}\le\cdots\le y_2\le y_1\le i\Big\}$$

and then reformulated by reversing the integral order as

$$\begin{aligned}\frac{(i-x)^{m+k+\ell-1}}{(m+k+1)_{\ell-1}}&=\int_x^i dy_{\ell-1}\int_{y_{\ell-1}}^i dy_{\ell-2}\cdots\int_{y_3}^i dy_2\int_{y_2}^i(i-y_1)^{m+k}dy_1\\&=\int_x^i(i-y_1)^{m+k}dy_1\int_x^{y_1}dy_2\cdots\int_x^{y_{\ell-3}}dy_{\ell-2}\int_x^{y_{\ell-2}}dy_{\ell-1}\\&=\int_x^i(i-y_1)^{m+k}\frac{(y_1-x)^{\ell-2}}{(\ell-2)!}dy_1.\end{aligned}$$

According to the binomial theorem, we get the expression

$$\begin{aligned}\Phi_\ell(m,n) &= \sum_{i=1}^{n-1}\sum_{k=0}^{m}\binom{m}{k}(2x-n)^{m-k}\int_x^i (i-y_1)^{m+k}\frac{(y_1-x)^{\ell-2}}{(\ell-2)!}dy_1\\ &= \sum_{i=1}^{n-1}\int_x^i \frac{(y_1-x)^{\ell-2}}{(\ell-2)!}(i-y_1)^m(2x-n+i-y_1)^m dy_1. \end{aligned} \tag{5}$$

Under the change of variable by $y_1 = i - T(i-x)$, or equivalently $T = \frac{i-y_1}{i-x}$, the last integral becomes

$$\begin{aligned}&\int_x^i (y_1-x)^{\ell-2}(i-y_1)^m(2x-n+i-y_1)^m dy_1\\ =&(i-x)^{m+\ell-1}\int_0^1 T^m(1-T)^{\ell-2}\big\{2x-n+T(i-x)\big\}^m dT.\end{aligned}$$

Expanding the binomial in the braces "$\{\cdots\}$"

$$\begin{aligned}&\big\{2x-n+T(i-x)\big\}^m = \big\{x-n+i-(1-T)(i-x)\big\}^m\\ =&\sum_{j=0}^{m}(-1)^j\binom{m}{j}(1-T)^j(i-x)^j(x-n+i)^{m-j}\end{aligned}$$

and then evaluating the beta integral by

$$\int_0^1 T^m(1-T)^{\ell+j-2}dT = \frac{m!(\ell+j-2)!}{(m+\ell+j-1)!}$$

we find the following expression for the afore-displayed integral:

$$\begin{aligned}&\int_x^i (y_1-x)^{\ell-2}(i-y_1)^m(2x-n+i-y_1)^m dy_1\\ =&\sum_{j=0}^{m}(-1)^j\binom{m}{j}(i-x)^{m+j+\ell-1}(x-n+i)^{m-j}\int_0^1 T^m(1-T)^{\ell+j-2}dT\\ =&\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{(\ell+j-2)!}{(m+1)_{\ell+j-1}}(i-x)^{m+j+\ell-1}(x-n+i)^{m-j}.\end{aligned}$$

By substituting this into (5), we get another double sum expression

$$\Phi_\ell(m,n) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{(\ell-1)_j}{(m+1)_{\ell+j-1}}\Omega_n(m+j+\ell-1,m-j), \tag{6}$$

where $\Omega_n(\lambda,\mu)$ denotes the convolution of arithmetic progressions:

$$\Omega_n(\lambda,\mu) = \sum_{i=1}^{n-1}(i-x)^\lambda(i+x-n)^\mu.$$

Summing up, we have established the following theorem.

**Theorem 2.1.** *For any variable $x$ and three integer parameters $m,n,\ell$ with $m,n$ being nonnegative, the following algebraic identity holds:*

$$\begin{aligned}\Phi_\ell(m,n) = 2\chi(\ell\equiv_2 1)&\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k}\\ &-\frac{m!^2\chi(\ell>0)}{(2m+\ell)!}\sum_{k=1}^{\ell}\binom{2m+\ell}{\ell-k}\binom{-k}{m}B_{\ell-k}(n-x)(2x-n)^{2m+k}.\end{aligned}$$

When $\ell<1$, Theorem 2.1 gives a simpler identity.

**Corollary 2.1** ($\ell<1$: $m\geq 0$ and $n>0$).

$$\Phi_\ell(m,n) = 2\chi(\ell\equiv_2 1)\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k}.$$

When $\ell = 1$, the double sum $\Phi_1(m, n)$ reduces to a single term in view of (6). In this case, we recover from Theorem 2.1 the following identity.

**Corollary 2.2** ($\ell = 1$: $m \geq 0$ and $n > 0$)**.**

$$\Omega_n(m, m) = \frac{(-1)^m (n-2x)^{2m+1}}{(2m+1)\binom{2m}{m}} + 2\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+1}(x)}{m+k+1}(n-2x)^{m-k}.$$

It is obvious that the formula due to Barbero [3, Theorem 1] is equivalent to Corollary 2.2 under the replacement $x \to \frac{n-y}{2}$. However, our formula looks more elegant.

When $\ell = 2$, we find from Theorem 2.1, by taking into account that

$$B_0(x) = 1 \quad \text{and} \quad B_1(x) = x - \frac{1}{2},$$

the following unusual double sum evaluation.

**Corollary 2.3** ($\ell = 2$: $m \geq 0$ and $n > 0$)**.**

$$\Phi_2(m, n) = \sum_{j=0}^{m}\frac{(-1)^j \langle m\rangle_j}{(m+1)_{j+1}}\Omega_n(m+j+1, m-j) = (-1)^m\frac{(n-1)(n-2x)^{2m+1}}{2(2m+1)\binom{2m}{m}}.$$

## 3. Euler polynomials

Analogously, making first the replacements $n \to m$, $y \to n-x$ and then specifying $A_n(x)$ to Euler polynomial in Lemma 1.1, we have another equality (cf. [9])

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k} - \sum_{k=0}^{m}\binom{m}{k}\frac{E_{m+k+\ell}(n-x)}{(m+k+1)_\ell}(2x-n)^{m-k}\\ &= \frac{m!^2\chi(\ell>0)}{(2m+\ell)!}\sum_{k=1}^{\ell}\binom{2m+\ell}{\ell-k}\binom{-k}{m}E_{\ell-k}(n-x)(2x-n)^{2m+k}. \end{aligned} \tag{7}$$

By iterating the recurrence relation

$$E_m(1+x) = 2x^m - E_m(x),$$

we can reformulate the polynomial

$$\begin{aligned}E_m(n-x) &= 2(n-x-1)^m - E_m(n-1-x)\\ &= 2(n-x-1)^m - 2(n-x-2)^m + E_m(n-2-x)\\ &= 2\sum_{i=1}^{n-1}(-1)^{i-1}(n-x-i)^m - (-1)^n E_m(1-x).\end{aligned}$$

According to the reciprocal relation

$$E_m(1-x) = (-1)^m E_m(x),$$

we deduce further the expression

$$E_m(n-x) = 2\sum_{i=1}^{n-1}(-1)^{1+n-i}(i-x)^m - (-1)^{m+n}E_m(x).$$

Substituting this into (7) and then simplifying the resultant equation, we get the following counterpart identity of that in Theorem 2.1 for Euler polynomials.

**Theorem 3.1.** *For any variable $x$ and three integer parameters $m, n, \ell$ with $m, n$ being nonnegative, the following algebraic identity holds:*

$$\begin{aligned}\Psi_\ell(m, n) &= 2\chi(n+\ell \equiv_2 0)\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k}\\ &\quad - \frac{m!^2\chi(\ell>0)}{(2m+\ell)!}\sum_{k=1}^{\ell}\binom{2m+\ell}{\ell-k}\binom{-k}{m}E_{\ell-k}(n-x)(2x-n)^{2m+k},\end{aligned} \tag{8}$$

*where $\Psi_\ell(m, n)$ is a double sum defined by*

$$\Psi_\ell(m, n) = 2\sum_{k=0}^{m}\binom{m}{k}(2x-n)^{m-k}\sum_{i=1}^{n-1}(-1)^{n-i+1}\frac{(i-x)^{m+k+\ell}}{(m+k+1)_\ell}. \tag{9}$$

By carrying out exactly the same procedure as that from (4) to (6), we can write $\Psi_\ell(m,n)$ in terms of a multiple integral

$$\Psi_\ell(m,n) = 2\sum_{i=1}^{n-1}\sum_{k=0}^{m}(-1)^{n-i+1}\binom{m}{k}(2x-n)^{m-k} \times \int_x^i dy_\ell \int_{y_\ell}^i dy_{\ell-1}\cdots\int_{y_3}^i dy_2\int_{y_2}^i (i-y_1)^{m+k}dy_1$$

and then derive the following alternative expression

$$\Psi_\ell(m,n) = 2\sum_{j=0}^{m}(-1)^{n+j+1}\binom{m}{j}\frac{(\ell)_j}{(m+1)_{\ell+j}}\bar{\Omega}_n(m+j+\ell,m-j), \tag{10}$$

where $\bar{\Omega}_n(\lambda,\mu)$ stands for the alternating convolution of arithmetic progressions:

$$\bar{\Omega}_n(\lambda,\mu) = \sum_{i=1}^{n-1}(-1)^i(i-x)^\lambda(i+x-n)^\mu.$$

Theorem 3.1 contains the following three interesting special cases.

**Corollary 3.1** ($\ell < 1$: $m \geq 0$ and $n > 0$)**.**

$$\Psi_\ell(m,n) = 2\chi(n+\ell \equiv_2 0)\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m+k+\ell}(x)}{(m+k+1)_\ell}(n-2x)^{m-k}.$$

**Corollary 3.2** ($\ell = 1$: $m \geq 0$ and $n > 0$)**.**

$$\Psi_1(m,n) = \frac{(-1)^m(n-2x)^{2m+1}}{(2m+1)\binom{2m}{m}} + \begin{cases} 0, & n \equiv_2 0; \\ 2\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m+k+1}(x)}{m+k+1}(n-2x)^{m-k}, & n \equiv_2 1. \end{cases}$$

**Corollary 3.3** ($\ell = 2$: $m \geq 0$ and $n > 0$)**.**

$$\Psi_2(m,n) = \frac{(-1)^m(n-1)(n-2x)^{2m+1}}{2(2m+1)\binom{2m}{m}} + \begin{cases} 0, & n \equiv_2 1; \\ 2\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m+k+2}(x)}{(m+k+1)_2}(n-2x)^{m-k}, & n \equiv_2 0. \end{cases}$$

## 4. Hermite polynomials

The Hermite polynomials are an important class of orthogonal polynomials (cf. Rainville [8, Chapter 11]). They are defined by the exponential generating function

$$e^{2x\tau-\tau^2} = \sum_{n=0}^{\infty}H_n(x)\frac{\tau^n}{n!}.$$

- Explicit expression

$$H_n(x) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n}{2k}\frac{(2k)!}{k!}(2x)^{n-2k}.$$

- Reciprocal relation

$$H_n(-x) = (-1)^n H_n(x).$$

- Expansion formula

$$H_n(x+y) = \sum_{k=0}^{n}(2y)^k\binom{n}{k}H_{n-k}(x).$$

Comparing (1) with the explicit formula of $H_n(x)$, we can see that there exists a reciprocal relation corresponding to Lemma 1.1, where $A_n(x)$ is specified by $H_n(x/2)$. Under the replacements $x \to 2x$ and $y \to 2y$, this reciprocity is stated in the following theorem.

**Theorem 4.1.** *For two variables $x, y$ and three integer parameters $m, n, \ell$ with $m, n$ being nonnegative, the following algebraic identity holds:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\frac{H_{n+k+\ell}(x)}{(n+k+1)_\ell}(2y-2x)^{m-k}-\sum_{k=0}^{n}\binom{n}{k}\frac{H_{m+k+\ell}(y)}{(m+k+1)_\ell}(2x-2y)^{n-k}\\&=\frac{m!n!\chi(\ell>0)}{(m+n+\ell)!}\sum_{k=1}^{\ell}\binom{m+n+\ell}{\ell-k}\binom{-k}{m}H_{\ell-k}(y)(2x-2y)^{m+n+k}.\end{aligned}\tag{11}$$

When $m = n$ and $x = -y$, this theorem gives the simpler expression below:

$$0=2\chi(\ell\equiv_2 1)\sum_{k=0}^{n}(-4y)^{n-k}\binom{n}{k}\frac{H_{n+k+\ell}(y)}{(n+k+1)_\ell}+\frac{n!^2\chi(\ell>0)}{(2n+\ell)!}\sum_{k=1}^{\ell}(-4y)^{2n+k}\binom{2n+\ell}{\ell-k}\binom{-k}{n}H_{\ell-k}(y).\tag{12}$$

In particular, the following summation formulae are believed to be new.

- $\ell \equiv_2 0$ with $\ell > 0$:

$$\sum_{k=1}^{\ell}(-4y)^{2n+k}\binom{2n+\ell}{\ell-k}\binom{-k}{n}H_{\ell-k}(y)=0.$$

- $\ell \equiv_2 1$ with $\ell \le 0$:

$$\sum_{k=0}^{n}(-4y)^{n-k}\binom{n}{k}\frac{H_{n+k+\ell}(y)}{(n+k+1)_\ell}=0.$$

- $\ell \equiv_2 1$ with $\ell > 0$:

$$0=\sum_{k=1}^{\ell}(-4y)^{2n+k}\binom{2n+\ell}{\ell-k}\binom{-k}{n}H_{\ell-k}(y)+\frac{2(2n+\ell)!}{n!^2}\sum_{k=0}^{n}(-4y)^{n-k}\binom{n}{k}\frac{H_{n+k+\ell}(y)}{(n+k+1)_\ell}.$$

Alternatively, for $n = m$ and $y = n - x$, the corresponding relation in Theorem 4.1 becomes

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\frac{H_{m+k+\ell}(x)}{(m+k+1)_\ell}(2n-4x)^{m-k}-\sum_{k=0}^{m}\binom{m}{k}\frac{H_{m+k+\ell}(n-x)}{(m+k+1)_\ell}(4x-2n)^{m-k}\\&=\frac{m!^2\chi(\ell>0)}{(2m+\ell)!}\sum_{k=1}^{\ell}\binom{2m+\ell}{\ell-k}\binom{-k}{m}H_{\ell-k}(n-x)(4x-2n)^{2m+k}.\end{aligned}\tag{13}$$

Unlike Bernoulli and Euler polynomials, the last expression cannot further be reduced unfortunately. Even though by making use of the expansion

$$H_m(n-x)=\sum_{j=0}^{m}(2n)^{m-j}\binom{m}{j}H_j(-x)=\sum_{j=0}^{m}(-1)^j(2n)^{m-j}\binom{m}{j}H_j(x),$$

we can reformulate the second sum with respect to $k$ in (13) as

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\frac{H_{m+k+\ell}(n-x)}{(m+k+1)_\ell}(4x-2n)^{m-k}\\&=\sum_{k=0}^{m}\binom{m}{k}\frac{(4x-2n)^{m-k}}{(m+k+1)_\ell}\sum_{j=0}^{m+k+\ell}(-1)^j(2n)^{m+k+\ell-j}\binom{m+k+\ell}{j}H_j(x)\\&=\sum_{k=0}^{m}\binom{m}{k}\frac{(4x-2n)^{m-k}}{(m+k+1)_\ell}\sum_{i=-m-\ell}^{k}(-1)^{m+i+\ell}(2n)^{k-i}\binom{m+k+\ell}{m+i+\ell}H_{m+i+\ell}(x)\\&=\sum_{i=-m-\ell}^{m}(-1)^{m+i+\ell}H_{m+i+\ell}(x)\sum_{k=\max\{0,i\}}^{m}\binom{m}{k}\binom{m+k+\ell}{m+i+\ell}\frac{(2n)^{k-i}(4x-2n)^{m-k}}{(m+k+1)_\ell}.\end{aligned}$$

However, it is not plausible to simplify this last double sum further.

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