On the normalized (distance) Laplacian spectrum of linear dependence graph of a finite-dimensional vector space

Jinu Mary Jameson\textsuperscript{1,2}, Gopalapilla Indulal\textsuperscript{3,*}

\textsuperscript{1}Department of Mathematics, St. Berchmans College, Changanassery, Kottayam, India
\textsuperscript{2}Department of Mathematics, Marthoma College, Thiruvalla, Pathanamthitta, India
\textsuperscript{3}Department of Mathematics, St. Aloysius College, Edathua, Alappuzha, India

(Received: 11 November 2020. Received in revised form: 17 February 2021. Accepted: 3 March 2021. Published online: 9 March 2021.)

© 2021 the authors. This is an open access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/)

Abstract

Let \( W \) be a finite-dimensional vector space over a finite field \( F \). The linear dependence graph \( \Gamma(W) \) of \( W \) is a simple graph whose vertices are the elements of the vector space \( W \) and two distinct vertices \( u, v \) of the graph \( \Gamma(W) \) are adjacent if and only if \( u \) and \( v \) are linearly dependent in \( W \). In this paper, we describe the normalized Laplacian spectrum and normalized distance Laplacian spectrum of the graph \( \Gamma(W) \).

Keywords: linear dependence graph; normalized Laplacian matrix; normalized Laplacian spectrum; normalized distance Laplacian matrix; normalized distance Laplacian spectrum.

2020 Mathematics Subject Classification: 05C12, 05C50.

1. Introduction

The existing literature on algebraic graph theory gives different kinds of spectra associated with a graph. Spectral analysis of graphs has been the subject of considerable research in algebraic graph theory from the pioneering work of Hückel [5]. The study of different algebraic structures, using the properties of graphs associated to them, is also a well-studied topic. In this paper, we associate a finite-dimensional vector space over a finite field to a simple graph by using the linear dependency of two vectors in the considered space.

Let \( F \) be a finite field with \( q \) elements and \( W \) be a finite-dimensional vector space over \( F \) with dimension \( n \). Hence, the vector space \( W \) consists of \( q^n \) elements. Two finite-dimensional vector spaces over \( F \) are isomorphic if and only if they have the same dimension. The linear dependence graph \( \Gamma(W) \) associated to the vector space \( W(F) \) is defined [6] as the graph with vertex set \( W \) where two vertices \( u, v \) of the graph \( \Gamma(W) \) are adjacent if and only if \( u \) and \( v \) are linearly dependent in \( W \). The linear dependence graph of the 2-dimensional vector space \( W = \{ (0, 0), (0, 1), (1, 0), (2, 0), (0, 2), (1, 1), (2, 2), (1, 2), (2, 1) \} \) over the finite field \( F_3 = \{ 0, 1, 2 \} \) is shown in Figure 1.

The adjacency spectrum of \( \Gamma(W) \) was found in [6]. The study of graphs related to algebraic structures along the spectral aspect gained momentum through the work of Maity and Bhuniya [6]. For some recent works along these lines, see [4,8,9]. In this paper, we obtain the normalized Laplacian and normalized distance Laplacian spectra of \( \Gamma(W) \).

Let \( G = (V, E) \) be a simple graph with vertex set \( V(G) = \{ v_1,v_2,\ldots,v_n \} \). Denote by \( D(G) \) the diagonal matrix of \( G \) (also known as the degree matrix of \( G \)). The adjacency matrix of \( G \) is the matrix \( A(G) = [a_{ij}] \), \( 1 \leq i, j \leq n \), where \( a_{ij} = 1 \) if \( v_i \) adjacent to \( v_j \) and \( a_{ij} = 0 \) otherwise. The Laplacian matrix \( L(G) \) is defined as \( L(G) = D(G) - A(G) \). The normalized Laplacian matrix \( \mathcal{L}(G) \) was introduced in [3] as the square matrix whose \((i,j)\)-entry \( \mathcal{L}_{ij} \), \( 1 \leq i, j \leq n \), is given as

\[
\mathcal{L}_{ij} = \begin{cases} 
1 & \text{if } v_i = v_j \text{ and } d_{v_i} \neq 0, \\
-\frac{1}{\sqrt{d_{v_i}d_{v_j}}} & \text{if } v_i \text{ and } v_j \text{ are adjacent}, \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( d_{v_i} \) and \( d_{v_j} \) are degrees of \( v_i \) and \( v_j \), respectively. We can also write

\[
\mathcal{L}(G) = D^{-1/2}(G)L(G)D^{-1/2}(G) = I - D^{-1/2}(G)A(G)D^{-1/2}(G)
\]

*Corresponding author (indulalgopal@aloysiuscollege.ac.in).
Figure 1: The linear dependence graph of the 2-dimensional vector space \( W \) over the finite field \( F_3 = \{0, 1, 2\} \) where \( W = \{(0, 0), (0, 1), (1, 0), (2, 0), (0, 2), (1, 1), (2, 1), (1, 2), (2, 1)\} \).

where \( A(G), L(G) \) and \( D(G) \) are the adjacency matrix, Laplacian matrix and degree matrix respectively of the graph \( G \) with the convention that \( D(G)^{-1}(u, u) = 0 \) if \( d_u = 0 \). The eigenvalues of \( L(G) \) are called normalized Laplacian eigenvalues of \( G \) and forms the \( \ell \)-spectrum of \( G \). Two graphs are said to be \( \ell \)-cospectral if they have the same \( \ell \)-spectrum. In [3], Chung and Graham showed that \( L(G) \) is a positive definite symmetric matrix and all its eigenvalues lie within the interval [0, 2] and that 0 is always a normalized Laplacian eigenvalue of any graph.

Let \( P(G) = D(G)^{-1}A(G) \). Then, \( L(G) = D(G)^{1/2}(I - D(G)^{-1}A(G))D(G)^{-1/2} = D(G)^{1/2}(I - P(G))D(G)^{-1/2} \). Hence, if the spectrum of \( P(G) \) is \( \{\mu_1, \mu_2, \ldots, \mu_n\} \), then the spectrum of \( L(G) \) is \( \{(1 - \mu_1), (1 - \mu_2), \ldots, (1 - \mu_n)\} \).

The distance matrix \( D(G) \) is the matrix whose \( (i, j) \)-entry is equal to \( d_G(v_i, v_j) \), the distance between the vertices \( v_i \) and \( v_j \) of \( G \). The eigenvalues of \( D(G) \) are said to be the \( D \)-eigenvalues of \( G \). The transmission matrix \( T(v) \) of a vertex \( v \) of \( G \) is the sum of the distances from \( v \) to all other vertices of \( G \) and the transmission matrix \( T(G) \) is the diagonal matrix defined as \( T(G) = \text{diag}(Tr(v_1), Tr(v_2), \ldots, Tr(v_n)) \). The distance Laplacian matrix is defined as \( D^L(G) = T(G) - D(G) \). The normalized distance Laplacian matrix is defined [7] as \( D^N(G) = T(G)^{-1/2}D^L(G)T(G)^{-1/2} \). Eigenvalues of \( D^N(G) \) are called normalized distance Laplacian eigenvalues of \( G \) and forms the \( D^N \)-spectrum of \( G \). Two graphs are said to be \( D^N \)-cospectral if they have same \( D^N \)-spectrum. A graph \( G \) is said to be a \( r \)-transmission regular graph if \( Tr(v) = r \) for every vertex \( v \) of \( G \). If \( G \) is \( r \)-transmission regular then,

\[
D^L(G) = rI - D(G) \quad \text{and} \quad D^N(G) = T(G)^{-1/2}(T(G) - D(G))T(G)^{-1/2} = I - \frac{D(G)}{r}.
\]

Let \( \hat{P}(G) = T(G)^{-1}D(G) \). Then \( D^N(G) = T(G)^{1/2}(I - T(G)^{-1}D(G))T(G)^{-1/2} = T(G)^{1/2}(I - \hat{P}(G))T(G)^{-1/2} \). Hence if the spectrum of \( \hat{P}(G) \) is \( \{\mu'_1, \mu'_2, \ldots, \mu'_n\} \), then the spectrum of \( D^N(G) \) is \( \{(1 - \mu'_1), (1 - \mu'_2), \ldots, (1 - \mu'_n)\} \).

Since we are dealing with the graph \( \Gamma(W) \) throughout this paper \( A(\Gamma(W)), D(\Gamma(W)), P(\Gamma(W)), D(\Gamma(W)), T(\Gamma(W)), \hat{P}(\Gamma(W)) \) are denoted as simply \( A, D, P, D, T, \hat{P} \), respectively. Also, \( I_n \) denotes the zero matrix of order \( n \), \( J_n \) denotes all-one matrix of order \( n \) and \( J_{m \times n} \) denotes the all-one matrix of order \( m \times n \). All spectral graph theoretic terminology are taken from [3]. We use the following lemma in the next section.

**Lemma 1.1.** [2] Let \( M, N, P \) and \( Q \) be matrices such that \( M \) invertible. Let

\[
S = \begin{bmatrix}
M & N \\
P & Q
\end{bmatrix}.
\]

Then \( |S| = |M||Q - PM^{-1}N| \) and if \( M \) and \( P \) commutes, then \( |S| = |MQ - PN| \) where the symbol \(|.|\) denotes the determinant.

2. The normalized Laplacian spectrum of the linear dependence graph \( \Gamma(W) \)

In this section, we find the normalized Laplacian spectrum of \( \Gamma(W) \) in terms of \( q \) by using the spectrum of \( P \).

**Theorem 2.1.** Let \( F \) be a finite field of order \( q \geq 2 \) and \( W \) be an \( n \)-dimensional vector space over \( F \). Then, the spectrum of \( P \) is

\[
\begin{bmatrix}
n & -(q - 1)^{-1} & (q - 2)(q - 1)^{-1} \\
1 & k(q - 2) + 1 & k - 1
\end{bmatrix},
\]

where \( k = q^{n-1} + q^{n-2} + \ldots + q + 1 \).
Proof. In [6], the adjacency matrix of $\Gamma(W)$ was given as

$$A = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
0 & J_{1 \times (q^n-1)} \\
J_{(q^n-1) \times 1} & [J_{q-1} - I_{q-1}] \otimes I_k
\end{bmatrix}
$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$. The degree of the zero vector is $q^n - 1$ and degree of all other vectors in $W$ is $q - 1$. Hence the degree matrix of $\Gamma(W)$ is given by

$$D = \begin{bmatrix}
q^n-1 & 0 & 0 & \cdots & 0 \\
0 & q-1 & 0 & \cdots & 0 \\
0 & 0 & q-1 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & q-1
\end{bmatrix} \begin{bmatrix}
q^n-1 & 0_{1 \times (q^n-1)} \\
0_{(q^n-1) \times 1} & (q-1)^{-1} I_{q-1} \otimes I_k
\end{bmatrix}
= \begin{bmatrix}
(q^n-1)^{-1} & 0_{1 \times (q^n-1)} \\
[(q-1)^{-1} I_{q-1} \otimes I_k] & [(q-1)^{-1} I_{q-1} \otimes I_k] \otimes I_k
\end{bmatrix}
$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$. Hence,

$$P = D^{-1} A = \begin{bmatrix}
(q^n-1)^{-1} & 0_{1 \times (q^n-1)} \\
0_{(q^n-1) \times 1} & (q-1)^{-1} I_{q-1} \otimes I_k
\end{bmatrix} \begin{bmatrix}
0 & J_{1 \times (q^n-1)} \\
J_{(q^n-1) \times 1} & [J_{q-1} - I_{q-1}] \otimes I_k
\end{bmatrix}
= \begin{bmatrix}
0 & (q^n-1)^{-1} J_{1 \times (q^n-1)} \\
[(q-1)^{-1} I_{q-1} \otimes I_k] & [(q-1)^{-1} I_{q-1} \otimes I_k] \otimes I_k
\end{bmatrix}
= \begin{bmatrix}
0 & (q^n-1)^{-1} J_{1 \times (q^n-1)} \\
(q-1)^{-1} J_{(q^n-1) \times 1} & (q-1)^{-1} [J_{q-1} - I_{q-1}] \otimes I_k
\end{bmatrix}
$$

The characteristic polynomial of $P$ is

$$(\lambda I_{q^n} - P) = \begin{vmatrix}
\lambda & -(q^n-1)^{-1} J_{1 \times (q^n-1)} \\
-(q-1)^{-1} J_{(q^n-1) \times 1} & \lambda I_{q^n} - ((q-1)^{-1} [J_{q-1} - I_{q-1}] \otimes I_k)
\end{vmatrix}
= (q^n-1)^{-1} (q-1)^{-1} \begin{vmatrix}
\lambda & 0 \\
0 & -(q^n-1)^{-1} J_{1 \times (q^n-1)} \\
-J_{(q^n-1) \times 1} & \lambda I_{q^n} - ((q-1)^{-1} [J_{q-1} - I_{q-1}] \otimes I_k)
\end{vmatrix}
$$

Multiplying the first row by $\lambda (q-1) - (q - 2)$ and $R_1^r \mapsto R_1^r + R_2 + \ldots + R_{q^n}$ and applying Lemma 1.1 yield

$$\det(\lambda I_{q^n} - P) = \frac{(q^n-1)^{-1} (q-1)^{-1} (q^n-1)^{-1}}{\lambda (q-1) - (q - 2)} \begin{vmatrix}
\lambda & -J_{(q^n-1) \times 1} \\
-J_{(q^n-1) \times 1} & \lambda (q-1) - (q - 2)
\end{vmatrix}
= \frac{(q^n-1)^{-1} (q-1)^{-1} (q^n-1)^{-1}}{\lambda (q-1) - (q - 2)} \begin{vmatrix}
\lambda (q^n-1) - (q - 2) & -(q^n-1) \det([\lambda (q^n-1) - (q - 2)])^k
\end{vmatrix}
$$

If $A_1 = \det(\lambda (q^n-1) - (q - 2))$ then

$$A_1 = \begin{vmatrix}
\lambda (q-1) & -1 & \cdots & -1 \\
-1 & \lambda (q-1) & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & \lambda (q-1)
\end{vmatrix}_{(q^n-1) \times (q^n-1)}$$
If the spectrum of \( L \) be an \( n \)-dimensional vector space over \( F \). Then the normalized Laplacian spectrum of the linear dependence graph \( \Gamma(W) \) is

\[
\begin{pmatrix}
0 & (q-1)^{-1} & (q-1)^{-1} \\
1 & k(q-2) + 1 & k - 1
\end{pmatrix}
\]

Proof. If the spectrum of \( P \) is \( \{\mu_1, \mu_2, \ldots, \mu_r\} \), then the spectrum of \( L(\Gamma(W)) \) is \( \{(1 - \mu_1), (1 - \mu_2), \ldots, (1 - \mu_r)\} \). Thus the \( q^n \) normalized Laplacian eigenvalues of \( \Gamma(W) \) are 0 with multiplicity 1, \( 1 + (q-1)^{-1} = q(q-1)^{-1} \) with multiplicity \( k(q-2) + 1 \) and \( 1 - (q-2)(q-1)^{-1} = (q-1)^{-1} \) with multiplicity \( k-1 \).
3. The normalized distance Laplacian spectrum of the linear dependence graph $\Gamma(W)$

In this section, we find the normalized distance Laplacian spectrum of $\Gamma(W)$ in terms of $q$ by using the spectrum of $\bar{P}$.

**Theorem 3.1.** Let $F$ be a finite field of order $q \geq 2$ and $W$ be an $n$-dimensional vector space over $F$. Then, the spectrum of $\bar{P}$ is

$$
\begin{pmatrix}
1 & -(2q^n - q - 1)^{-1} & -q(2q^n - q - 1)^{-1} \\
1 & k(q - 2) + 1 & k - 1
\end{pmatrix},
$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$.

**Proof.** In [6], the distance matrix of $\Gamma(W)$ was given as

$$D = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & 2 \\
1 & 1 & 0 & \cdots & 1 & 2 & 2 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 2 & 2 & \cdots & 2 & 2 \\
1 & 2 & 2 & \cdots & 2 & 0 & 1 & \cdots & 1 & 2 \\
1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 1 \\
1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 1 \\
\end{bmatrix},
$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$. The transmission matrix of $\Gamma(W)$ is given by

$$T = \begin{bmatrix}
q^n - 1 & 0_{1 \times (q^n - 1)} \\
0_{(q^n - 1) \times 1} & (2q^n - q - 1)I_{q-1} \otimes I_k
\end{bmatrix},$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$. Hence,

$$\bar{P} = T^{-1}D = \begin{bmatrix}
(q^n - 1)^{-1} & 0_{1 \times (q^n - 1)} \\
0_{(q^n - 1) \times 1} & (2q^n - q - 1)^{-1}I_{q-1} \otimes I_k
\end{bmatrix},$$

which is

$$\left(\begin{array}{cc}
0 & J_{1 \times (q^n - 1)} \\
J_{(q^n - 1) \times 1} & 2J_{(q^n - 1)} - ([J_{q-1} + I_{q-1}] \otimes I_k)
\end{array}\right).$$

The characteristic polynomial of $\bar{P}$ is

$$\det(\lambda I_{q^n} - \bar{P}) = \begin{vmatrix}
\lambda & -(q^n - 1)^{-1}J_{1 \times (q^n - 1)} \\
-(2q^n - q - 1)^{-1}J_{(q^n - 1) \times 1} & \lambda I_{q^n} - 2(2q^n - q - 1)^{-1}J_{q-1} + ([2q^n - q - 1]^{-1}(J_{q-1} + I_{q-1}) \otimes I_k)
\end{vmatrix},$$

$$= (q^n - 1)^{-1}((2q^n - q - 1)^{-1}[q^n - 1]) \begin{vmatrix}
\lambda(q^n - 1) & -J_{(q^n - 1) \times 1} \\
-J_{(q^n - 1) \times 1} & \lambda I_{q^n} - 2(2q^n - q - 1)^{-1}J_{q-1} + ([2q^n - q - 1]^{-1}(J_{q-1} + I_{q-1}) \otimes I_k)
\end{vmatrix}.$$

Now, apply the elementary transformations of subtracting from each column $C_i$, twice of $C_i$ for $i = 2, 3, \ldots, q^n$, result in

$$\det(\lambda I_{q^n} - \bar{P}) = (q^n - 1)^{-1}((2q^n - q - 1)^{-1}[q^n - 1]) \begin{vmatrix}
\lambda(q^n - 1) & -[1 + 2\lambda(q^n - 1)]J_{(q^n - 1) \times 1} \\
-J_{(q^n - 1) \times 1} & ([\lambda(2q^n - q - 1) + 1]J_{q-1} + J_{q-1}) \otimes I_k
\end{vmatrix},$$

$$= (q^n - 1)^{-1}((2q^n - q - 1)^{-1}[q^n - 1]) \begin{vmatrix}
\lambda(q^n - 1) & -J_{(q^n - 1) \times 1} \\
-J_{(q^n - 1) \times 1} & ([\lambda(2q^n - q - 1) + 1]J_{q-1} + J_{q-1}) \otimes I_k
\end{vmatrix}.$$

Multiplying the first row by $\lambda(2q^n - q - 1) + q$ and $R_i' \rightarrow R_i' + R_2 + \ldots + R_{q^n}$ yield

$$\det(\lambda I_{q^n} - \bar{P}) = \begin{vmatrix}
\lambda(q^n - 1) & \lambda(2q^n - q - 1) + q \\
-J_{(q^n - 1) \times 1} & 0_{1 \times (q^n - 1)}
\end{vmatrix}.$$
\[
\frac{\lambda(q^n-1)[\lambda(2q^n-q-1)+q]-(q^n-1)(1+2\lambda(q^n-1))}{(q^n-1)(2q^n-q-1)^{(q^n-1)}} \det([\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1})^k \text{ using Lemma 1.1.}
\]

If \( B_1 = \det[\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1} \) then
\[
B_1 = \begin{vmatrix}
\lambda(2q^n-q-1)+2 & 1 & \cdots & 1 \\
1 & \lambda(2q^n-q-1)+2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & \lambda(2q^n-q-1)+2 \\
\end{vmatrix}_{(q-1) \times (q-1)}
\]

Multiply first row by \( \lambda(2q^n-q-1)+(q-1) \) and then \( R_1 \rightarrow R_1-R_2-\ldots-R_{q-1}, \) we have
\[
B_1 = \begin{vmatrix}
\lambda(2q^n-q-1)+2 & 1 & \cdots & 1 \\
1 & \lambda(2q^n-q-1)+2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & \lambda(2q^n-q-1)+2 \\
\end{vmatrix}_{(q-2) \times (q-2)}
\]

Proceeding like this, we have \( B_1 = [\lambda(2q^n-q-1)+1]^{(q-2)}[\lambda(2q^n-q-1)+q] \) and hence,
\[
\det(\lambda I_{q^n} - \bar{P}) = \frac{\lambda(q^n-1)\lambda(2q^n-q-1)+q-(q^n-1)(1+2\lambda(q^n-1))}{(q^n-1)(2q^n-q-1)^{(q^n-1)}}[\lambda(2q^n-q-1)+1]^{(q-2)}[\lambda(2q^n-q-1)+q]^k.
\]

The right hand side of the above equation is equal to
\[
\frac{\lambda^2(q^n-1)(2q^n-q-1)+\lambda(q^n-1)(q-2(q^n-1))-(q^n-1)}{(q^n-1)(2q^n-q-1)^{(q^n-1)}}[\lambda(2q^n-q-1)+1]^{(q-2)}[\lambda(2q^n-q-1)+q]^{k-1}.
\]

Thus, the eigenvalues of \( \bar{P} \) are given by the following equations
\[
\lambda^2(q^n-1)(2q^n-q-1)+\lambda(q^n-1)(q-2(q^n-1))-(q^n-1) = 0,
\]
\[
[\lambda(2q^n-q-1)+1]^{(q-2)}[\lambda(2q^n-q-1)+q]^{k-1} = 0.
\]

After solving the above equations, we conclude that the \( q^n \) eigenvalues of \( \bar{P} \) are 1 with multiplicity \( k(q-2)+1 \) and \( -(2q^n-q-1)^{-1} \) with multiplicity \( k-1 \) respectively.

\[ \square \]

**Corollary 3.1.** Let \( F \) be a finite field of order \( q \geq 2 \) and \( W \) be an \( n \)-dimensional vector space over \( F \). Then the normalized distance Laplacian spectrum of the linear dependence graph \( \Gamma(W) \) is
\[
\begin{pmatrix}
0 & 1 + (2q^n-q-1)^{-1} & 1 + q(2q^n-q-1)^{-1} \\
1 & k(q-2) + 1 & k - 1 \\
\end{pmatrix}.
\]

**Proof.** If the spectrum of \( \bar{P} \) is \( \{\mu_1', \mu_2', \ldots, \mu_r'\} \), then the spectrum of \( D^N(\Gamma(W)) \) is \( \{(1-\mu_1'), (1-\mu_2'), \ldots, (1-\mu_r')\} \). Thus, the \( q^n \) normalized distance Laplacian eigenvalues of \( \Gamma(W) \) are 0 with multiplicity 1, \( 1 + (2q^n-q-1)^{-1} \) with multiplicity \( k(q-2) + 1 \) and \( 1 + q(2q^n-q-1)^{-1} \) with multiplicity \( k-1 \) respectively.

\[ \square \]

4. **Conclusion**

In this paper, we obtained the normalized Laplacian and normalized distance Laplacian spectra of the linear dependence graph \( \Gamma(W) \) of a finite-dimensional vector space \( W \) over a finite field \( F \). These spectra are the new addition to the existing literature of spectral graph theory. A similar treatment can be applied to obtain other spectra associated with other matrices of a linear dependence graph.
References


