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Research Article

On the normalized (distance) Laplacian spectrum of linear dependence graph of a finite-dimensional vector space

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Abstract

Let W be a finite-dimensional vector space over a finite field F. The linear dependence graph $\Gamma(W)$ of W is a simple graph whose vertices are the elements of the vector space W and two distinct vertices u, v of the graph $\Gamma(W)$ are adjacent if and only if u and v are linearly dependent in W. In this paper, we describe the normalized Laplacian spectrum and normalized distance Laplacian spectrum of the graph $\Gamma(W)$.

Keywords: linear dependence graph; normalized Laplacian matrix ; normalized Laplacian spectrum; normalized distance Laplacian matrix; normalized distance Laplacian spectrum.

2020 Mathematics Subject Classification: 05C12, 05C50.

1. Introduction

The existing literature on algebraic graph theory gives different kinds of spectra associated with a graph. Spectral analysis of graphs has been the subject of considerable research in algebraic graph theory from the pioneering work of Hückel [5]. The study of different algebraic structures, using the properties of graphs associated to them, is also a well-studied topic. In this paper, we associate a finite-dimensional vector space over a finite field to a simple graph by using the linear dependency of two vectors in the considered space.

Let F be a finite field with q elements and W be a finite-dimensional vector space over F with dimension n. Hence, the vector space W consists of q^n elements. Two finite-dimensional vector spaces over F are isomorphic if and only if they have the same dimension. The linear dependence graph $\Gamma(W)$ associated to the vector space $W(F)$ is defined [6] as the graph with vertex set W where two vertices u, v of the graph $\Gamma(W)$ are adjacent if and only if u and v are linearly dependent in W. The linear dependence graph of the 2-dimensional vector space $W = \{(0,0),(0,1),(1,0),(2,0),(0,2),(1,1),(2,2),(1,2),(2,1)\}$ over the finite field $F_3 = \{0,1,2\}$ is shown in Figure 1.

The adjacency spectrum of $\Gamma(W)$ was found in [6]. The study of graphs related to algebraic structures along the spectral aspect gained momentum through the work of Maity and Bhuniya [6]. For some recent works along these lines, see [4,8,9]. In this paper, we obtain the normalized Laplacian and normalized distance Laplacian spectra of $\Gamma(W)$.

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Denote by $D(G)$ the diagonal matrix of G (also known as the degree matrix of G). The adjacency matrix of G is the matrix $A(G) = [a_{ij}]$, $1 \le i, j \le n$, where $a_{ij} = 1$ if v_i adjacent to v_j and $a_{ij} = 0$ otherwise. The Laplacian matrix $L(G)$ is defined as $L(G) = D(G) - A(G)$. The normalized Laplacian matrix $\mathcal{L}(G)$ was introduced in [3] as the square matrix whose (i,j)-entry $\mathcal{L}_{ij}$, $1 \le i, j \le n$, is given as

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } d_{v_i} \neq 0, \\ \dfrac{-1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

where d_{v_i} and d_{v_j} are degrees of v_i and v_j, respectively. We can also write

$$\mathcal{L}(G) = D^{-1/2}(G)L(G)D^{-1/2}(G) = I - D^{-1/2}(G)A(G)D^{-1/2}(G)$$

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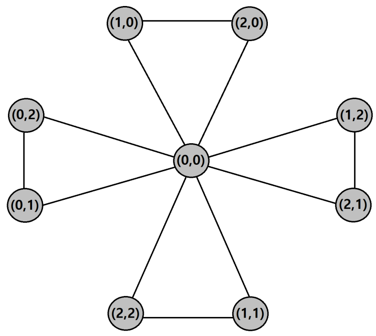

Figure 1: The linear dependence graph of the 2-dimensional vector space W over the finite field $F_3 = \{0,1,2\}$ where $W = \{(0,0),(0,1),(1,0),(2,0),(0,2),(1,1),(2,2),(1,2),(2,1)\}$.

where $A(G)$, $L(G)$ and $D(G)$ are the adjacency matrix, Laplacian matrix and degree matrix respectively of the graph G with the convention that $D(G)^{-1}(u,u) = 0$ if $d_u = 0$. The eigenvalues of $\mathcal{L}(G)$ are called normalized Laplacian eigenvalues of G and forms the $\mathcal{L}$-spectrum of G. Two graphs are said to be $\mathcal{L}$-cospectral if they have the same $\mathcal{L}$-spectrum. In [3], Chung and Graham showed that $\mathcal{L}(G)$ is a positive definite symmetric matrix and all its eigenvalues lie within the interval $[0,2]$ and that 0 is always a normalized Laplacian eigenvalue of any graph G.

Let $P(G) = D(G)^{-1}A(G)$. Then, $\mathcal{L}(G) = D(G)^{1/2}(I - D(G)^{-1}A(G))D(G)^{-1/2} = D(G)^{1/2}(I - P(G))D(G)^{-1/2}$. Hence, if the spectrum of $P(G)$ is $\{\mu_1, \mu_2, \ldots, \mu_n\}$, then the spectrum of $\mathcal{L}(G)$ is $\{(1-\mu_1),(1-\mu_2),\ldots,(1-\mu_n)\}$.

The distance matrix $\mathcal{D}(G)$ is the matrix whose (i,j)-entry is equal to $d_G(v_i, v_j)$, the distance between the vertices v_i and v_j of G. The eigenvalues of $\mathcal{D}(G)$ are said to be the $\mathcal{D}$-eigenvalues of G. The transmission $Tr(v)$ of a vertex v of G is the sum of the distances from v to all other vertices of G and the transmission matrix $T(G)$ is the diagonal matrix defined as $T(G) = diag(Tr(v_1), Tr(v_2), \ldots, Tr(v_n))$. The distance Laplacian matrix is defined as $\mathcal{D}^L(G) = T(G) - \mathcal{D}(G)$. The normalized distance Laplacian matrix is defined [7] as $\mathcal{D}^N(G) = T(G)^{-1/2}\mathcal{D}^L(G)T(G)^{-1/2}$. Eigenvalues of $\mathcal{D}^N(G)$ are called normalized distance Laplacian eigenvalues of G and forms the $\mathcal{D}^N$-spectrum of G. Two graphs are said to be $\mathcal{D}^N$-cospectral if they have same $\mathcal{D}^N$-spectrum. A graph G is said to be a r-transmission regular graph if $Tr(v) = r$ for every vertex v of G. If G is r-transmission regular then,

$$\mathcal{D}^L(G) = rI - \mathcal{D}(G) \quad \text{and} \quad \mathcal{D}^N(G) = T(G)^{-1/2}(T(G) - \mathcal{D}(G))T(G)^{-1/2} = I - \frac{\mathcal{D}(G)}{r}.$$

Let $\bar{P}(G) = T(G)^{-1}\mathcal{D}(G)$. Then $\mathcal{D}^N(G) = T(G)^{1/2}(I - T(G)^{-1}\mathcal{D}(G))T(G)^{-1/2} = T(G)^{1/2}(I - \bar{P}(G))T(G)^{-1/2}$. Hence if the spectrum of $\bar{P}(G)$ is $\{\mu'_1, \mu'_2, \ldots, \mu'_n\}$, then the spectrum of $\mathcal{D}^N(G)$ is $\{(1-\mu'_1),(1-\mu'_2),\ldots,(1-\mu'_n)\}$.

Since we are dealing with the graph $\Gamma(W)$ throughout this paper $A(\Gamma(W))$, $D(\Gamma(W))$, $P(\Gamma(W))$, $\mathcal{D}(\Gamma(W))$, $T(\Gamma(W))$, $\bar{P}(\Gamma(W))$ are denoted as simply A, D, P, $\mathcal{D}$, T,$\bar{P}$, respectively. Also, I_n denotes the identity matrix of order n, O_n denotes the zero matrix of order n, J_n denotes all-one matrix of order n and $J_{m\times n}$ denotes the all-one matrix of order $m \times n$. All spectral graph theoretic terminology are taken from [3]. We use the following lemma in the next section.

Lemma 1.1. [2] *Let M, N, P and Q be matrices such that M invertible. Let*

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}.$$

Then $|S| = |M|\left|Q - PM^{-1}N\right|$ and if M and P commutes, then $|S| = |MQ - PN|$ where the symbol $|.|$ denotes the determinant.

2. The normalized Laplacian spectrum of the linear dependence graph $\Gamma(W)$

In this section, we find the normalized Laplacian spectrum of $\Gamma(W)$ in terms of q by using the spectrum of P.

Theorem 2.1. *Let F be a finite field of order $q \geq 2$ and W be an n-dimensional vector space over F. Then, the spectrum of P is*

$$\begin{pmatrix} 1 & -(q-1)^{-1} & (q-2)(q-1)^{-1} \\ 1 & k(q-2)+1 & k-1 \end{pmatrix},$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$.

Proof. In [6], the adjacency matrix of $\Gamma(W)$ was given as

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & J_{1\times(q^n-1)} \\ J_{(q^n-1)\times 1} & [J_{q-1} - I_{q-1}] \otimes I_k \end{bmatrix}$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$. The degree of the zero vector is $q^n - 1$ and degree of all other vectors in W is $q - 1$. Hence the degree matrix of $\Gamma(W)$ is given by

$$D = \begin{bmatrix} q^n-1 & 0 & 0 & \cdots & 0 \\ 0 & q-1 & 0 & \cdots & 0 \\ 0 & 0 & q-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q-1 \end{bmatrix} = \begin{bmatrix} q^n-1 & 0_{1\times(q^n-1)} \\ 0_{(q^n-1)\times 1} & (q-1)I_{q-1} \otimes I_k \end{bmatrix}$$

where $k = q^{n-1} + q^{n-2} + \ldots + q + 1$. Hence,

$$\begin{aligned} P = D^{-1}A &= \begin{bmatrix} (q^n-1)^{-1} & 0_{1\times(q^n-1)} \\ 0_{(q^n-1)\times 1} & (q-1)^{-1}I_{q-1} \otimes I_k \end{bmatrix} \begin{bmatrix} 0 & J_{1\times(q^n-1)} \\ J_{(q^n-1)\times 1} & [J_{q-1} - I_{q-1}] \otimes I_k \end{bmatrix} \\ &= \begin{bmatrix} 0 & (q^n-1)^{-1}J_{1\times(q^n-1)} \\ ((q-1)^{-1}I_{q-1} \otimes I_k)J_{(q^n-1)\times 1} & [(q-1)^{-1}I_{q-1} \otimes I_k]([J_{q-1} - I_{q-1}] \otimes I_k) \end{bmatrix} \\ &= \begin{bmatrix} 0 & (q^n-1)^{-1}J_{1\times(q^n-1)} \\ (q-1)^{-1}J_{(q^n-1)\times 1} & (q-1)^{-1}[J_{q-1} - I_{q-1}] \otimes I_k \end{bmatrix}. \end{aligned}$$

The characteristic polynomial of P is

$$\begin{aligned} (\lambda I_{q^n} - P) &= \begin{vmatrix} \lambda & -(q^n-1)^{-1}J_{1\times(q^n-1)} \\ -(q-1)^{-1}J_{(q^n-1)\times 1} & \lambda I_{q^n-1} - ((q-1)^{-1}[J_{q-1} - I_{q-1}] \otimes I_k) \end{vmatrix} \\ &= (q^n-1)^{-1}(q-1)^{-(q^n-1)} \begin{vmatrix} \lambda(q^n-1) & -J_{1\times(q^n-1)} \\ -J_{(q^n-1)\times 1} & ([\lambda(q-1)+1]I_{q-1} - J_{q-1}) \otimes I_k \end{vmatrix}. \end{aligned}$$

Multiplying the first row by $\lambda(q-1) - (q-2)$ and $R_1' \mapsto R_1' + R_2 + \ldots + R_{q^n}$ and applying Lemma 1.1 yield

$$\begin{aligned} \det(\lambda I_{q^n} - P) &= \frac{(q^n-1)^{-1}(q-1)^{-(q^n-1)}}{\lambda(q-1)-(q-2)} \begin{vmatrix} \lambda(q^n-1)[\lambda(q-1)-(q-2)] - (q^n-1) & 0_{1\times(q^n-1)} \\ -J_{(q^n-1)\times 1} & ([\lambda(q-1)+1]I_{q-1} - J_{q-1}) \otimes I_k \end{vmatrix} \\ &= \frac{(q^n-1)^{-1}(q-1)^{-(q^n-1)}}{\lambda(q-1)-(q-2)}[\lambda(q^n-1)[\lambda(q-1)-(q-2)] - (q^n-1)][\det([\lambda(q-1)+1]I_{q-1} - J_{q-1})]^k. \end{aligned}$$

If $A_1 = \det([\lambda(q-1)+1]I_{q-1} - J_{q-1})$ then

$$A_1 = \begin{vmatrix} \lambda(q-1) & -1 & \cdots & -1 \\ -1 & \lambda(q-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda(q-1) \end{vmatrix}_{(q-1)\times(q-1)}$$

$$= \frac{1}{\lambda(q-1)-(q-3)} \begin{vmatrix} \lambda(q-1)[\lambda(q-1)-(q-3)]-(q-2) & 0 & \cdots & 0 \\ -1 & \lambda(q-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda(q-1) \end{vmatrix}_{(q-1)\times(q-1)}$$

$$= \frac{\lambda(q-1)[\lambda(q-1)-(q-3)]-(q-2)}{\lambda(q-1)-(q-3)} \begin{vmatrix} \lambda(q-1) & -1 & \cdots & -1 \\ -1 & \lambda(q-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda(q-1) \end{vmatrix}_{(q-2)\times(q-2)}$$

$$= \frac{[\lambda(q-1)+1][\lambda(q-1)-(q-2)]}{\lambda(q-1)-(q-3)} \begin{vmatrix} \lambda(q-1) & -1 & \cdots & -1 \\ -1 & \lambda(q-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda(q-1) \end{vmatrix}_{(q-2)\times(q-2)}$$

$$= \frac{[\lambda(q-1)+1][\lambda(q-1)-(q-2)]}{\lambda(q-1)-(q-3)} \cdot \frac{[\lambda(q-1)+1][\lambda(q-1)-(q-3)]}{\lambda(q-1)-(q-4)} \begin{vmatrix} \lambda(q-1) & -1 & \cdots & -1 \\ -1 & \lambda(q-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda(q-1) \end{vmatrix}_{(q-3)\times(q-3)}$$

$$= \frac{[\lambda(q-1)+1]^2[\lambda(q-1)-(q-2)]}{\lambda(q-1)-(q-4)} \begin{vmatrix} \lambda(q-1) & -1 & \cdots & -1 \\ -1 & \lambda(q-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda(q-1) \end{vmatrix}_{(q-3)\times(q-3)}.$$

Proceeding like this, we have $A_1 = [\lambda(q-1)+1]^{q-2}[\lambda(q-1)-(q-2)]$ and hence

$$\det(\lambda I_{q^n} - P) = \frac{(q^n-1)^{-1}(q-1)^{-(q^n-1)}}{\lambda(q-1)-(q-2)}[\lambda(q^n-1)[\lambda(q-1)-(q-2)]-(q^n-1)]\{[\lambda(q-1)+1]^{q-2}[\lambda(q-1)-(q-2)]\}^k$$

$$= (q^n-1)^{-1}(q-1)^{-(q^n-1)}[\lambda(q^n-1)[\lambda(q-1)-(q-2)]-(q^n-1)][\lambda(q-1)+1]^{k(q-2)}[\lambda(q-1)-(q-2)]^{k-1}.$$

Eigenvalues of P are given by the equations,

$$\lambda^2(q^n-1)(q-1) - \lambda(q^n-1)(q-2) - (q^n-1) = 0,$$
$$\lambda(q-1)+1 = 0,$$
$$\lambda(q-1)-(q-2) = 0.$$

Thus, the q^n eigenvalues of P are 1 with multiplicity 1, $-(q-1)^{-1}$ with multiplicity $k(q-2)+1$ and $(q-2)(q-1)^{-1}$ with multiplicity $k-1$.

□

Corollary 2.1. *Let F be a finite field of order q and W be an n-dimensional vector space over F. Then the normalized Laplacian spectrum of the linear dependence graph $\Gamma(W)$ is*

$$\begin{pmatrix} 0 & q(q-1)^{-1} & (q-1)^{-1} \\ 1 & k(q-2)+1 & k-1 \end{pmatrix}.$$

Proof. If the spectrum of P is $\{\mu_1, \mu_2, \ldots, \mu_n\}$, then the spectrum of $\mathcal{L}(\Gamma(W))$ is $\{(1-\mu_1), (1-\mu_2), \ldots, (1-\mu_n)\}$. Thus the q^n normalized Laplacian eigenvalues of $\Gamma(W)$ are 0 with multiplicity 1, $1+(q-1)^{-1} = q(q-1)^{-1}$ with multiplicity $k(q-2)+1$ and $1-(q-2)(q-1)^{-1} = (q-1)^{-1}$ with multiplicity $k-1$. □

3. The normalized distance Laplacian spectrum of the linear dependence graph $\Gamma(W)$

In this section, we find the normalized distance Laplacian spectrum of $\Gamma(W)$ in terms of q by using the spectrum of $\bar{P}$.

Theorem 3.1. *Let F be a finite field of order $q \geq 2$ and W be an n-dimensional vector space over F. Then, the spectrum of $\bar{P}$ is*

$$\begin{pmatrix} 1 & -(2q^n-q-1)^{-1} & -q(2q^n-q-1)^{-1} \\ 1 & k(q-2)+1 & k-1 \end{pmatrix},$$

where $k = q^{n-1}+q^{n-2}+\ldots+q+1$.

Proof. In [6], the distance matrix of $\Gamma(W)$ was given as

$$\mathcal{D} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 \\ 1 & 1 & 0 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 2 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 \\ 1 & 2 & 2 & \cdots & 2 & 0 & 1 & \cdots & 1 & \cdots & 2 & 2 & \cdots & 2 \\ 1 & 2 & 2 & \cdots & 2 & 1 & 0 & \cdots & 1 & \cdots & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 2 & \cdots & 2 & 1 & 1 & \cdots & 0 & \cdots & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & \cdots & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & \cdots & 1 & 1 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & J_{1\times(q^n-1)} \\ J_{(q^n-1)\times 1} & 2J_{(q^n-1)} - [J_{q-1}+I_{q-1}]\otimes I_k \end{bmatrix}$$

where $k = q^{n-1}+q^{n-2}+\ldots+q+1$. The transmission matrix of $\Gamma(W)$ is given by

$$T = \begin{bmatrix} q^n-1 & 0_{1\times(q^n-1)} \\ 0_{(q^n-1)\times 1} & (2q^n-q-1)I_{q-1}\otimes I_k \end{bmatrix}$$

where $k = q^{n-1}+q^{n-2}+\ldots+q+1$. Hence,

$$\bar{P} = T^{-1}\mathcal{D} = \begin{bmatrix} (q^n-1)^{-1} & 0_{1\times(q^n-1)} \\ 0_{(q^n-1)\times 1} & (2q^n-q-1)^{-1}I_{q-1}\otimes I_k \end{bmatrix} \begin{bmatrix} 0 & J_{1\times(q^n-1)} \\ J_{(q^n-1)\times 1} & 2J_{(q^n-1)} - ([J_{q-1}+I_{q-1}]\otimes I_k) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (q^n-1)^{-1}J_{1\times(q^n-1)} \\ (2q^n-q-1)^{-1}J_{(q^n-1)\times 1} & (2q^n-q-1)^{-1}[2J_{(q^n-1)} - ([J_{q-1}+I_{q-1}]\otimes I_k)] \end{bmatrix}.$$

The characteristic polynomial of $\bar{P}$ is

$$\det(\lambda I_{q^n} - \bar{P}) = \begin{vmatrix} \lambda & -(q^n-1)^{-1}J_{1\times(q^n-1)} \\ -(2q^n-q-1)^{-1}J_{(q^n-1)\times 1} & \lambda I_{q^n-1} - 2(2q^n-q-1)^{-1}J_{q^n-1} + [(2q^n-q-1)^{-1}(J_{q-1}+I_{q-1})\otimes I_k] \end{vmatrix}$$

$$= (q^n-1)^{-1}[(2q^n-q-1)^{-1}]^{(q^n-1)} \begin{vmatrix} \lambda(q^n-1) & -J_{1\times(q^n-1)} \\ -J_{(q^n-1)\times 1} & \lambda(2q^n-q-1)I_{q^n-1} - 2J_{q^n-1} + [(J_{q-1}+I_{q-1})\otimes I_k] \end{vmatrix}.$$

Now, apply the elementary transformations of subtracting from each column C_i, twice of C_1 for $i = 2,3,\ldots,q^n$, result in

$$\det(\lambda I_{q^n} - \bar{P}) = (q^n-1)^{-1}[(2q^n-q-1)^{-1}]^{(q^n-1)} \begin{vmatrix} \lambda(q^n-1) & -[1+2\lambda(q^n-1)]J_{1\times(q^n-1)} \\ -J_{(q^n-1)\times 1} & ([\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1})\otimes I_k \end{vmatrix}$$

$$= (q^n-1)^{-1}[(2q^n-q-1)^{-1}]^{(q^n-1)}[1+2\lambda(q^n-1)] \begin{vmatrix} \frac{\lambda(q^n-1)}{(1+2\lambda(q^n-1))} & -J_{1\times(q^n-1)} \\ -J_{(q^n-1)\times 1} & ([\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1})\otimes I_k \end{vmatrix}.$$

Multiplying the first row by $\lambda(2q^n-q-1)+q$ and $R_1' \mapsto R_1' + R_2 + \ldots + R_{q^n}$ yield

$\det(\lambda I_{q^n} - \bar{P}) =$

$$\frac{(q^n-1)^{-1}[(2q^n-q-1)^{-(q^n-1)}][1+2\lambda(q^n-1)]}{[\lambda(2q^n-q-1)+q]} \begin{vmatrix} \frac{[\lambda(q^n-1)][\lambda(2q^n-q-1)+q]}{(1+2\lambda(q^n-1))} - (q^n-1) & 0_{1\times(q^n-1)} \\ -J_{(q^n-1)\times 1} & ([\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1})\otimes I_k \end{vmatrix}$$

$$= \frac{\lambda(q^n-1)[\lambda(2q^n-q-1)+q]-(q^n-1)(1+2\lambda(q^n-1))}{(q^n-1)(2q^n-q-1)^{(q^n-1)}[\lambda(2q^n-q-1)+q]}\det([\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1})^k \text{ using Lemma 1.1.}$$

If $B_1 = \det[\lambda(2q^n-q-1)+1]I_{q-1}+J_{q-1}$ then

$$B_1 = \begin{vmatrix} \lambda(2q^n-q-1)+2 & 1 & \cdots & 1 \\ 1 & \lambda(2q^n-q-1)+2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \lambda(2q^n-q-1)+2 \end{vmatrix}_{(q-1)\times(q-1)}$$

Multiply first row by $\lambda(2q^n-q-1)+(q-1)$ and then $R_1' \mapsto R_1' - R_2 - \ldots - R_{q-1}$, we have

$$B_1 = \frac{[\lambda(2q^n-q-1)+2][\lambda(2q^n-q-1)+(q-1)]-(q-2)}{\lambda(2q^n-q-1)+(q-1)}\begin{vmatrix} \lambda(2q^n-q-1)+2 & 1 & \cdots & 1 \\ 1 & \lambda(2q^n-q-1)+2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \lambda(2q^n-q-1)+2 \end{vmatrix}_{(q-2)\times(q-2)}$$

$$= \frac{[\lambda(2q^n-q-1)+1][\lambda(2q^n-q-1)+q]}{\lambda(2q^n-q-1)+(q-1)}\begin{vmatrix} \lambda(2q^n-q-1)+2 & 1 & \cdots & 1 \\ 1 & \lambda(2q^n-q-1)+2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \lambda(2q^n-q-1)+2 \end{vmatrix}_{(q-2)\times(q-2)}$$

$$= \frac{[\lambda(2q^n-q-1)+1]^2[\lambda(2q^n-q-1)+q]}{\lambda(2q^n-q-1)+(q-2)}\begin{vmatrix} \lambda(2q^n-q-1)+2 & 1 & \cdots & 1 \\ 1 & \lambda(2q^n-q-1)+2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \lambda(2q^n-q-1)+2 \end{vmatrix}_{(q-3)\times(q-3)}.$$

Proceeding like this, we have $B_1 = [\lambda(2q^n-q-1)+1]^{(q-2)}[\lambda(2q^n-q-1)+q]$ and hence,

$$\det(\lambda I_{q^n} - \bar{P}) = \frac{\lambda(q^n-1)[\lambda(2q^n-q-1)+q]-(q^n-1)(1+2\lambda(q^n-1))}{(q^n-1)(2q^n-q-1)^{(q^n-1)}[\lambda(2q^n-q-1)+q]}[\lambda(2q^n-q-1)+1]^{k(q-2)}[\lambda(2q^n-q-1)+q]^k$$

The right hand side of the above equation is equal to

$$\frac{[\lambda^2(q^n-1)(2q^n-q-1)+\lambda(q^n-1)(q-2(q^n-1))-(q^n-1)]}{(q^n-1)(2q^n-q-1)^{(q^n-1)}}[\lambda(2q^n-q-1)+1]^{k(q-2)}[\lambda(2q^n-q-1)+q]^{k-1}.$$

Thus, the eigenvalues of $\bar{P}$ are given by the following equations

$$\begin{aligned} \lambda^2(q^n-1)(2q^n-q-1)+\lambda(q^n-1)(q-2(q^n-1))-(q^n-1) &= 0, \\ [\lambda(2q^n-q-1)+1]^{k(q-2)} &= 0, \\ [\lambda(2q^n-q-1)+q]^{k-1} &= 0. \end{aligned}$$

After solving the above equations, we conclude that the q^n eigenvalues of $\bar{P}$ are 1 with multiplicity 1, $-(2q^n-q-1)^{-1}$ with multiplicity $k(q-2)+1$ and $-q(2q^n-q-1)^{-1}$ with multiplicity $k-1$ respectively.

□

Corollary 3.1. *Let F be a finite field of order $q \geq 2$ and W be an n-dimensional vector space over F. Then the normalized distance Laplacian spectrum of the linear dependence graph $\Gamma(W)$ is*

$$\begin{pmatrix} 0 & 1+(2q^n-q-1)^{-1} & 1+q(2q^n-q-1)^{-1} \\ 1 & k(q-2)+1 & k-1 \end{pmatrix}.$$

Proof. If the spectrum of $\bar{P}$ is $\{\mu_1', \mu_2', \ldots, \mu_n'\}$, then the spectrum of $\mathcal{D}^N(\Gamma(W))$ is $\{(1-\mu_1'), (1-\mu_2'), \ldots, (1-\mu_n')\}$. Thus, the q^n normalized distance Laplacian eigenvalues of $\Gamma(W)$ are 0 with multiplicity 1, $1+(2q^n-q-1)^{-1}$ with multiplicity $k(q-2)+1$ and $1+q(2q^n-q-1)^{-1}$ with multiplicity $k-1$ respectively. □

4. Conclusion

In this paper, we obtained the normalized Laplacian and normalized distance Laplacian spectra of the linear dependence graph $\Gamma(W)$ of a finite-dimensional vector space W over a finite field F. These spectra are the new addition to the existing literature of spectral graph theory. A similar treatment can be applied to obtain other spectra associated with other matrices of a linear dependence graph.

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