Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **5** (2021) 29–33
DOI: 10.47443/dml.2020.0037

Research Article

Computing 2-twinless blocks

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(Received: 22 June 2020. Received in revised form:† 30 January 2021. Accepted: 16 February 2021. Published online: 20 February 2021.)

Abstract

Let $G = (V, E)$ be a directed graph. A 2-twinless block in G is a maximal subset $B \subseteq V$ of size at least 2 such that for every pair of distinct vertices $x, y \in B$, and for every vertex $w \in V \setminus \{x, y\}$, the vertices x, y are in the same twinless strongly connected component of $G \setminus \{w\}$. In this paper, algorithms for computing the 2-twinless blocks of a directed graph are presented.

Keywords: directed graphs; connectivity; graph algorithms; 2-blocks; twinless strongly connected graphs.

2020 Mathematics Subject Classification: 05C85, 05C20.

1. Introduction

Let $G = (V, E)$ be a directed graph. The graph G is *twinless strongly connected* if it contains a strongly connected spanning subgraph (V, E^t) such that E^t does not contain any pair of antiparallel edges. A *twinless strongly connected component* of G is a maximal subset $C_t \subseteq V$ such that the induced subgraph on C_t is twinless strongly connected. A *strong articulation point* of G is a vertex whose removal increases the number of strongly connected components of G. A *strong bridge* of G is an edge whose deletion increases the number of strongly connected components of G. A strongly connected graph is 2-vertex-connected if it has at least 3 vertices and it has no strong articulation points. A 2-vertex-connected component of G is a maximal vertex subset $C^v \subseteq V$ such that the induced subgraph on C^v is 2-vertex-connected. A 2*-directed block* in G is a maximal vertex subset $B^d \subseteq V$ with $|B^d| > 1$ such that for any distinct vertices $x, y \in B^d$, the graph G contains two vertex-disjoint paths from x to y and two vertex-disjoint paths from y to x. A 2*-edge block* in G is a maximal subset $B^{eb} \subseteq V$ with $|B^{eb}| > 1$ such that for any distinct vertices $v, w \in B^{eb}$, there are two edge-disjoint paths from v to w and two edge-disjoint paths from w to v in G. A 2*-strong block* in G is a maximal vertex subset $B^s \subseteq V$ with $|B^s| > 1$ such that for each pair of distinct vertices $x, y \in B^s$ and for every vertex $u \in V \setminus \{x, y\}$, the vertices x and y are in the same strongly connected component of the graph $G \setminus \{u\}$. A *twinless articulation point* of G is a vertex whose removal increases the number of twinless strongly connected components of G. A 2-twinless block in G is a maximal vertex set $B \subseteq V$ of size at least 2 such that for each pair of distinct vertices $x, y \in B$, and for each vertex $w \in V \setminus \{x, y\}$, the vertices x, y are in the same twinless strongly connected component of $G \setminus \{w\}$. Notice that 2-strong blocks are not necessarily 2-twinless blocks (see Figure 1).

A twinless strongly connected graph G is said to be 2-vertex-twinless-connected if it has at least three vertices and it does not contain any twinless articulation point [19]. A 2-vertex-twinless-connected component is a maximal subset $U^{2vt} \subseteq V$ such that the induced subgraph on U^{2vt} is 2-vertex-twinless-connected. While 2-vertex-twinless-connected components have at least linear number of edges, the subgraphs induced by 2-twinless blocks do not necessarily contain edges.

Strongly connected components can be found in linear time [25]. In 2006, Raghavan [22] showed that the twinless strongly connected component of a directed graph can be found in linear time. In 2010, Georgiadis [7] presented an algorithm to check whether a strongly connected graph is 2-vertex-connected in linear time. Italiano et al. [15] gave linear time algorithms for identifying all the strong articulation points and strong bridges of a directed graph. Their algorithms are based on dominators [1–4, 6, 21]. In 2014, Jaberi [17] presented algorithms for computing the 2-vertex-connected components of directed graphs in $O(nm)$ time (published in [16]). Henzinger et al. [14] gave algorithms for calculating the 2-vertex-connected components in $O(n^2)$ time. Jaberi [18] presented algorithms for computing 2-blocks in directed graphs. Georgiadis et al. [9, 10] gave linear time algorithms for determining 2-edge blocks. Georgiadis et al. [11, 12] also gave linear time algorithms for calculating 2-directed blocks and 2-strong blocks. Georgiadis et al. [13] and Luigi et al. [20] performed

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†First decision made: 28 July 2020.

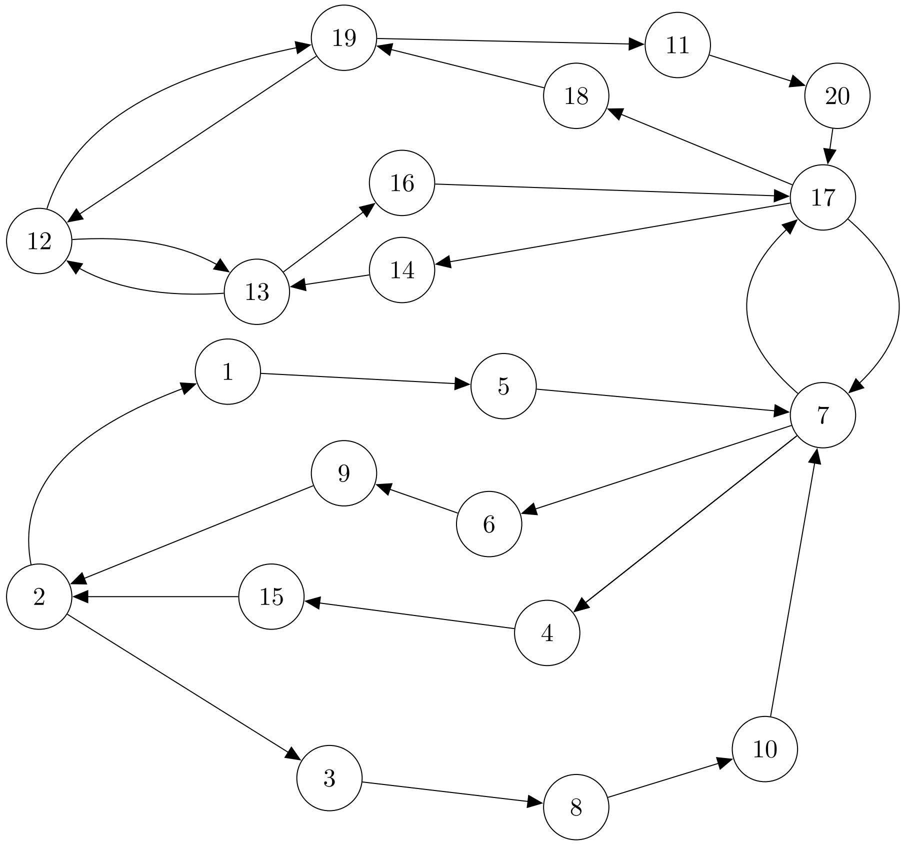

Figure 1: A strongly connected graph G, which contains two 2-strong blocks $C_1 = \{2, 7\}, C_2 = \{12, 13, 17, 19\}$, and one 2-twinless block $B = \{2, 7\}$. Notice that the vertices 12 and 17 do not belong to the same twinless strongly connected component of $G \setminus \{13\}$.

experimental studies of recent algorithms that calculate 2-blocks and 2-connected components in directed graphs. In 2019, Jaberi [19] presented an algorithm for computing 2-vertex-twinless-connected components. Georgiadis and Kosinas [8] gave a linear time algorithm for calculating twinless articulation points.

In the next section, we show that the 2-twinless blocks of a directed graph can be calculated in $O(n^3)$ time.

2. Algorithm for computing 2-twinless blocks

In this section we present an algorithm for computing the 2-twinless blocks of a twinless strongly connected graph. Since twinless strongly connected components do not share vertices of the same 2-twinless block, we consider only twinless strongly connected graphs. Let $G = (V, E)$ be a twinless strongly connected graph. We define a relation $\overset{2t}{\leftrightsquigarrow}$ as follows. For any distinct vertices $x, y \in V$, we write $x \overset{2t}{\leftrightsquigarrow} y$ if for all vertices $w \in V \setminus \{x, y\}$, the vertices x, y are in the same twinless strongly connected component of $G \setminus \{w\}$. By definition, a 2-twinless block in G is a maximal subset $B^{2t} \subseteq V$ with $|B^{2t}| > 1$ such that for every two vertices $x, y \in B^{2t}$, we have $x \overset{2t}{\leftrightsquigarrow} y$.

The next lemma shows that 2-twinless blocks share at most one vertex.

Lemma 2.1. *Let $G = (V, E)$ be a twinless strongly connected graph. Let B_1^{2t}, B_2^{2t} be distinct 2-twinless blocks in G. Then $|B_1^{2t} \cap B_2^{2t}| \leq 1$.*

Proof. Suppose for the sake of contradiction that B_1^{2t} and B_2^{2t} have at least two vertices in common. Clearly, $B_1^{2t} \cup B_2^{2t}$ is not a 2-twinless block in G. Let x and y be vertices belonging to B_1^{2t} and B_2^{2t}, respectively, such that $x, y \notin B_1^{2t} \cap B_2^{2t}$. Let z be any vertex in $V \setminus \{x, y\}$. Since $|B_1^{2t} \cap B_2^{2t}| > 1$, there is a vertex v in $(B_1^{2t} \cap B_2^{2t}) \setminus \{z\}$. Note that x, v are in the same twinless strongly connected component of $G \setminus \{z\}$ since $x, v \in B_1^{2t}$. Moreover, v and y lie in the same twinless strongly connected component of $G \setminus \{z\}$. By Lemma 1 of [22], x and y are in the same twinless strongly connected component of $G \setminus \{z\}$. Therefore, x, y belong to the same 2-twinless block. □

The following lemma shows an interesting property of the relation $\overset{2t}{\leftrightsquigarrow}$.

Lemma 2.2. *Let $G = (V, E)$ be a twinless strongly connected graph and let $\{v_0, v_1, \ldots, v_l\}$ be set of vertices of G such that $v_l \overset{2t}{\leftrightsquigarrow} v_0$ and $v_{i-1} \overset{2t}{\leftrightsquigarrow} v_i$ for $i \in \{1, 2 \ldots, l\}$. Then $\{v_0, v_1, \ldots, v_l\}$ is a subset of a 2-twinless block in G.*

Proof. Assume for the purpose of contradiction that there are two vertices v_r and v_q in G such that v_r and v_q are in distinct 2-twinless blocks of G and $r, q \in \{0, 1, \ldots, l\}$. Suppose without loss of generality that $r < q$. Then there is a vertex $z \in V \setminus \{v_r, v_q\}$ such that v_r and v_q are in distinct twinless strongly connected components of $G \setminus \{z\}$. We distinguish two cases.

1. $z \in \{v_{r+1}, v_{r+2}, \ldots, v_{q-1}\}$. In this case, the vertices v_{i-1}, v_i belong to the same twinless strongly connected component of $G \setminus \{z\}$ for each $i \in \{1, 2, \ldots, r\} \cup \{q+1, q+2, \ldots, l\}$. Moreover, the vertices v_0, v_l are in the same twinless strongly connected component of $G \setminus \{z\}$ because $v_0 \overset{2t}{\rightsquigarrow} v_l$. Therefore, the vertices v_r, v_q are in the same twinless strongly connected component of the graph $G \setminus \{z\}$, a contradiction.

2. $z \notin \{v_{r+1}, v_{r+2}, \ldots, v_{q-1}\}$. Then, for each $i \in \{r+1, r+2, \ldots, q\}$, the vertices v_{i-1}, v_i lie in the same twinless strongly connected component of $G \setminus \{z\}$. Consequently, the vertices v_r, v_q belong to the same twinless strongly connected component of the graph $G \setminus \{z\}$, a contradiction.

□

Let $G = (V, E)$ be a twinless strongly connected graph. We construct the 2-*twinless block graph* $G^{2t} = (V^{2t}, E^{2t})$ of G as follows. For every 2-twinlesss block B_i, we add a vertex v_i to V^{2t}. Moreover, for each vertex $v \in V$, if v belongs to at least two distinct 2-twinless blocks, we add a vertex v to V^{2t}. For any distinct 2-twinless blocks B_i, B_j with $B_i \cap B_j = \{v\}$, we put two undirected edges $(v_i, v), (v, v_j)$ into E^{2t}.

Lemma 2.3. *The* 2*-twinless block graph of a twinless strongly connected graph is a forest.*

Proof. This result follows from Lemma 2.2 and Lemma 2.1. □

Lemma 2.4. *Let $G = (V, E)$ be a twinless strongly connected graph and let x, y be distinct vertices in G. Suppose that $v \in V \setminus \{x, y\}$ is not a twinless articulation point. Then x, y are in the same twinless strong connected component of $G \setminus \{v\}$.*

Proof. Immediate from the definition. □

Now, we give an algorithm for computing the 2-twinless blocks of a twinless strongly connected graph $G = (V, E)$.

Algorithm 2.1.
Input: A twinless strongly connected graph $G = (V, E)$.
Output: The 2-twinless blocks of G.
1 **if** G is 2-vertex-twinless connected **then**
2 Output V.
3 **else**
4 Let S be an $n \times n$ matrix.
5 Initialize S with 1s.
6 determine the twinless articulation points of G.
7 **for** each twinless articulation point z of G **do**
8 Identify the twinless strongly connected components of $G \setminus \{z\}$.
9 **for** each pair $(v, w) \in (V \setminus \{z\}) \times (V \setminus \{z\})$ **do**
10 **if** v, w in different twinless strongly connected components of $G \setminus \{z\}$ **then**
11 $S[v, w] \leftarrow 0$.
12 $E^b \leftarrow \emptyset$.
13 **for** each pair $(v, u) \in V \times V$ **do**
14 **if** $S[v, u] = 1$ and $S[u, v] = 1$ **then**
15 $E^b \leftarrow E^b \cup \{(v, u)\}$.
16 calculate the blocks of size ≥ 2 of $G^b = (V, E^b)$ and output them.

The correctness of Algorithm 2.1 follows from the following lemma.

Lemma 2.5. *A vertex subset $B \subseteq V$ is a* 2*-twinless block of G if and only if B is a block of the undirected graph $G^b = (V, E^b)$ which is constructed in lines* 12–15 *of Algorithm 2.1*

Proof. It follows from Lemma 2.2 and Lemma 2.4. □

Theorem 2.1. *Algorithm 2.1 runs in $O(n^3)$ time.*

Proof. Georgiadis and Kosinas [8] showed that the twinless articulation points can be computed in linear time. The initialization of matrix S takes $O(n^2)$ time. The number of iterations of the for-loop in lines 7–11 is at most n because the number of twinless articulation points is at most n. Consequently, lines 7–11 require $O(n^3)$. Furthermore, the blocks of an undirected graph can be found in linear time [24, 25]. □

The following lemma shows an important property of G^b.

Lemma 2.6. *The graph G^b which is constructed in lines 12–15 of Algorithm 2.1 is chordal.*

Proof. It follows from Lemma 2.2. □

By Lemma 2.6, one can calculate the maximal cliques of G^b instead of blocks. The maximal cliques of a chordal graph can be calculated in linear time [5, 23].

3. An improved version of Algorithm 2.1

In this section, we present an improved version of Algorithm 2.1.

The following lemma shows a connection between 2-twinless blocks and 2-strong blocks.

Lemma 3.1. *Let $G = (V, E)$ be a twinless strongly connected graph. Suppose that B_t is a 2-twinless block in G. Then B_t is a subset of a 2-strong block in G.*

Proof. Let v and w be distinct vertices in B_t, and let $x \in V \setminus \{v, w\}$. By definition, the vertices v, w belong to the same twinless strongly connected component C of $G \setminus \{x\}$. Since C is a subset of a strongly connected component of G, the vertices v, w also lie in the same strongly connected component of $G \setminus \{x\}$. Consequently, v, w are in the same 2-strong block in G. □

The next algorithm describes an improved version of Algorithm 2.1 which is based on Lemma 3.1 and Lemma 2.2.

Algorithm 3.1.
Input: A twinless strongly connected graph $G = (V, E)$.
Output: The 2-twinless blocks of G.

```
if G is 2-vertex-twinless connected then
  Output V.
else
  find the 2-strong blocks of G
  Let S be an n × n matrix.
  Initialize S with 0.
  A ← ∅.
  for each 2-strong block B of G do
    for each pair of vertices v, w ∈ B do
      S[v, w] ← 1
      S[w, v] ← 1
    for each vertex v ∈ B do
      if v ∉ A then
          add v to A
  determine the twinless articulation points of G.
  for each twinless articulation point z of G do
      Identify the twinless strongly connected components of G \ {z}.
      for each pair (v, w) ∈ (A \ {z}) × (A \ {z}) do
        if v, w in different twinless strongly connected components of G \ {z} then
          S[v, w] ← 0.
  E^b ← ∅.
  for each pair (v, u) ∈ A × A do
    if S[v, u] = 1 and S[u, v] = 1 then
      E^b ← E^b ∪ {(v, u)}.
  calculate the blocks of size ≥ 2 of G^b = (A, E^b) and output them.
```

Theorem 3.1. *The running time of Algorithm 3.1 is $O(t(s^2 + m) + n^2)$, where $s = |A|$ and t is the number of twinless articulation points of G.*

Proof. The 2-strong blocks of a directed graph can be computed in linear time [11]. Furthermore, the twinless articulation points of a directed graph can be identified in linear time using the algorithm of Georgiadis and Kosinas [8]. Since the number of iterations of the for-loop in lines 16–20 is at most t, lines 16–20 take $O(t(s^2 + m))$ time. □

Let $G = (V, E)$ be a twinless strongly connected graph. If the refine operation defined in [11, 20] is used to refine the 2-strong blocks of G for all twinless articulation points, then the 2-twinless blocks of a directed graph $G = (V, E)$ can be computed in $O(tm)$ time, where t is the number of twinless articulation points of G.

We leave as an open problem whether the 2-twinless blocks of a directed graph can be calculated in linear time.

Acknowledgement

The author would like to thank the anonymous reviewers for their helpful comments and suggestions.

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