Computing 2-twinless blocks

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Abstract

Let \( G = (V, E) \) be a directed graph. A 2-twinless block in \( G \) is a maximal subset \( B \subseteq V \) of size at least 2 such that for every pair of distinct vertices \( x,y \in B \), and for every vertex \( w \in V \setminus \{x,y\} \), the vertices \( x,y \) are in the same twinless strongly connected component of \( G \setminus \{w\} \). In this paper, algorithms for computing the 2-twinless blocks of a directed graph are presented.

Keywords: directed graphs; connectivity; graph algorithms; 2-blocks; twinless strongly connected graphs.

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1. Introduction

Let \( G = (V, E) \) be a directed graph. The graph \( G \) is twinless strongly connected if it contains a strongly connected spanning subgraph \( (V, E') \) such that \( E' \) does not contain any pair of antiparallel edges. A twinless strongly connected component of \( G \) is a maximal subset \( C \subseteq V \) such that the induced subgraph on \( C \) is twinless strongly connected. A strong articulation point of \( G \) is a vertex whose removal increases the number of strongly connected components of \( G \). A strong bridge of \( G \) is an edge whose deletion increases the number of strongly connected components of \( G \). A strongly connected graph is 2-vertex-connected if it has at least 3 vertices and it has no strong articulation points. A 2-vertex-connected component of \( G \) is a maximal vertex subset \( C' \subseteq V \) such that the induced subgraph on \( C' \) is 2-vertex-connected. A 2-directed block in \( G \) is a maximal vertex subset \( B' \subseteq V \) with \(|B'| > 1\) such that for any distinct vertices \( x,y \in B' \), the graph \( G \) contains two vertex-disjoint paths from \( x \) to \( y \) and two vertex-disjoint paths from \( y \) to \( x \). A 2-edge block in \( G \) is a maximal subset \( B^{e} \subseteq V \) with \(|B^{e}| > 1\) such that for any distinct vertices \( u,v \in B^{e} \), there are two edge-disjoint paths from \( u \) to \( v \) and two edge-disjoint paths from \( v \) to \( u \) in \( G \). A 2-strong block in \( G \) is a maximal vertex subset \( B^{s} \subseteq V \) with \(|B^{s}| > 1\) such that for each pair of distinct vertices \( x,y \in B^{s} \) and for every vertex \( u \in V \setminus \{x,y\} \), the vertices \( x \) and \( y \) are in the same strongly connected component of the graph \( G \setminus \{u\} \). A twinless articulation point of \( G \) is a vertex whose removal increases the number of twinless strongly connected components of \( G \). A 2-twinless block in \( G \) is a maximal vertex set \( B \subseteq V \) of size at least 2 such that for each pair of distinct vertices \( x,y \in B \), and for each vertex \( w \in V \setminus \{x,y\} \), the vertices \( x \) and \( y \) are in the same twinless strongly connected component of \( G \setminus \{w\} \). Notice that 2-strong blocks are not necessarily 2-twinless blocks (see Figure 1).

A twinless strongly connected graph \( G \) is said to be 2-vertex-twinless-connected if it has at least three vertices and it does not contain any twinless articulation point \([19]\). A 2-vertex-twinless-connected component is a maximal subset \( U^{2vt} \subseteq V \) such that the induced subgraph on \( U^{2vt} \) is 2-vertex-twinless-connected. While 2-vertex-twinless-connected components have at least linear number of edges, the subgraphs induced by 2-twinless blocks do not necessarily contain edges.

Strongly connected components can be found in linear time \([25]\). In 2006, Raghavan \([22]\) showed that the twinless strongly connected component of a directed graph can be found in linear time. In 2010, Georgiadis \([7]\) presented an algorithm to check whether a strongly connected graph is 2-vertex-connected in linear time. Italiano et al. \([15]\) gave linear time algorithms for identifying all the strong articulation points and strong bridges of a directed graph. Their algorithms are based on dominators \([1–4, 6, 21]\). In 2014, Jaberi \([17]\) presented algorithms for computing the 2-vertex-connected components of directed graphs in \( O(nm) \) time (published in \([16]\)). Henzinger et al. \([14]\) gave algorithms for calculating the 2-vertex-connected components in \( O(n^2) \) time. Jaberi \([18]\) presented algorithms for computing 2-blocks in directed graphs. Georgiadis et al. \([9, 10]\) gave linear time algorithms for determining 2-edge blocks. Georgiadis et al. \([11, 12]\) also gave linear time algorithms for calculating 2-directed blocks and 2-strong blocks. Georgiadis et al. \([13]\) and Luigi et al. \([20]\) performed

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† First decision made: 28 July 2020.
Figure 1: A strongly connected graph $G$, which contains two 2-strong blocks $C_1 = \{2, 7\}, C_2 = \{12, 13, 17, 19\}$, and one 2-twinless block $B = \{2, 7\}$. Notice that the vertices 12 and 17 do not belong to the same twinless strongly connected component of $G \setminus \{13\}$.


In the next section, we show that the 2-twinless blocks of a directed graph can be calculated in $O(n^3)$ time.

2. Algorithm for computing 2-twinless blocks

In this section we present an algorithm for computing the 2-twinless blocks of a twinless strongly connected graph. Since twinless strongly connected components do not share vertices of the same 2-twinless block, we consider only twinless strongly connected graphs. Let $G = (V, E)$ be a twinless strongly connected graph. We define a relation $\sim_{2t}$ as follows. For any distinct vertices $x, y \in V$, we write $x \sim_{2t} y$ if for all vertices $w \in V \setminus \{x, y\}$, the vertices $x, y$ are in the same twinless strongly connected component of $G \setminus \{w\}$. By definition, a 2-twinless block in $G$ is a maximal subset $B^{2t} \subseteq V$ with $|B^{2t}| > 1$ such that for every two vertices $x, y \in B^{2t}$, we have $x \sim_{2t} y$.

The next lemma shows that 2-twinless blocks share at most one vertex.

**Lemma 2.1.** Let $G = (V, E)$ be a twinless strongly connected graph. Let $B^{2t}_1, B^{2t}_2$ be distinct 2-twinless blocks in $G$. Then $|B^{2t}_1 \cap B^{2t}_2| \leq 1$.

**Proof.** Suppose for the sake of contradiction that $B^{2t}_1$ and $B^{2t}_2$ have at least two vertices in common. Clearly, $B^{2t}_1 \cup B^{2t}_2$ is not a 2-twinless block in $G$. Let $x$ and $y$ be vertices belonging to $B^{2t}_1$ and $B^{2t}_2$, respectively, such that $x, y \notin B^{2t}_1 \cap B^{2t}_2$. Let $z$ be any vertex in $V \setminus \{x, y\}$. Since $|B^{2t}_1 \cap B^{2t}_2| > 1$, there is a vertex $v \in (B^{2t}_1 \cap B^{2t}_2) \setminus \{z\}$. Note that $x, v$ are in the same twinless strongly connected component of $G \setminus \{z\}$ since $x, v \in B^{2t}_1$. Moreover, $v$ and $y$ lie in the same twinless strongly connected component of $G \setminus \{z\}$. By Lemma 1 of [22], $x$ and $y$ are in the same twinless strongly connected component of $G \setminus \{z\}$. Therefore, $x, y$ belong to the same 2-twinless block. 

The following lemma shows an interesting property of the relation $\sim_{2t}$.

**Lemma 2.2.** Let $G = (V, E)$ be a twinless strongly connected graph and let $\{v_0, v_1, \ldots, v_l\}$ be set of vertices of $G$ such that $v_i \sim_{2t} v_0$ and $v_{i-1} \sim_{2t} v_i$ for $i \in \{1, 2, \ldots, l\}$. Then $\{v_0, v_1, \ldots, v_l\}$ is a subset of a 2-twinless block in $G$. 

Proof. Assume for the purpose of contradiction that there are two vertices $v_r$ and $v_q$ in $G$ such that $v_r$ and $v_q$ are in distinct 2-twinless blocks of $G$ and $r, q \in \{0, 1, \ldots, l\}$. Suppose without loss of generality that $r < q$. Then there is a vertex $z \in V \setminus \{v_r, v_q\}$ such that $v_r$ and $v_q$ are in distinct twinless strongly connected components of $G \setminus \{z\}$. We distinguish two cases.

1. $z \in \{v_{r+1}, v_{r+2}, \ldots, v_{q-1}\}$. In this case, the vertices $v_{i+1}, v_i$ belong to the same twinless strongly connected component of $G \setminus \{z\}$ for each $i \in \{1, 2, \ldots, r \} \cup \{q + 1, q + 2, \ldots, l\}$. Moreover, the vertices $v_0, v_1$ are in the same twinless strongly connected component of $G \setminus \{z\}$ because $v_0 \leftrightarrow v_1$. Therefore, the vertices $v_r, v_q$ are in the same twinless strongly connected component of the graph $G \setminus \{z\}$, a contradiction.

2. $z \notin \{v_{r+1}, v_{r+2}, \ldots, v_{q-1}\}$. Then, for each $i \in \{r + 1, r + 2, \ldots, q\}$, the vertices $v_{i-1}, v_i$ are in the same twinless strongly connected component of $G \setminus \{z\}$. Consequently, the vertices $v_r, v_q$ belong to the same twinless strongly connected component of the graph $G \setminus \{z\}$, a contradiction.

Let $G = (V, E)$ be a twinless strongly connected graph. We construct the 2-twinless block graph $G^{2t} = (V^{2t}, E^{2t})$ of $G$ as follows. For every 2-twinless block $B_i$, we add a vertex $v_i$ to $V^{2t}$. Moreover, for each vertex $v \in V$, if $v$ belongs to at least two distinct 2-twinless blocks, we add a vertex $v$ to $V^{2t}$. For any distinct 2-twinless blocks $B_i, B_j$ with $B_i \cap B_j = \{v\}$, we put two undirected edges $(v_i, v), (v, v_j)$ into $E^{2t}$.

**Lemma 2.3.** The 2-twinless block graph of a twinless strongly connected graph is a forest.

Proof. This result follows from Lemma 2.2 and Lemma 2.1.

**Lemma 2.4.** Let $G = (V, E)$ be a twinless strongly connected graph and let $x, y$ be distinct vertices in $G$. Suppose that $v \in V \setminus \{x, y\}$ is not a twinless articulation point. Then $x, y$ are in the same twinless strongly connected component of $G \setminus \{v\}$.

Proof. Immediate from the definition.

Now, we give an algorithm for computing the 2-twinless blocks of a twinless strongly connected graph $G = (V, E)$.

**Algorithm 2.1.**

**Input:** A twinless strongly connected graph $G = (V, E)$.

**Output:** The 2-twinless blocks of $G$.

1. if $G$ is 2-vertex-twinless connected then
2. Output $V$.
3. else
4. Let $S$ be an $n \times n$ matrix.
5. Initialize $S$ with 1s.
6. determine the twinless articulation points of $G$.
7. for each twinless articulation point $z$ of $G$ do
8. Identify the twinless strongly connected components of $G \setminus \{z\}$.
9. for each pair $(v, w) \in (V \setminus \{z\}) \times (V \setminus \{z\})$ do
10. if $v, w$ in different twinless strongly connected components of $G \setminus \{z\}$ then
11. $S_{v, w} \leftarrow 0$.
12. $E^b \leftarrow \emptyset$.
13. for each pair $(v, u) \in V \times V$ do
14. if $S_{v, u} = 1$ and $S_{u, v} = 1$ then
15. $E^b \leftarrow E^b \cup \{v, u\}$.
16. calculate the blocks of size $\geq 2$ of $G^b = (V, E^b)$ and output them.

The correctness of Algorithm 2.1 follows from the following lemma.

**Lemma 2.5.** A vertex subset $B \subseteq V$ is a 2-twinless block of $G$ if and only if $B$ is a block of the undirected graph $G^b = (V, E^b)$ which is constructed in lines 12–15 of Algorithm 2.1.

Proof. It follows from Lemma 2.2 and Lemma 2.4.

**Theorem 2.1.** Algorithm 2.1 runs in $O(n^2)$ time.
Proof. Georgiadis and Kosinas [8] showed that the twinless articulation points can be computed in linear time. The initialization of matrix $S$ takes $O(n^2)$ time. The number of iterations of the for-loop in lines 7–11 is at most $n$ because the number of twinless articulation points is at most $n$. Consequently, lines 7–11 require $O(n^3)$. Furthermore, the blocks of an undirected graph can be found in linear time [24,25].

The following lemma shows an important property of $G^b$.

**Lemma 2.6.** The graph $G^b$ which is constructed in lines 12–15 of Algorithm 2.1 is chordal.

**Proof.** It follows from Lemma 2.2.

By Lemma 2.6, one can calculate the maximal cliques of $G^b$ instead of blocks. The maximal cliques of a chordal graph can be calculated in linear time [5,23].

### 3. An improved version of Algorithm 2.1

In this section, we present an improved version of Algorithm 2.1.

The following lemma shows a connection between 2-twinless blocks and 2-strong blocks.

**Lemma 3.1.** Let $G = (V, E)$ be a twinless strongly connected graph. Suppose that $B_t$ is a 2-twinless block in $G$. Then $B_t$ is a subset of a 2-strong block in $G$.

**Proof.** Let $v$ and $w$ be distinct vertices in $B_t$, and let $x \in V \setminus \{v, w\}$. By definition, the vertices $v, w$ belong to the same twinless strongly connected component $C$ of $G \setminus \{x\}$. Since $C$ is a subset of a strongly connected component of $G$, the vertices $v, w$ also lie in the same strongly connected component of $G \setminus \{x\}$. Consequently, $v, w$ are in the same 2-strong block in $G$.

The next algorithm describes an improved version of Algorithm 2.1 which is based on Lemma 3.1 and Lemma 2.2.

**Algorithm 3.1.**

**Input:** A twinless strongly connected graph $G = (V, E)$.

**Output:** The 2-twinless blocks of $G$.

1. if $G$ is 2-vertex-twinless connected then
2. Output $V$.
3. else
4. find the 2-strong blocks of $G$
5. Let $S$ be an $n \times n$ matrix.
6. Initialize $S$ with 0.
7. $A \leftarrow \emptyset$.
8. for each 2-strong block $B$ of $G$ do
9. for each pair of vertices $v, w \in B$ do
10. $S[v, w] \leftarrow 1$
11. $S[w, v] \leftarrow 1$
12. for each vertex $v \in B$ do
13. if $v \notin A$ then
14. add $v$ to $A$
15. determine the twinless articulation points of $G$.
16. for each twinless articulation point $z$ of $G$ do
17. Identify the twinless strongly connected components of $G \setminus \{z\}$.
18. for each pair $(v, w) \in (A \setminus \{z\}) \times (A \setminus \{z\})$ do
19. if $v, w$ in different twinless strongly connected components of $G \setminus \{z\}$ then
20. $S[v, w] \leftarrow 0$.
21. $E^b \leftarrow \emptyset$.
22. for each pair $(v, u) \in A \times A$ do
23. if $S[v, u] = 1$ and $S[u, v] = 1$ then
24. $E^b \leftarrow E^b \cup \{(v, u)\}$.
25. calculate the blocks of size $\geq 2$ of $G^b = (A, E^b)$ and output them.

**Theorem 3.1.** The running time of Algorithm 3.1 is $O(t(s^2 + m) + n^2)$, where $s = |A|$ and $t$ is the number of twinless articulation points of $G$. 
Proof: The 2-strong blocks of a directed graph can be computed in linear time [11]. Furthermore, the twinless articulation points of a directed graph can be identified in linear time using the algorithm of Georgiadis and Kosinas [8]. Since the number of iterations of the for-loop in lines 16–20 is at most $t$, lines 16–20 take $O(t(s^2 + m))$ time.

Let $G = (V, E)$ be a twinless strongly connected graph. If the refine operation defined in [11, 20] is used to refine the 2-strong blocks of $G$ for all twinless articulation points, then the 2-twinless blocks of a directed graph $G = (V, E)$ can be computed in $O(tmn)$ time, where $t$ is the number of twinless articulation points of $G$.

We leave as an open problem whether the 2-twinless blocks of a directed graph can be calculated in linear time.

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