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*Research Article*

# Counting clusters in a coloring grid

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**Abstract**

In this note, finite automata tools are used to count the number of the coloring grids according to the number of clusters colored white/black.

**Keywords:** coloring grids; finite automata; number of clusters.

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## 1. Introduction

By a *cell* $[a, b]$ we mean a unit square in the Cartesian plane $\mathbb{Z}^2$ with its sides parallel to the coordinate axes whose vertices are at integer points $(a, b), (a + 1, b), (a, b + 1), (a + 1, b + 1) \in \mathbb{Z}^2$. Two cells are *edge-connected* if they share a common edge exists. A *squared lattice polyomino* (or just *polyomino*) $\pi$, also known as an *animals*, is a finite edge-connected set of cells, where for each distinct pair of cells $X$ and $Y$ there is a finite consecutive sequence of edge-adjacent cells in $\pi$ connecting $X$ and $Y$. A *cluster* is a squared lattice polyomino such that all its cells have the same color (either black or white). By a *coloring (finite) grid* we mean the set $G_{m,n} = [0, m] \times [0, n]$ of $m \times n$ cells in $\mathbb{Z}^2$, where each cell is colored either black or white. For each $\pi \in G_{m,n}$, we denote the number of clusters colored black (white) by $c_b(\pi)$ ($c_w(\pi)$). For instance, Figure 2 presents all possible coloring grids $\pi$ in $G_{2,2}$.

$\pi$ :

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_w(\pi)$ : | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 0 |
| $c_b(\pi)$ : | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |

Figure 1: All possible coloring grids in $G_{2,2}$.

In this paper, our interest is to count the number of coloring grids $G_{m,n}$ according to the number of clusters. This counting problem is motivated by the dynamics of site percolation on the $G_{m,n}$, which have been extensively considered (see [1] and references therein). The core question in this area is bounding the expected number of clusters in $G_{m,n}$, which is inspired by the following Putnam exam problem in 2005:

*"An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares $x, y$ are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at $x$ and ending at $y$, in which successive squares in the same sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$."*

Richey [1] showed few ways to answer this problem and noticed that determining the exact expectation formula is much more difficult. More precisely, he showed that the limit $\lambda = \lim_{n,m\to\infty} e(m, n)/mn$ exists and it is finite, where $e(m, n)$ is the expected number of clusters in $G_{m,n}$. Moreover, he showed that $\frac{29}{448} \leq \lambda \leq \frac{1}{12}$.

In this paper, we use finite automata to count the number of coloring grids in $G_{m,n}$ according to the statistics $c_w$ and $c_b$, namely, the number of clusters colored white and the number of clusters colored black. In particular, we obtain exact formulas in cases $m = 1, 2, 3$.

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## 2. Results

Let $F_m(x,p,q)=\sum_{n\geq 0}F_{m,n}(p,q)x^n$ be the generating function for the number of coloring grids in $G_{m,n}$ according to the statistics $c_w$ and $c_b$, that is,

$$F_m(x,p,q)=\sum_{n\geq 0}F_{m,n}(p,q)x^n=\sum_{n\geq 0}\left(\sum_{\pi\in G_{m,n}}p^{c_w(\pi)}q^{c_b(\pi)}\right)x^n.$$

In order to find the generating function $F_m(x,p,q)$, we use finite automata tools as described in [2].

We identify the coloring grid in $G_{m,n}$ as a binary matrix $\mathbf{G}_{m,n}$ with $n$ columns and $m$ rows, where $0$ represents the white color and $1$ represents the black color. In order to count the number of coloring grids in $G_{m,n}$, we need to define the following.

Let $\mathcal{B}_m$ be the set of all binary vectors with $m$ coordinates and $\mathcal{M}_m=\{\epsilon\}\cup\cup_{n\geq 1}M_{m,n}$ be the set of all binary matrices with $m$ rows and $n$ columns, where $\epsilon$ denotes the empty matrix with $m$ rows and zero columns. We define an equivalence relation on $\mathcal{M}_m$ as follows. We say that two matrices $v$ and $v'$ in $\mathcal{M}_m$ are equivalent and write $v\sim v'$ if the following condition holds for all $u\in\mathcal{M}_m$,

$$(c_w(vu),c_b(vu))-(c_w(v'u),c_b(v'u))=constant, \tag{1}$$

where $vu$ denotes the concatenation of $v$ and $u$ (sometimes, we write the a vector in a line notation to save spaces). For instance, if

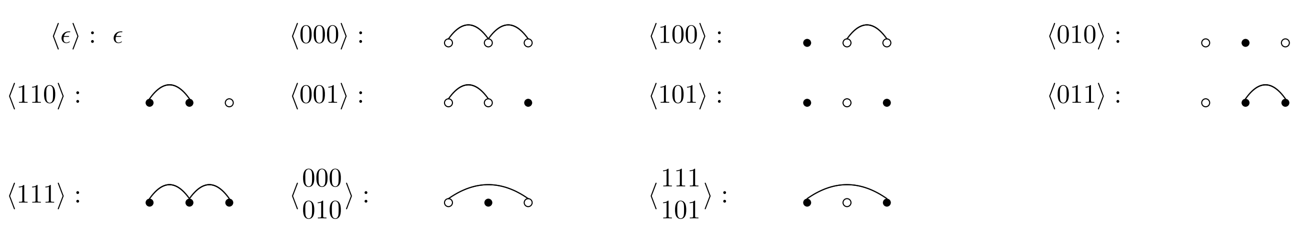
$$u_1=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad u_2=\begin{bmatrix}1\\0\\1\end{bmatrix},\quad u_3=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad u_4=\begin{bmatrix}1\\1\\1\end{bmatrix},$$

then $u_1u_2\sim u_2$ and $u_3u_4\sim u_4$. On other hand, $u_3\not\sim u_4$, since $(c_w(u_3u_1),c_b(u_3u_1))=(1,1)$, $(c_w(u_4u_1),c_b(u_4u_1))=(1,1)$, $(c_w(u_3u_2),c_b(u_3u_2))=(1,2)$ and $(c_w(u_4u_2),c_b(u_4u_2))=(1,1)$, so $(c_w(u_3u),c_b(u_3u))-(c_w(u_4u),c_b(u_4u))$ is not a constant for all $u\in\mathcal{M}_3$.

The following lemma shows that the equivalence of any two matrices in $\mathcal{M}_m$ can be checked with a finite number of steps. The proof is followed immediately from the definition of the clusters.

**Lemma 2.1.** *Let $v$ and $v'$ be two matrices in $\mathcal{M}_m$. Then $v\sim v'$ if and only if (1) holds for all $u\in\mathcal{B}_m\cup\{\epsilon\}$.*

Let $\mathcal{C}_m$ be the set of all equivalence classes of $\sim$ including the empty matrix $\epsilon$. We denote the equivalence class of a matrix $v\in\mathcal{M}_m$ by $\langle v\rangle$. Clearly, $\langle\epsilon\rangle=\{\epsilon\}$. Now, let us give a geometric representation for each equivalence classes. Give an equivalence class $\langle v\rangle\in\mathcal{C}_m$ such that $v=v^{(1)}\cdots v^{(k)}$ has exactly $k\geq 1$ columns. Since we are interested in counting clusters, we have to focus on what are the zeros (white cells) in $v^{(k)}$ that are in the same cluster, and what are the ones (black cells) in $v^{(k)}$ that are in the same cluster. The graph representation of $v$ is given by $k$ dots on a horizontal line $\ell$ where each dot is colored either white or black, and two zeros (ones) are connected by an arc above the line $\ell$ if they belong to the same cluster in $v$. For the equivalence class $\langle\epsilon\rangle$, we denote its graphical representation by $\epsilon$. For example, the graphical representation of the equivalence classes in $\mathcal{C}_3$ are given by




where we denote the concatenation of vectors $v_1,\ldots,v_s$ as a matrix with the lines $v_1^t,\ldots,v_s^t$ and the binary word $abc$ represents the column vector $(a,b,c)^t$ (see the last two equivalence classes).

Recall that a *partition* of the set $[n]=\{1,2,\ldots,n\}$ is a collection $\{B_1,\ldots,B_k\}$ of nonempty disjoint subsets of $[n]$ whose union equals $[n]$. Thus, by Lemma 2.1, the number of equivalence classes in $\mathcal{C}_m$ is at most number of set partitions of $A=\{a_1,a_2,\ldots,a_m\}\subseteq[n]$ for the white dots times the number of set partitions of $[n]\backslash A$ for the black dots. Hence, the cardinality of $\mathcal{C}_m$ is bounded by $Bell_m^2$, where $Bell_m$ is the $m$th Bell number which is given by

$$\sum_{m\geq 0}Bell_m\frac{x^m}{m!}=e^{e^x-1}.$$

**Proposition 2.1.** *Let $m \geq 0$. Then the number of equivalence classes in $\mathcal{C}_m$ is finite.*

We next introduce the key tool in our finding formulas for $F_m(x,p,q)$.

**Definition 2.1.** *We denote by $\mathcal{A}$ a* finite automaton *[2] such that*

- *The set of states of the automaton is $\mathcal{C}_m$;*
- *The input alphabet is $\mathcal{B}_m$;*
- *The transition function $\delta : \mathcal{C}_m \times \mathcal{B}_m \to \mathcal{C}_m$ is given by the rule $\delta(\langle v\rangle, u) = \langle vu\rangle$;*
- *The* initial state *is $\langle\epsilon\rangle$;*
- *All states are final states.*

We identify the automaton $\mathcal{A}_m$ with a (labeled) directed graph with vertices in $\mathcal{C}_m$ such that there is a labeled edge $\stackrel{p^aq^b}{\longrightarrow}$ from $\langle v\rangle$ to $\langle v'\rangle$ if and only if

$$(c_w(vu), c_b(vu)) - (c_w(v'u), c_n(v'u)) = (a, b), \tag{2}$$

for all $u \in \mathcal{B}_m \cup \{\epsilon\}$.

The *transition matrix* $T_m$ of $\mathcal{A}_m$ is the matrix with coefficients of the form $\sum_{(a,b)} p^a q^b$ defined by

$$[T_m]_{vv'} = \sum_{\delta(\langle v\rangle,u)=v' \ and\ (2)\ holds} p^a q^b.$$

Thus, the generating function for the number of binary matrices in $M_m$ with $n$ columns according to the statistics $c_w$ and $c_n$ is given by the generating function for weighted paths (the weight of the path is the sum of total weights of its edges) of length $n$ starting at $\langle\epsilon\rangle$ in the automaton $\mathcal{A}_m$. Hence, by Proposition 2.1, we can state our main result.

**Theorem 2.1.** *For all $m \geq 1$, the generating function $F_m(x,p,q)$ is a rational generating function and it is given by*

$$F_m(x,p,q) = e_1^t \cdot (1 - xT_m)^{-1} \cdot 1,$$

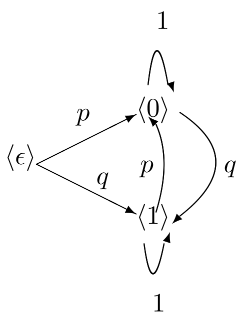
*where $e_1 = (1,0,0,\ldots)^t$ and $1 = (1,1,1,\ldots)^t$.*

Now, we are ready to find an explicit formula for $F_m(x,p,q)$, where $m = 1,2,3$.

**Case** $\mathrm{m} = 1$. Starting with the equivalence class $\langle\epsilon\rangle$, we get two other classes $\langle 0\rangle$ and $\langle 1\rangle$. By Lemma 2.1, we see that

$$\mathcal{C}_1 = \{\langle\epsilon\rangle, \langle 0\rangle, \langle 1\rangle\}.$$

Figure 2 represents the automaton $\mathcal{A}_1$:




$$T_1 = \begin{bmatrix} 0 & p & q \\ 0 & 1 & q \\ 0 & p & 1 \end{bmatrix}$$

Figure 2: Directed graph $A_1$ and the matrix $T_1$

Thus, Theorem 2.1 gives

$$\begin{aligned} F_1(x,p,q) &= (1,0,0) \cdot \sum_{n\geq 0} T_1^n x^n \cdot (1,1,1)^T = (1,0,0)(1 - xT_1)^{-1}(1,1,1)^T \\ &= \frac{(1-(1-p)x)(1-(1-q)x)}{(1-x)^2 - pqx^2} \\ &= ((1-x)^2 + (p+q)x(1-x) + pqx^2) \sum_{j\geq 0} \frac{p^j q^j x^j}{(1-x)^{2j+2}} \end{aligned}$$

$$= \sum_{j\geq 0} \frac{p^j q^j x^j}{(1-x)^{2j}} + (p+q)\sum_{j\geq 0} \frac{p^j q^j x^{j+1}}{(1-x)^{2j+1}} + \sum_{j\geq 0} \frac{p^{j+1} q^{j+1} x^{j+2}}{(1-x)^{2j+2}}.$$

Hence, by using the fact that $\frac{1}{(1-x)^d} = \sum_{n\geq 0}\binom{d-1+n}{n}x^n$ for all $d\geq 1$, we have

$$F_{1,n}(p,q) = \sum_{j=1}^{n}(pq)^j\binom{n+j-1}{n-j} + (p+q)\sum_{j=0}^{n-1}(pq)^j\binom{n+j-1}{n-1-j} + \sum_{j=0}^{n-2}(pq)^{j+1}\binom{n+j-1}{n-2-j}.$$

In particular,

$$F_1(x,q,q) = \frac{1-x+qx}{1-x-qx} = \sum_{j\geq 0}(1-x+qx)x^j(1+q)^j,$$

which leads to the following result.

**Theorem 2.2.** *The generating function for the number of coloring grids in $G_{1,n}$, $n\geq 1$, according to the number of clusters is given by*

$$F_{1,n}(q,q) = 2q(1+q)^{n-1}.$$

Thus, the total number of clusters over all coloring grids in $G_{1,n}$ is given by $(n+1)2^{n-1}$.

**Case m = 2.** In case $m=2$, the equivalence classes are $\langle\epsilon\rangle$, $\langle 00\rangle$, $\langle 01\rangle$, $\langle 10\rangle$ and $\langle 11\rangle$, where the binary word $ab$ represents the column vector $(a,b)^t$. The automaton $\mathcal{A}_2$ is given by the matrix

$$T_2 = \begin{bmatrix} 0 & p & pq & pq & q \\ 0 & 1 & q & q & q \\ 0 & 1 & 1 & pq & 1 \\ 0 & 1 & pq & 1 & 1 \\ 0 & p & p & p & 1 \end{bmatrix}.$$

So, Theorem 2.1 for $m=2$ gives that the generating function $F_2(x,p,q)-1$ is given by

$$\frac{(2pq+p+q)x + (pq(6-p-q)-2(p+q))x^2 + (pq(5p+5q-4pq)+(p+q)(1-2p-2q))x^3}{1-(pq+3)x+(pq-2p-2q+3)x^2+(p^2q^2-4pq+2p+2q-1)x^3}.$$

Note that $F_2(x,1,1) = \frac{1}{1-4x}$, as expected. So, by finding the coefficient of $x^n$ in the generating function $\frac{\partial}{\partial q}F_2(x,q,q)\mid_{q=1}$, we can state the following result.

**Theorem 2.3.** *The total number of clusters over all coloring grids in $G_{2,n}$ is given by* $\frac{5n+7}{8}4^n$.

**Case m = 3.** In case $m=3$, the equivalence classes are $\langle\epsilon\rangle$, $\langle 000\rangle$, $\langle 100\rangle$, $\langle 010\rangle$, $\langle 110\rangle$, $\langle 001\rangle$, $\langle 101\rangle$, $\langle 011\rangle$, $\langle 111\rangle$, $\left\langle \begin{smallmatrix} 000 \\ 010 \end{smallmatrix}\right\rangle$ and $\left\langle \begin{smallmatrix} 111 \\ 101 \end{smallmatrix}\right\rangle$. The automaton $\mathcal{A}_3$ is given by the matrix

$$T_3 = \begin{bmatrix} 0 & p & pq & p^2q & pq & pq & pq^2 & pq & q & 0 & 0 \\ 0 & 1 & q & 0 & q & q & q^2 & q & q & q & 0 \\ 0 & 1 & 1 & pq & 1 & q & q & pq & 1 & 0 & 0 \\ 0 & \frac{1}{p} & q & 1 & 1 & q & pq^2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & p & 1 & pq & pq & p & 1 & 0 & 0 \\ 0 & 1 & q & pq & pq & 1 & q & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & p^2q & p & 1 & 1 & p & \frac{1}{q} & 0 & 0 \\ 0 & 1 & pq & p & p & 1 & pq & 1 & 1 & 0 & 0 \\ 0 & p & p & p^2 & p & p & 0 & p & 1 & 0 & p \\ 0 & 1 & q & 0 & 1 & q & pq^2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & p^2q & p & 1 & 0 & p & 1 & 0 & 1 \end{bmatrix}.$$

So, Theorem 2.1 for $m=3$ gives that the generating function $F_3(x,p,q) = e_1(1-xT_3)^{-1}\cdot 1$ (the formula is too long to present it here). Hence, by finding the coefficient of $x^n$ in the generating function $\frac{\partial}{\partial q}F_3(x,q,q)\mid_{q=1}$, we can state the following result.

**Theorem 2.4.** *The total number of clusters over all coloring grids in $G_{3,n}$ is given by*

$$\left(\frac{169n}{224} + \frac{1945}{1568}\right)8^n + \frac{2}{49}.$$

## References


[1] J. Richey, *Counting Clusters on a Grid*, Undergraduate Honors Thesis, Dartmouth College, Hanover, 2014.

[2] J. E. Hopcroft, R. Motwani, J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation*, 3rd Edition, Pearson, London, 2006.