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Research Article

Excellent graphs with respect to domination: subgraphs induced by minimum dominating sets

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Abstract

A graph $G = (V(G), E(G))$ is γ-excellent if $V(G)$ is the union of all γ-sets of G, where γ stands for the domination number of G and a γ-set is a dominating set of cardinality γ. Let $\mathcal{I}$ be a set of all mutually nonisomorphic graphs and let $\emptyset \neq \mathcal{H} \subsetneq \mathcal{I}$. In this paper, the study of the $\mathcal{H}$-γ-excellent graphs is initiated. A graph G is $\mathcal{H}$-γ-excellent if the following conditions hold: (i) for every $H \in \mathcal{H}$ and for each $x \in V(G)$ there exists an induced subgraph H_x of G such that H and H_x are isomorphic, $x \in V(H_x)$ and $V(H_x)$ is a subset of some γ-set of G, and (ii) the vertex set of every induced subgraph H of G, which is isomorphic to some element of $\mathcal{H}$, is a subset of some γ-set of G. We consider some well-known graphs, including cycles, trees and some cartesian products of two graphs, and for every considered graph we describe its largest set $\mathcal{H} \subsetneq \mathcal{I}$ for which the graph is $\mathcal{H}$-γ-excellent. Results on γ-excellent regular graphs and on a generalized lexicographic product of graphs are presented. Several open problems and questions are also posed.

Keywords: domination number; excellent graph; graph product.

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1. Introduction

All graphs in this paper will be finite, simple, and undirected. We use [8] as a reference for terminology and notation which are not explicitly defined here. In a graph $G = (V(G), E(G))$, for a subset $S \subseteq V(G)$ the *subgraph induced* by S is the graph $\langle S \rangle$ with vertex set S and two vertices in $\langle S \rangle$ are adjacent if and only if they are adjacent in G. The *complement* $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. The union of two disjoint graphs G and H is denoted by $G \cup H$. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of x is $deg_G(x) = |N_G(x)|$. The *minimum* and *maximum* degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S \subseteq V(G)$, let $N_G[S] = \cup_{v \in S} N_G[v]$. Let $X \subseteq V(G)$ and $x \in X$. The X-*private neighborhood* of x, denoted by $pn_G[x, X]$ or simply by $pn[x, X]$ (if the graph is clear from the context), is the set $\{y \in V(G) \mid N[y] \cap X = \{x\}\}$. A *leaf* is a vertex of degree one and a *support vertex* is a vertex adjacent to a leaf. A vertex which separates two other vertices of the same component is a *cut-vertex*, and an edge separating its ends is a *bridge*. The *distance* $dist_G(x, y)$ in G of two vertices x, y is the length of a shortest $x - y$ path in G; if no such path exists, we set $dist_G(x, y) := \infty$. An *isomorphism* of two graphs G and H is a bijection $f \colon V(G) \to V(H)$ between the vertex sets of G and H such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If an isomorphism exists between two graphs, then the graphs are called isomorphic and denoted as $G \simeq H$. We write K_n for the *complete graph* of order n and P_n for the *path* on n vertices. Let C_m denote the *cycle* of length m. A complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ is a graph whose vertex set can be partitioned into r parts, say $V_1, V_2, \ldots, V_r$, such that (a) no two vertices within the same part are adjacent, (b) there is an edge between every two vertices of different parts of the partition, and (c) $|V_i| = n_i$, $i = 1, 2, \ldots, r$. The 1-*corona*, denoted $cor(U)$, of a graph U is the graph obtained from U by adding a degree-one neighbor to every vertex of U. We use the notation $[k]$ for $\{1, 2, .., k\}$.

An *independent set* is a set of vertices in a graph, no two of which are adjacent. The *independence number* of G, denoted $\beta_0(G)$, is the maximum size of an independent set in G. A subset $D \subseteq V(G)$ is called a *dominating set* (or a *total dominating set*) in G, if for each $x \in V(G) - D$ (or for each $x \in V(G)$, respectively) there exists a vertex $y \in D$ adjacent to x. A dominating set R of a graph G is a *restrained dominating set* (or an *outer-connected dominating set*) in G, if every vertex in $V(G) - R$ is adjacent to a vertex in $V(G) - R$ (or $V(G) - R$ induces a connected graph, respectively). The minimum number of vertices of a dominating set in a graph G is the *domination number* $\gamma(G)$ of G. Analogously the *total domination number* $\gamma_t(G)$, the *restrained domination number* $\gamma_r(G)$ and the *outer-connected domination number* $\gamma^{oc}(G)$ are defined. The minimum

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cardinality of a set S which is simultaneously total dominating and restrained dominating in G is called the *total restrained domination number* $\gamma_{tr}(G)$ of G. The minimum cardinality of a set S which is simultaneously total dominating and outer-connected dominating in G is called the *total outer-connected domination number* $\gamma_t^{oc}(G)$ of G. The *independent domination number* of G, denoted by $i(G)$, is the minimum size of an independent dominating set.

For a graph G, let π be a graphical property that can be possessed, or satisfied by the subsets of V. For example, being a maximal complete subgraph, a maximal independent set, acyclic, a closed/open neighborhood, a minimal dominating set, etc. Suppose that f_π and F_π are the associated graph invariants: the minimum and maximum cardinalities of a set with property π. Let $\mu \in \{f_\pi, F_\pi\}$. For a graph G, denote by $\mathtt{M}_\mu(G)$ the family of all subsets of $V(G)$ each of which has property π and cardinality $\mu(G)$. Each element of $\mathtt{M}_\mu(G)$ is called a μ*-set* of G. Fricke et al. [6] define a graph G to be μ*-excellent* if each its vertex belongs to some μ-set. Perhaps historically the first results on μ-excellent graphs were published by Berge [1] who introduced the class of B-graphs consisting of all graphs in which every vertex is in a maximum independent set. Of course all B-graphs form the class of β_0-excellent graphs. The study of excellent graphs with respect to the some domination related parameters was initiated by Fricke et al. [6] and continued e.g. in [3,9,10,14,18,20,23].

In this paper we focus on the following subclass of the class of μ-excellent graphs.

Definition 1.1. *Let $\mathcal{I}$ be a set of all mutually nonisomorphic graphs and $\emptyset \neq \mathcal{H} \subsetneq \mathcal{I}$. We say that a graph G is $\mathcal{H}$-μ-excellent if the following hold:*

(i) For each $H \in \mathcal{H}$ and for each $x \in V(G)$ there exists an induced subgraph H_x of G such that H and H_x are isomorphic, $x \in V(H_x)$ and $V(H_x)$ is a subset of some μ-set of G.

(ii) For each induced subgraph H of G, which is isomorphic to some element of $\mathcal{H}$, there is a μ-set of G having $V(H)$ as a subset.

By the above definition it immediately follows that each $\mathcal{H}$-μ-excellent graph is μ-excellent. If a graph G is $\mathcal{H}$-μ-excellent and $\mathcal{H}$ contains only one element, e.g. $\mathcal{H} = \{H\}$, we sometimes omit the brackets and say that a graph G is H-μ-excellent. Define the *μ-excellent family of induced subgraphs* of a μ-excellent graph G, denoted by $G\langle\mu\rangle$, as the family of all graphs $H \in \mathcal{I}$ for which G is H-μ-excellent. The next two observations are obvious.

Observation 1.1. *If G is a μ-excellent graph, then $\{K_1\} \subseteq G\langle\mu\rangle$ and $\mu(G) \geq \max\{|V(H)| \mid H \in G\langle\mu\rangle\}$.*

Observation 1.2. *Let a graph G be both μ-excellent and ν-excellent. If the set of all μ-sets and the set of all ν-sets of G coincide, then $G\langle\mu\rangle = G\langle\nu\rangle$.*

As first examples of $\mathcal{H}$-μ-excellent graphs let us consider the case $\mu = \beta_0$. Clearly, any β_0-excellent graph G is $\{\overline{K_1}, \overline{K_{\beta_0(G)}}\}$-$\beta_0$-excellent. A graph is r*-extendable* if every independent set of size r is contained in a maximum independent set (Dean and Zito [4]). Clearly, a graph is $\{\overline{K_1}, \overline{K_2}, .., \overline{K_r}\}$-$\beta_0$-excellent if and only if it is s-extendable for all $s = 1, 2, .., r$. Plummer [15] define a graph G to be *well covered* whenever G is k-extendable for every integer k. In other words, a graph G is well covered if and only if $G\langle\beta_0\rangle = \{\overline{K_1}, \overline{K_2}, .., \overline{K_{\beta_0(G)}}\}$.

In this paper we concentrate mainly on excellent graphs with respect to the domination number γ. We give basic terminologies and notations in the rest of this section. In Section 2 we describe the γ-excellent family of induced subgraphs for some well known graphs. In Section 3 we show that, under appropriate restrictions, the generalized lexicographic product of graphs has the same excellent family of induced subgraphs with respect to six domination-related parameters. Section 4 contains results on γ-excellent regular graphs and trees. We conclude in Section 5 with some open problems.

2. Examples

Here we find the γ-excellent family of induced subgraphs of some well known graphs.

Example 2.1. *Let G be a connected graph with $\gamma(G) = 2$. In [11] it is proved that (in our terminology) G is K_2-γ-excellent if and only if G is a complete r-partite graph $K_{n_1,n_2,\dots,n_r}$, $n_i \geq 2$, $i = 1, 2, \dots, r \geq 2$. Clearly $K_{2,2,\dots,2}\langle\gamma\rangle = \{K_1, K_2, \overline{K_2}\}$ and $K_{n_1,n_2,\dots,n_r}\langle\gamma\rangle = \{K_1, K_2\}$ when $n_s \geq 3$ for some $s \in [r]$.*

Example 2.2. *Let $\nu \in \{\gamma, i\}$. Then all the following hold:*

(i) (folklore) $\nu(P_n) = \lceil n/3 \rceil$ and $\nu(C_r) = \lceil r/3 \rceil$. C_r is ν-excelent for all $r \geq 3$. P_n is ν-excellent if and only if $n = 2$ or $n \equiv 1 \pmod 3$.

(ii) $P_n\langle\nu\rangle = \{K_1\}$ when $n \in \{1, 2\} \cup \{7, 10, \dots\}$ and $P_4\langle\nu\rangle = \{K_1, \overline{K_2}\}$

(iii) $C_5 \langle \nu \rangle = \{K_1, \overline{K_2}\}$ *and* $C_{3r} \langle \nu \rangle = C_{5+3r} \langle \nu \rangle = \{K_1\}$, $r \geq 1$.

(iv) $C_7 \langle \gamma \rangle = \{K_1, K_2, \overline{K_2}, \overline{K_3}\}$, *and* $C_{3r+1} \langle \gamma \rangle = \{K_1, K_2, \overline{K_2}\}$ *for* $r \neq 2$.

(v) $C_7 \langle i \rangle = \{K_1, \overline{K_2}, \overline{K_3}\}$ *and* $C_{3r+1} \langle i \rangle = \{K_1, \overline{K_2}\}$ *for* $r \neq 2$.

The proof is straightforward and hence we omit it.

Denote by (CEA) the class of all graphs G such that $\gamma(G+e) \neq \gamma(G)$ for all $e \in E(\overline{G})$.

Example 2.3. *Let a noncomplete graph G be in (CEA). It is well known fact that any two nonadjacent vertices of G belong to some γ-set of G (Sumner and Blitch [21]). In other words, G is $\{K_1, \overline{K_2}\}$-γ-excellent graph.*

Proposition 2.1. *Let G be a graph with $\beta_0(G) = \gamma(G) = s$. Then G is $\{\overline{K_1}, \overline{K_2}, ..., \overline{K_s}\}$-$\gamma$-excellent and $G\langle i \rangle = G\langle \beta_0 \rangle = \{\overline{K_1}, ..., \overline{K_s}\}$ (for the second conclusion, see [15]).*

Proof. Every independent set of G is a subset of a maximal independent set. Since each maximal independent set is always a dominating set and $\beta_0(G) = \gamma(G) = s$, the result immediately follows. □

The *Cartesian product* of two graphs G and H is the graph $G\Box H$ whose vertex set is the Cartesian product of the sets $V(G)$ and $V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G\Box H$ precisely when either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. It is clear from this definition that $G\Box H \simeq H\Box G$ and if G or H is not connected then $G\Box H$ is not connected.

Example 2.4. *Let $G = K_m \Box K_n$, $n \geq m \geq 2$. Then $G\langle i \rangle = G\langle \beta_0 \rangle = \{\overline{K_1}, ..., \overline{K_m}\}$. If $n > m$, then $G\langle \gamma \rangle = \{\overline{K_1}, ..., \overline{K_m}\}$. If $n = m$, then $G\langle \gamma \rangle = \{\overline{K_1}, ..., \overline{K_m}\} \cup \{K_1, K_2, ..., K_m\} \cup \{K_p \cup \overline{K_q} \mid (p \geq 2) \wedge (q \geq 1) \wedge (p+q \leq m)\}$.*

Proof. Let $G = K_m \Box K_n$, $n \geq m \geq 2$. We consider G as an $m \times n$ array of vertices $\{x_{i,j} \mid (1 \leq i \leq m) \wedge (1 \leq j \leq n)\}$, where the closed neighborhood of $x_{i,j}$ is the union of the sets $A_i = \{x_{i,1}, x_{i,2}, ..., x_{i,n}\}$ and $B_j = \{x_{1,j}, x_{2,j}, ..., x_{m,j}\}$. Then $\langle A_i \rangle \simeq K_n$ and $\langle B_j \rangle \simeq K_m$. It is well-known that [7] (a) $\gamma(G) = i(G) = \beta_0(G) = m$, (b) $A_1, A_2, ..., A_m$ are γ-sets of G, and if $m = n$, $B_1, B_2, ..., B_n$ are also γ-sets of G. Hence, by Proposition 2.1, G is $\{\overline{K_1}, \overline{K_2}, ..., \overline{K_m}\}$-$\gamma$-excellent and $G\langle i \rangle = G\langle \beta_0 \rangle = \{\overline{K_1}, ..., \overline{K_m}\}$. Suppose that G is H-γ-excellent. Then there is a γ-set D of G such that $\langle D \rangle$ has an induced subgraph $H_1 \simeq H$. Assume that H has at least one edge.

Case 1: $m < n$. Clearly $|A_i \cap D| = 1$ for all $i = 1, 2, .., m$. Because of symmetry, we assume without loss of generality that $D \cap B_j$ is empty for all $j > m$. Define now the set $D^t = \{x_{r,s} \mid x_{s,r} \in D\}$. Since H is not edgeless, $|D \cap B_j| > 1$ for some $j \leq m$. But then $|D^t \cap A_j| > 1$, which means that D^t is not a γ-set of G. Since $\langle D \rangle \simeq \langle D^t \rangle$, G is not H-γ-excellent. Thus, $G\langle \gamma \rangle = \{\overline{K_1}, ..., \overline{K_s}\}$.

Case 2: $m = n$. Obviously in this case exactly one of $|A_i \cap D| = 1$ for all $i = 1, 2, .., m$ and $|B_j \cap D| = 1$ for all $j = 1, 2, ..., m$ holds. Say the first is valid. Let R_1 be a l-order component of $\langle H \rangle$ for some $l \geq 2$. For the sake of symmetry, we can assume that all elements of R_1 are in B_1 and $D \subset \cup_{s=1}^{p} B_s$, where $D \cap B_s$ is not empty for all $s \in [p]$. Clearly $p \leq m - l + 1$. Suppose that $\langle D \rangle$ has another nontrivial component. Then the difference $m - p$ is not less than l. Define the set $D_1 = (D - V(R_1)) \cup \{x_{1,p+1}, x_{1,p+2}, ..., x_{1,p+l}\}$. Clearly D_1 is not a γ-set of G and $\langle D_1 \rangle \simeq \langle D \rangle$. Thus R_1 is the only nontrivial component of $\langle D \rangle$. Hence H is either a complete graph or a union of complete and edgeless graph. Finally, it is easy to see that for each such a graph H, G is H-γ-excellent. □

We need the following negative result.

Theorem 2.1. *There is no P_3-γ-excellent graph G with $\gamma(G) = 3$.*

Proof. Assume that G is a P_3-γ-excellent graph, $\gamma(G) = 3$ and x_1, x_2, x_3 is an induced path in G. Since $X = \{x_1, x_2, x_3\}$ is a γ-set of G, there is $y_i \in pn[x_i, X]$, $i = 1, 2, 3$. Then $\{x_1, x_2, y_2\}$ is a γ-set of G, which implies $y_2y_3 \in E(G)$. But now no vertex of the induced path y_2, y_3, x_3 is adjacent to x_1, a contradiction. □

Example 2.5. $\overline{K_3 \Box K_n}\langle \gamma \rangle = \{K_1, K_2, \overline{K_2}, K_1 \cup K_2, \overline{K_3}, K_3\}$ *when* $n \geq 3$, *and* $\overline{K_p \Box K_n}\langle \gamma \rangle = \{K_1, K_2, \overline{K_2}, K_1 \cup K_2, K_3\}$ *when* $n \geq p \geq 4$.

Proof. First note that $\overline{K_3 \Box K_3} \simeq K_3 \Box K_3$ and by Example 2.4 it immediately follows that $\overline{K_3 \Box K_3}\langle \gamma \rangle = \{K_1, K_2, \overline{K_2}, K_1 \cup K_2, \overline{K_3}, K_3\}$. So, let $n \geq 4$ and m be an integer such that $n \geq m \geq 3$. It is well known that [7] $\gamma(\overline{K_m \Box K_n}) = 3 \leq m = i(\overline{K_m \Box K_n})$. Let us consider the graph $G_{m,n} = \overline{K_m \Box K_n}$ as a $m \times n$ array of vertices $\{a_{i,j} \mid (1 \leq i \leq m) \wedge (1 \leq j \leq n)\}$, with an adjacency $N(a_{i,j}) = V(G_{m,n}) - (Y_i \cup Z_j)$, where $Y_i = \cup_{k=1}^{n}\{a_{i,k}\}$ and $Z_j = \cup_{r=1}^{m}\{a_{r,j}\}$. Remark now that:

(a) $\langle\{a_{i,j}, a_{k,l}, a_{r,s}\}\rangle \simeq K_3$ if and only if both 3-tuples (i,k,r) and (j,l,s) consist of paired distinct integers. The vertices of each triangle of $G_{m,n}$ form a γ-set. Every two adjacent vertices $a_{i,j}$ and $a_{k,l}$ belong to a triangle.

(b) All induced subgraphs isomorphic to $K_1 \cup K_2$ are $\langle\{a_{i,j}, a_{k,l}, a_{i,l}\}\rangle$ and $\langle\{a_{i,j}, a_{k,l}, a_{k,j}\}\rangle$, where $i \neq k$ and $j \neq l$. The vertices of each such a subgraph form a γ-set. Every two vertices belong to an induced subgraph isomorphic to $K_1 \cup K_2$.

(c) Each 3-cardinality subset of Z_j is independent and it is not dominating.

Theorem 2.1 together with (a)-(c) immediately lead to the required. □

To continue we need the following theorem and definitions.

Theorem 2.2. [5] $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$ *for any two arbitrary graphs G and H.*

A *G-layer* of the Cartesian product $G\Box H$ is the set $\{(u,y) \mid u \in V(G)\}$,where $y \in V(H)$. Analogously an *H-layer* is the set $\{(x,v) \mid v \in V(H)\}$, where $x \in V(G)$. A subgraph of $G\Box H$ induced by a G-layer or an H-layer is isomorphic to G or H, respectively.

Theorem 2.3. *Let H be a connected noncomplete n-order graph and $p \geq n \geq 3$. If each induced subgraph of $K_p\Box H$ which is isomorphic to H has as a vertex set some H-layer, then $\gamma(K_p\Box H) = n$ and $K_p\Box H$ is a H-γ-excellent graph.*

Proof. Each H-layer of $K_p\Box H$ is a dominating set of $K_p\Box H$. Hence $\gamma(K_p\Box H) \leq |V(H)| = n$. Since $p \geq n$, by Theorem 2.2 we have that each H-layer is a γ-set of $K_p\Box H$. It remains to note that clearly each vertex of $K_p\Box H$ belongs to some H-layer. □

The next example serves as an illustration of the above theorem.

Example 2.6. *If $p \geq n \geq 5$, then the graph $K_p\Box C_n$ is C_n-γ-excellent.*

Proof. Let H be an induced subgraph of $K_p\Box C_n$ which is isomorphic to C_r. It is easy to see that if the vertex set of H is not a C_n-layer, then either $r \in \{3,4\}$ or $r \geq n+2$. The required immediately follows by Theorem 2.3. □

3. Generalized lexicographic product

Let G be a graph with vertex set $V(G) = \{\mathbf{1}, \mathbf{2}, .., \mathbf{n}\}$ and let $\Phi = (F_1, F_2, ..., F_n)$ be an ordered n-tuple of paired disjoint graphs. Denote by $G[\Phi]$ the graph with vertex set $\cup_{i=1}^n V(F_i)$ and edge set defined as follows: (a) $F_1, F_2, ..., F_n$ are induced subgraphs of $G[\Phi]$, and (b) if $x \in V(F_i)$, $y \in V(F_j)$, $i,j \in [n]$ and $i \neq j$, then $xy \in E(G[\Phi])$ if and only if $\mathbf{ij} \in E(G)$. A graph $G[\Phi]$ is called the *generalized lexicographic product* of G and Φ. If $F_i \simeq F$ for every $i = 1, 2, ..., n$, then $G[\Phi]$ becomes the standard lexicographic product $G[F]$. Each subset $U = \{u_1, u_2, ..., u_n\} \subseteq V(G[\Phi])$ such that $u_i \in V(F_i)$, for every $i \in [n]$, is called a G-layer. From the definition of $G[\Phi]$ it immediately follow:

(A) (folklore) $G[\Phi] \simeq G$ if and only if $G[\Phi] = G[K_1]$. $G[F] \simeq F$ if and only if $G \simeq K_1$. If G has at least two vertices, then $G[\Phi]$ is connected if and only if G is connected. If G is edgeless, then $G[\Phi] = \cup_{i=1}^n F_i$. For any G-layer $U = \{u_1, u_2, ..., u_n\}$ the bijection $f\colon V(G) \to U$ defined by $f(\mathbf{i}) = u_i \in V(F_i)$ is an isomorphism between G and $\langle U\rangle$. For any $x \in V(F_i)$ and $y \in V(F_j)$, $i \neq j$, is fulfilled $dist_{G[\Phi]}(x,y) = dist_G(\mathbf{i},\mathbf{j})$.

The equality $dist_{G[\Phi]}(x,y) = dist_G(\mathbf{i},\mathbf{j})$ will be used in the sequel without specific references.

Theorem 3.1. *Given a graph $G[\Phi]$, where G is connected of order $n \geq 2$ and $|V(F_k)| \geq 3$ for all $k \in [n]$. Then $G[\Phi]\langle\gamma\rangle = G[\Phi]\langle\gamma_r\rangle = G[\Phi]\langle\gamma^{oc}\rangle$ and $G[\Phi]\langle\gamma_t\rangle = G[\Phi]\langle\gamma_{tr}\rangle = G[\Phi]\langle\gamma_t^{oc}\rangle$. If $\gamma(F_k) \geq 3$ for all $k \in [n]$, then $G[\Phi]\langle\gamma\rangle = G[\Phi]\langle\gamma_r\rangle = G[\Phi]\langle\gamma^{oc}\rangle = G[\Phi]\langle\gamma_t\rangle = G[\Phi]\langle\gamma_{tr}\rangle = G[\Phi]\langle\gamma_t^{oc}\rangle$.*

Proof. Let $\mu \in \{\gamma, \gamma_t\}$ and D a μ-set of $G[\Phi]$. Assume there is $i \in [n]$ such that $V(F_i) \cap D = \{v_1, v_2, ..., v_r\}$, where $r \geq 2$. Then clearly for each $\mathbf{j} \in N(\mathbf{i})$, $V(F_j) \cap D$ is empty and for any $u_j \in V(F_j)$ the set $(D - \{v_2, ..., v_r\}) \cup \{u_j\}$ is a dominating set of $G[\Phi]$ or a total dominating set of $G[\Phi]$ depending on whether $\mu = \gamma$ or $\mu = \gamma_t$, respectively. Hence $r = 2$. Since G is connected of order $n \geq 2$ and $|V(F_i)| \geq 3$ for all $i \in [n]$, the graph $\langle V(G[\Phi]) - D\rangle$ is connected. Therefore the first two equality chains are correct.

Finally, let D_1 be a γ-set of $G[\Phi]$ and $\gamma(F_k) \geq 3$ for all $k \in [n]$. Then clearly for every $i \in [n]$ the sets D and $V(F_i)$ must have no more than one element in common. But this immediately implies that D_1 is a total dominating set of $G[\Phi]$. Thus, the last equality chain holds. □

Theorem 3.2. *Given a graph $G[\Phi]$, where G is connected of order $n \geq 2$ and F_k is complete with $|V(F_k)| \geq 2$ for all $k \in [n]$. Then $G[\Phi]$ is $\overline{K_s}$-γ-excellent if and only if G is $\overline{K_s}$-γ-excellent.*

Proof. Recall that any G-layer of $G[\Phi]$ induces a graph isomorphic to G. We need the following claim.

Claim 1. (i) Each γ-set D of $G[\Phi]$ is contained in a G-layer of $G[\Phi]$; moreover, D is a γ-set of each subgraph of $G[\Phi]$ that is induced by a G-layer containing D. (ii) If D^* is a γ-set of some subgraph of $G[\Phi]$ that is induced by a G-layer, then D^* is a γ-set of $G[\Phi]$.

Proof of Claim 1. If D is a γ-set of $G[\Phi]$, then since all F_i's are complete $|D \cap V(F_i)| \leq 1$ for all $i \in [n]$. But then D is a dominating set of any subgraph of $G[\Phi]$ that is induced by a G-layer containing D. In particular this leads to $\gamma(G[\Phi]) \leq \gamma(G)$.

If D^* is a γ-set of some subgraph of $G[\Phi]$ that is induced by a G-layer, then again by the fact that all F_i's are complete, it follows that D^* is a dominating set of $G[\Phi]$. This clearly leads to $\gamma(G[\Phi]) \geq \gamma(G)$.

Thus $\gamma(G[\Phi]) = \gamma(G)$ implying the required. □

$\Leftarrow$ Choose $u \in V(G[\Phi])$ arbitrarily. Then there is a G-layer U containing u. Since G is $\overline{K_s}$-γ-excellent, there is a γ-set D^* of $\langle U \rangle$ that contains s paired nonadjacent vertices one of which is u. By Claim 1, D^* is a γ-set of $G[\Phi]$.

If R is a s-vertex independent set in $G[\Phi]$, then since all F_i's are complete graphs, R is a subset of some G-layer. The rest is as above.

$\Rightarrow$ Let $L = \{l_1, l_2, ..., l_n\}$ be a G-layer of $G[\Phi]$, where $l_i \in V(F_i)$, $i \in [n]$. Choose $l_r \in L$ arbitrarily. Since $G[\Phi]$ is $\overline{K_s}$-γ-excellent, there is an s-vertex independent set I_s of $G[\Phi]$ and a γ-set D of $G[\Phi]$ such that $u \in I_s \subseteq D$. By Claim 1, D is a γ-set of some subgraph induced by a G-layer of $G[\Phi]$. Since all F_i's are complete, without loss of generality, we can assume that $D \subseteq L$.

Let R be a s-vertex independent set of L. Then there is a γ-set D_1 of $G[\Phi]$ which has R as a subset. By Claim 1 D_1 is a γ-set of a graph induced by some G-layer and as above we can assume that $D_1 \subseteq L$. □

4. Regular graphs and trees

To present the next results on regular graphs, we need the following theorem.

Theorem 4.1. *Let G be a n-order graph with minimum degree δ. Then $\gamma(G) \leq n\delta/(3\delta - 1)$ when $\delta \in \{3, 4, 5\}$ (see [16], [19] and [22], respectively).*

For any 5-regular graph G with $\gamma(G) = 3$, the bound stated in Theorem 4.1 can be improved by 3.

Proposition 4.1. *Let G be a 5-regular graph with $\gamma(G) = 3$. Then $n \geq 12$.*

Proof. By Theorem 4.1 we have $n \geq 9$. Since there is no 5-regular graphs of odd order, $n \geq 10$ is even. Note that there are exactly sixty 5-regular graphs of order 10 [12, 13]. Their adjacency lists can be found in [13]. A simple verification shows that each of these graphs has the domination number equals to 2. □

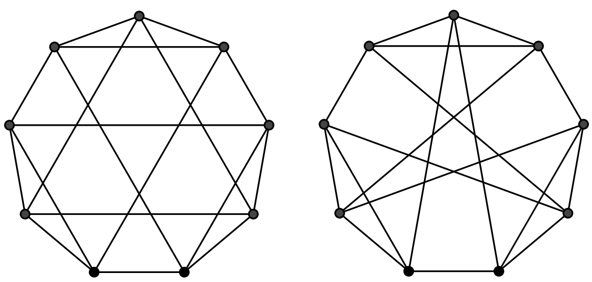

Figure 1: The two 4-regular K_3-γ-excellent graphs of order 9. The graph on the right is $K_3\Box K_3$.

Theorem 4.2. *Let G be a s-regular K_r-γ-excellent n-order connected graph with $\gamma(G) = r$, where $n > s \geq r \geq 3$. Then the following assertions hold.*

(i) $n \leq r(s - r + 2)$.

(ii) *If $r = 3$, then $s \geq 4$ with equality if and only if $n = 9$ and G is one of the graphs depicted in Fig. 1.*

(iii) *If $r = 3$ and $s = 5$, then $n = 12$.*

Proof. (i) Let $H \simeq K_r$ be a subgraph of G. Each vertex of H is adjacent to $s - r + 1$ vertices outside $V(H)$. Hence $n \leq r + r(s - r + 1) = r(s - r + 2)$.

(ii) Since $r = 3$, we have $\gamma(G) = 3$ and $n \leq 3s - 3$. By Theorem 4.1 we obtain $8 \leq n$ when $s = 3$ and $9 \leq n$ when $s \geq 4$. Thus $s \geq 4$ and if the equality holds, then $n = 9$. There are exactly 16 4-regular graphs of order 9 [13]. An immediate verification shows that among them only the graphs depicted in Fig.1 are K_3-γ-excellent.

(iii) By (i), $n \leq 12$ and by Proposition 4.1 , $n \geq 12$. □

Note that the connected 5-regular K_3-γ-excellent graph depicted in Fig. 2 has order 12.

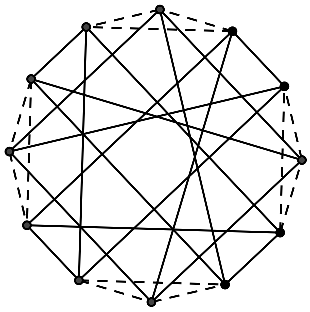

Figure 2: A 5-regular K_3-γ-excellent connected graph on 12 vertices.

Now we concentrate on graphs having cut-vertices.

Let $G_1, G_2, ..., G_k$ be pairwise disjoint connected graphs of order at least 2 and $v_i \in V(G_i)$, $i = 1, 2, ..., k$. Then the *coalescence* $(G_1{\cdot}G_2{\cdot}...{\cdot}G_k)(v_1, v_2, ..., v_k : v)$ of $G_1, G_2, ..., G_k$ via $v_1, v_2, ..., v_k$, is the graph obtained from the union of $G_1, G_2, ..., G_k$ by identifying $v_1, v_2, ..., v_k$ in a vertex labeled v. If for graphs $G_1, G_2, ..., G_k$ is fulfilled $V(G_i) \cap V(G_j) = \{x\}$ when $i, j = 1, 2, ..., k$ and $i \neq j$, then the *coalescence* $(G_1{\cdot}G_2{\cdot}...{\cdot}G_k)(x)$ of $G_1, G_2, ..., G_k$ via x is the union of $G_1, G_2, ..., G_k$.

Define $V^-(G) = \{x \in V(G) \mid \gamma(G - x) < \gamma(G)\}$ and $V^=(G) = \{x \in V(G) \mid \gamma(G - x) = \gamma(G)\}$. It is well known that $V^-(G) = \{x \in V(G) \mid \gamma(G - x) + 1 = \gamma(G)\}$. To continue we need the following result:

Lemma 4.1. [2] *Let $G = (F \cdot H)(x)$. Then $x \in V^-(G)$ if and only if $x \in V^-(F) \cap V^-(H)$. Furthermore, if $x \in V^-(G)$, then $\gamma(G) = \gamma(F) + \gamma(H) - 1$.*

Theorem 4.3. *Let $G = (G_1{\cdot}G_2{\cdot}...{\cdot}G_k)(x)$, $x \in V^-(G)$ and G_i is H-γ-exellent, $i = 1, 2, ..., k$, where H is connected and has no cut-vertex. Then G is also H-γ-excellent.*

Proof. Using induction on k we easily obtain from Lemma 4.1 that $\{x\} = V^-(G_1) \cap V^-(G_2) \cap ... \cap V^-(G_k)$ and $\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k) - k + 1$. Consider any induced subgraph R of G, which is isomorphic to H. Since H is connected and without cut-vertices, R is an induced subgraph of some G_i, say without loss of generality, $i = 1$. Then there is a γ-set D_1 of G_1 for which R is an induced subgraph of $\langle D_1 \rangle$. Let D_i be a γ-set of $G_i - x$, $i = 2, 3, ..., k$. Since $x \in V^-(G_i)$, $|D_i| = \gamma(G_i) - 1$. Then $D = \cup_{i=1}^k D_i$ is a γ-set of G and R is an induced subgraph of $\langle D \rangle$. □

Define a *vertex labeling* of a tree T as a function $S : V(T) \to \{0, 1\}$. A labeled tree T is denoted by a pair (T, S). Let $\mathbf{0}_T$ and $\mathbf{1}_T$ be the sets of vertices assigned the values 0 and 1, respectively. In a *labeled* 1-*corona tree* T of order at least four all its leaves are in $\mathbf{0}_T$ and all its support vertices form $\mathbf{1}_T$.

Let $\mathscr{T}$ be the family of labeled trees (T, S) that can be obtained from a sequence of labeled trees $\tau : (T^1, S^1), \ldots, (T^j, S^j)$, $(j \geq 1)$, such that (T^1, S^1) is a labeled 1-corona tree of order at least four and $(T, S) = (T^j, S^j)$, and, if $j \geq 2$, (T^{i+1}, S^{i+1}) can be obtained recursively from (T^i, S^i) by the following operation (a visual example of this operation is given in Figure 3):

Operation O. The labeled tree (T^{i+1}, S^{i+1}) is obtained from vertex disjoint (T^i, S^i) and a labeled 1-corona tree G_i in such a way that $T^{i+1} = (T^i \cdot G_i)(u, v : u)$, where (a) $u \in \mathbf{0}_{T^i}$, $v \in \mathbf{0}_{G_i}$ and $u \in \mathbf{0}_{T^{i+1}}$, and (b) $\mathbf{0}_{T^{i+1}} = \mathbf{0}_{T^i} \cup \mathbf{0}_{G_i} - \{v\}$ and $\mathbf{1}_{T^{i+1}} = \mathbf{1}_{T^i} \cup \mathbf{1}_{G_i}$.

Now we are in a position to present a (reformulated) constructive characterization of γ-excellent trees.

Theorem 4.4. [17] *For any tree T of order at least four the following are equivalent:*

(i) T is γ-excellent.

(ii) There is labeling $S : V(T) \to \{0, 1\}$ such that (T, S) is in $\mathscr{T}$.

Moreover, if (T, S) is in $\mathscr{T}$, then $\mathbf{0}_T = V^-(T)$, $\mathbf{0}_T$ is a γ-set of T and $\mathbf{1}_T = V^=(T)$. In particular, all leaves of T are in $V^-(T)$.

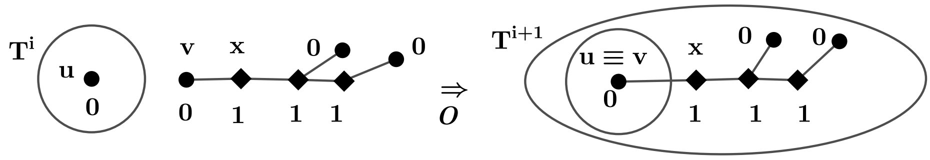

Figure 3: An example of Operation O.

Another constructive characterization of the γ-excellent trees can be found in [3]. To prove our last result we need the following lemma.

Lemma 4.2. *Let G be a connected graph and $x \in V^-(G)$.*

(i) If xy is a bridge in G, then no γ-set of G contains both x and y.

(ii) If xy and xz are bridges in G, then no γ-set of G contains both y and z.

Proof. (i) Clearly, we can consider G as a coalescence $(F \cdot H)(x)$, where without loss of generality, $y \in V(F)$ and x is a leaf of F. Suppose D is a γ-set of G and $x, y \in D$. Then $D \cap V(H)$ and $D \cap V(F)$ are dominating sets of H and F, respectively. Moreover, since x is a leaf in F, $D \cap V(F)$ is not a γ-set of F. Hence $|D| = |D \cap V(H)| + |D \cap V(F)| - 1 \geq \gamma(H) + (\gamma(F)+1) - 1$, a contradiction with Lemma 4.1.

(ii) Let as in (i), $G = (F \cdot H)(x)$, $y \in V(F)$ and x is a leaf of F. Hence $z \in V(H)$. Let D be a γ-set of G and $y, z \in D$. By (i), $x \notin D$ and then $D \cap V(H)$ and $D \cap V(F)$ are dominating sets of H and F, respectively. This implies $|D| = |D \cap V(H)| + |D \cap V(F)| \geq \gamma(H) + \gamma(F)$, a contradiction with Lemma 4.1. □

Theorem 4.5. *Let T be a γ-excellent tree of order at least four.*

(a) If T has a cut-vertex belonging to $V^-(T)$, then $T\langle\gamma\rangle = \{K_1\}$.

(b) If no cut-vertex of T is in $V^-(T)$, then T is a 1-corona tree and $T\langle\gamma\rangle = \{\overline{K_1}, ..., \overline{K_r}\}$, where $2r = |V(T)|$.

Proof. Suppose $H \in T\langle\gamma\rangle$ and H is not edgeless. Let D be a γ-set of T and $R \simeq H$ be an induced subgraph of $\langle D\rangle$. Choose arbitrarily an edge xy of R. Clearly both x and y are not leaves and by Lemma 4.2, neither x nor y is a cut-vertex belonging to $V^-(T)$. Hence $x, y \in V^=(T)$, because of Theorem 4.4. Now we choose xy so that x is a leaf in R. By Theorem 4.4, a vertex y has a neighbor $z \in V^-(T)$. Lemma 4.2 now implies $N[z] \cap D = \{y\}$. But then the graph $R_x = \langle V(R-x) \cup \{z\}\rangle$ is isomorphic to R. Since $z \in V^-(T)$ and $yz \in E(T)$, Lemma 4.2 shows that no γ-set of T contains both y and z.Thus, we arrive to a contradiction.

Therefore, $T\langle\gamma\rangle$ contains only edgeless graphs. By Theorem 4.4 $V^-(T)$ is a γ-set of T. Assume first that there is a cut-vertex $x \in V^-(T)$. Then for any two neighbors y and z of x the set $V_1 = (V^-(T) - \{x\}) \cup \{y, z\}$ is independent of cardinality $\gamma(T)+1$. Suppose T is $\overline{K_r}$-γ-excellent for some $r \geq 2$. Choose any cardinality r subset V_1 of $(V^-(T) - \{x\}) \cup \{y, z\}$ that contains both y and z. Now by Lemma 4.2, we conclude that no γ-set of T has V_1 as a subset. Thus, $T\langle\gamma\rangle = \{K_1\}$.

Finally, let $V^-(T)$ contains only leaves. By Theorem 4.4, T is a 1-corona tree. Clearly $\gamma(T) = i(T) = \beta_0(T) = r$ and then the required now follows by Proposition 2.1. □

5. Open problems and questions

We conclude the paper by listing some interesting problems and directions for further research.

- For which ordered pairs (r, s) there are s-regular K_r-excellent graphs of order $r(s-r+2)$ (see Theorem 4.2)? Find all 12-order 5-regular K_3-γ-excellent graphs.

- Characterize/describe all graphs F such that there is no F-μ-excellent graph G with $\mu(G) = |V(F)|$ (see Observation 1.1). Recall that there is no P_3-γ-excellent graph G with $\gamma(G) = 3$ (Theorem 2.1).

- Let b be a positive integer. Denote by $\mathscr{A}(\mu, b)$ the class of all μ-excellent connected graphs G for which $\mu(G) = b$ and $|G\langle\mu\rangle|$ is maximum. It might be interesting for the reader to investigate these classes at least when b is small. Note that we already know that $\mathscr{A}(\gamma, 1)$ consists of all complete graphs, and all connected graphs obtained from K_{2n}, $n \geq 2$, by removing a perfect matching form $\mathscr{A}(\gamma, 2)$ (Example 2.1). In addition, by Example 2.4 we have $\gamma(K_3 \Box K_3) = 3$, $K_3 \Box K_3\langle\gamma\rangle = \{K_1, \overline{K_2}, K_2, K_1 \cup K_2, \overline{K_3}, K_3\}$ and by Theorem 2.1 we know that there is no P_3-γ-excellent graph G with $\gamma(G) = 3$. Thus, $K_3 \Box K_3$ belongs to $\mathscr{A}(\gamma, 3)$ and $|K_3 \Box K_3\langle\gamma\rangle| = 6$. Find $\mathscr{A}(\gamma, 3)$.

- Find $T\langle\mu\rangle$ for each μ-excellent tree T, where $\mu \in \{i, \gamma_t, \gamma_R\}$ and γ_R stand for the Roman domination number (see [9], [10] and [18], respectively).

- Find graphs H such that each induced subgraph of $K_p \Box H$ which is isomorphic to H has as a vertex set some H-layer (see Theorem 2.3).

- Characterize/describe all connected $\overline{K_2}$-γ-excellent graphs G with $\gamma(G) = 2$.

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