**Discrete Mathematics Letters**
www.dmlett.com

*Discrete Math. Lett.* **5** (2021) 1–4
DOI: 10.47443/dml.2020.0046

*Research Article*

# On the pancyclicity of 1-tough graphs

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(Received: 14 August 2020. Received in revised form: 5 October 2020. Accepted: 13 November 2020. Published online: 23 November 2020.)



**Abstract**

Let $G$ be a graph of order $n \geq 6r-2$, size $e$, and minimum degree $\delta \geq r$, where $r$ is an integer greater than 1. The main result obtained in this note is that if $G$ is 1-tough with the degree sequence $(d_1, d_2, \cdots, d_n)$ and if $e \geq ((n-r)(n-r-1)+r(2r-1))/2$, then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that $d_1 = d_2 = \cdots = d_k = k$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n-k-1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n-1$, where $k < n/2$. This result implies that the following conjecture of Hoa, posed in 2002, is true under the conditions $n \geq 40$ and $\delta \geq 7$: every path-tough graph on $n$ vertices and with at least $((n-6)(n-7)+34)/2$ edges is Hamiltonian. Using the main result of this note, additional sufficient conditions for 1-tough graphs to be pancyclic are also obtained.

**Keywords:** pancyclic graph; 1-tough graph; Hamiltonian graph.

**2020 Mathematics Subject Classification:** 05C45.

## 1. Introduction

All graphs considered in this note are finite and undirected containing neither loops nor multiple edges. Terminology and notation not defined in this note follow those described in [1]. For a graph $G = (V, E)$, we take $V = \{v_1, v_2, \cdots, v_n\}$ and $|E| = e$. For a vertex $v_i \in V$, we use $d_i(G)$ to denote its degree in $G$. We use $(d_1(G), d_2(G), \cdots, d_n(G))$ to denote the degree sequence of $G$ where $\delta(G) = d_1(G) \leq d_2(G) \leq \cdots \leq d_n(G) = \Delta(G)$. Denote by $d_G(v_i, v_j)$ the distance between the two vertices $v_i, v_j \in V$. In a graph $G$, a cycle containing all the vertices of $G$ is known as a Hamilton cycle of $G$. A graph possessing a Hamilton cycle is called a Hamiltonian graph. A graph $G$ is called Hamiltonian bipartite if $G$ is both Hamiltonian and bipartite. Notice that a Hamiltonian bipartite graph must be a balanced bipartite graph. A graph containing cycles of all possible lengths is known as a pancyclic graph.

The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G)$ of a graph $G$ are the eigenvalues of its adjacency matrix $A(G)$. Let $D(G)$ be the diagonal matrix $diag(d_1, d_2, ..., d_n)$ of $G$. The Laplacian eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ of a graph $G$ are the eigenvalues of the matrix

$$L(G) := D(G) - A(G).$$

The signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq \cdots \geq q_{n-1}(G) \geq q_n(G) \geq 0$ of a graph $G$ are the eigenvalues of the matrix

$$Q(G) := D(G) + A(G).$$

The Wiener index [12] of a connected graph $G$ is denoted $W(G)$ and is defined as

$$\sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

The Harary index [10, 11] of a nontrivial connected graph $G$ is denoted $H(G)$ and is defined as

$$\sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u, v)}.$$

Chvátal [2] proposed the concept of the toughness of graphs. For a real number $t$, a graph $G$ is said to be a $t$-tough graph if for every vertex cut $S$, it holds that

$$t \cdot \omega(G - S) \leq |S|,$$

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where $\omega(G-S)$ denotes the number of components in $G-S$. The toughness of a graph $G$ is denoted by $\tau(G)$ and is defined as the maximum value of $t$ for which $G$ is $t$-tough (letting $\tau(K_n)=\infty$ for any positive integer $n$). Thus, if $G$ is different from the complete graph then

$$\tau(G)=\min\{|S|/\omega(G-S)\},$$

where the minimum is taken over all vertex cuts $S$ of $G$. It is a well-known fact that if $G$ is Hamiltonian then $G$ is also 1-tough. Dankelmann, Niessen, and Schiermeyer introduced the concept of path-tough graphs in [3]. The following definition of a path-tough graph is equivalent to its original definition given in [3]. A graph $G$ is path-tough if $G-v$ has a Hamiltonian path for every vertex $v\in V(G)$. It is observed in [3] that if $G$ is a path-tough graph then either $G$ is 1-tough or $G=K_2$.

In this note, we give the following sufficient condition for 1-tough pancyclic graphs.

**Theorem 1.1.** *For an integer $r\geq 2$, let $G$ be a graph of order $n\geq 6r-2$, size $e$, and minimum degree $\delta\geq r$. If $G$ is 1-tough and $e\geq((n-r)(n-r-1)+r(2r-1))/2$, then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that its degree sequence satisfies $d_1=d_2=\cdots=d_k=k$, $d_{k+1}=d_{k+2}=\cdots=d_{n-k+1}=n-k-1$, and $d_{n-k+2}=d_{n-k+3}=\cdots=d_n=n-1$, where $k<n/2$.*

## 2. Lemmas

We need the following results to prove Theorem 1.1. The following lemma is Proposition 1.3 of [2].

**Lemma 2.1.** *If $G$ is not complete, then $\tau(G)\leq\kappa(G)/2$, where $\kappa(G)$ is the vertex connectivity of $G$.*

The next result follows from Theorems 2 and 7 of [7].

**Lemma 2.2.** *Let $G$ be a 1-tough graph with degree sequence $d_1\leq d_2\leq\cdots\leq d_n$. If $d_i\leq i<\frac{n}{2}\Longrightarrow d_{n-i+1}\geq n-i$, then $G$ is pancyclic or Hamiltonian bipartite.*

The next lemma is Theorem 5 of [7].

**Lemma 2.3.** *Let $G$ be a 1-tough graph with degree sequence $d_1=d_2=\cdots=d_i=i$, $d_{i+1}=d_{i+2}=\cdots=d_{n-i+1}=n-i-1$, and $d_{n-i+2}=d_{n-i+3}=\cdots=d_n=n-1$, where $i<n/2$. Then $G$ is Hamiltonian.*

## 3. Proof of Theorem 1.1

**Proof of Theorem 1.1.** Note that if $G$ is complete then $G$ is pancyclic. In the remaining proof, we assume that $G$ contains at least one pair of non-adjacent vertices. From Lemma 2.1, we have $\kappa\geq 2$. Thus, $\delta\geq\kappa\geq 2$. Suppose $G$ is not pancyclic and not Hamiltonian bipartite. By Lemma 2.2, we have that there exists an integer $k$ satisfying

$$d_k\leq k<\frac{n}{2}\quad\text{and}\quad d_{n-k+1}\leq n-k-1.$$

Notice that

$$2\leq r\leq\delta=d_1\leq d_k\leq k<\frac{n}{2}.$$

Thus,

$$\begin{aligned}(n-r)(n-r-1)+r(2r-1)&\leq 2e\\&=\sum_{i=1}^{n}d_i\\&\leq k^2+(n-2k+1)(n-k-1)+(k-1)(n-1)=n^2-n-2kn+3k^2\\&=(n-r)(n-r-1)-(k-r)(2n-3k-3r)+r(2r-1).\end{aligned}$$

Thus, we have the following possible cases.

**Case 1.** $k=r$.

In this case, $d_1=d_2=\cdots=d_k=k<\frac{n}{2}$, $d_{k+1}=d_{k+2}=\cdots=d_{n-k+1}=n-k-1$, and $d_{n-k+2}=d_{n-k+3}=\cdots=d_n=n-1$. Lemma 2.3 implies that $G$ is Hamiltonian.

**Case 2.** $2n - 3k - 3r = 0$.

In this case, $d_1 = d_2 = \cdots = d_k = k < \frac{n}{2}$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n - k - 1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n - 1$. Lemma 2.3 again implies that $G$ is Hamiltonian.

**Case 3.** $k \geq r + 1$ and $2n - 3k - 3r < 0$.

In this case, we have $2n \leq 3k + 3r - 1 < (3n)/2 + 3r - 1$ and therefore $n < 6r - 2$, a contradiction.

□

## 4. Applications of Theorem 1.1

In this section, we present some applications of Theorem 1.1. Recall the following conjecture posed by Hoa on Page 142 in [6].

**Conjecture 4.1.** *Every path-tough graph on $n$ vertices and with at least $((n-6)(n-7)+34)/2$ edges is Hamiltonian.*

Notice that $((n-6)(n-7)+34)/2 > ((n-7)(n-7-1)+7(2*7-1))/2$ when $n \geq 40$. Letting $r = 7$ in Theorem 1.1 and noticing that every path-tough graph $G$ is 1-tough or $G = K_2$, we obtain the following result showing that Conjecture 4.1 is true when $n \geq 40$ and $\delta \geq 7$.

**Theorem 4.1.** *If $G$ is path-tough graph of order $n \geq 40$, size at least $((n-6)(n-7)+34)/2$, and minimum degree at least 7, then $G$ is Hamiltonian.*

Next, we will present several sufficient conditions based upon different graphical invariants for 1-tough pancyclic graphs. Recall the following result which is Theorem 1 of [8].

**Lemma 4.1.** *If $G$ is a connected graph of order $n$ with $e$ edges then $\lambda_1 \leq \sqrt{2e - n + 1}$ with equality if and only if $G = K_n$ or $G = K_{1,\,n-1}$.*

From Theorem 1.1 and Lemma 4.1, the next corollary follows.

**Corollary 4.1.** *Let $G$ be a graph of order $n \geq 6r - 2$, size $e$, and minimum degree $\delta \geq r$, where $r$ is an integer at least 2. If $G$ is 1-tough and $\lambda_1 \geq \sqrt{(n-r)(n-r-1) + r(2r-1) - n + 1}$, then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that its degree sequence is $d_1 = d_2 = \cdots = d_k = k$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n - k - 1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n - 1$, where $k < n/2$.*

Recall the following result which is Theorem 4.1 of [5].

**Lemma 4.2.** *Let $G$ be a non-complete graph. Then $\mu_{n-1} \leq \kappa$, where $\kappa$ is the vertex connectivity of $G$.*

Using Theorem 1.1, Lemma 4.2, and the fact $\kappa \leq \delta \leq (2e)/n$, we have the next result.

**Corollary 4.2.** *Let $G$ be a graph of order $n \geq 6r-2$, size $e$, and minimum degree $\delta \geq r$, where $r$ is an integer at least 2. If $G$ is 1-tough and $\mu_{n-1} \geq ((n-r)(n-r-1)+r(2r-1))/n$, then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that its degree sequence is $d_1 = d_2 = \cdots = d_k = k$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n - k - 1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n - 1$, where $k < n/2$.*

Recall the following result which is Lemma 2.4 of [4].

**Lemma 4.3.** *If $G$ is a connected graph of order $n$ and size $e$, then $q_1 \leq (2e)/(n-1) + n - 2$ with equality if and only if $G = K_n$ or $G = K_{1,\,n-1}$.*

From Theorem 1.1 and Lemma 4.3, the next result follows.

**Corollary 4.3.** *Let $G$ be a graph of order $n \geq 6r - 2$, size $e$, and minimum degree $\delta \geq r$, where $r$ is an integer at least 2. If $G$ is 1-tough and*

$$q_1 \geq \frac{(n-r)(n-r-1) + r(2r-1)}{n-1} + n - 2,$$

*then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that its degree sequence is $d_1 = d_2 = \cdots = d_k = k$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n - k - 1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n - 1$, where $k < n/2$.*

The next lemma follows from the proof of Theorem 2.2 of [13].

**Lemma 4.4.** *For a connected graph $G$ of order $n$ with $e$ edges, it holds that $W(G) \geq n(n-1) - e$.*

Using Theorem 1.1 and Lemma 4.4, we have the next result.

**Corollary 4.4.** *Let $G$ be a graph of order $n \geq 6r-2$, size $e$, and minimum degree $\delta \geq r$, where $r$ is an integer at least 2. If $G$ is 1-tough and $W(G) \leq n(n-1) - ((n-r)(n-r-1) + r(2r-1))/2$, then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that its degree sequence is $d_1 = d_2 = \cdots = d_k = k$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n-k-1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n-1$, where $k < n/2$.*

The next result follows from the proof of Theorem 2.2 of [9].

**Lemma 4.5.** *For a nontrivial connected graph $G$ order $n$ with $e$ edges, $H(G) \leq (n(n-1) + 2e)/4$.*

Using Theorem 1.1 and Lemma 4.5, we have the following result.

**Corollary 4.5.** *Let $G$ be a graph of order $n \geq 6r-2$, size $e$, and minimum degree $\delta \geq r$, where $r$ is an integer at least 2. If $G$ is 1-tough and $H(G) \geq (n(n-1) + (n-r)(n-r-1) + r(2r-1))/4$, then $G$ is pancyclic, Hamiltonian bipartite, or Hamiltonian such that its degree sequence is $d_1 = d_2 = \cdots = d_k = k$, $d_{k+1} = d_{k+2} = \cdots = d_{n-k+1} = n-k-1$, and $d_{n-k+2} = d_{n-k+3} = \cdots = d_n = n-1$, where $k < n/2$.*

## Acknowledgment

The author would like to thank the anonymous reviewers for their helpful comments and suggestions.

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