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Results on hyperbolicity in graphs: a survey

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Abstract

Let X be a geodesic metric space, and take $x_1, x_2, x_3 \in X$. A *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is said to be δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, that is, $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. In this paper, we collect some of the main theoretical results on the hyperbolicity constant in graphs.

Keywords: hyperbolic spaces; hyperbolicity constant; hyperbolic graph; geodesics, Gromov hyperbolicity.

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1. Introduction

Hyperbolic spaces, defined by Gromov in [34], play an important role in geometric group theory and in the geometry of negatively curved spaces (see [3, 11, 32, 34]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [3, 11, 32, 34]).

The initial works on Gromov hyperbolic spaces deal with finitely generated groups (see [34]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (for example, see [52]); indeed, hyperbolic groups are strongly geodesically automatic, that is, there is an automatic structure on the group [24].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [65] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension (formal proofs that the distortion is related to the hyperbolicity can be found in [68]); furthermore, it is evidenced that many real networks are hyperbolic (see, e.g., [1, 2, 43, 50]). A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [28, 31, 42]). Another important application of these spaces is the study of the spread of viruses through the internet (see [38, 40]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [38–40, 51]). The hyperbolicity has also been used extensively in the context of random graphs (see, e.g., [62–64]).

The study of Gromov hyperbolic graphs is a subject of increasing interest in graph theory; see, e.g., [4, 6–8, 12, 18, 25, 27, 35, 38–40, 43, 47, 48, 50, 51, 54, 58, 67, 70, 71] and the references therein.

We say that a curve $\gamma : [a, b] \to X$ in a metric space X is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L and d denote length and distance, respectively, and $\gamma|_{[t,s]}$ is the restriction of the curve γ to the interval $[t, s]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

In order to consider a graph G as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric space.

Throughout this paper, $G = (V, E) = (V(G), E(G))$ denotes a connected graph such that every edge has length 1 and $V \neq \emptyset$. These properties guarantee that any connected graph is a geodesic metric space. We will work both with simple

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and non-simple graphs. The difference between them is that the first type does not contain either loops or multiple edges. Although some operations on graphs, as edge contraction is naturally defined for non-simple graphs, simple graphs have a more usual context in the study of hyperbolicity.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$.

Definition 1.1. *We say that a* geodesic triangle T *is δ-thin if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T, i.e., $\delta(T) := \inf\{\delta \ge 0 : T \text{ is } \delta\text{-thin}\}$.*

Definition 1.2. *The space X is δ-*hyperbolic *(or satisfies the Rips condition with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is* hyperbolic *if X is δ-hyperbolic for some $\delta \ge 0$; then X is hyperbolic if and only if $\delta(X) < \infty$.*

If we have a triangle with two identical vertices, we call it a "bigon". Obviously, every bigon in a δ-hyperbolic space is δ-thin. If X has connected components $\{X_i\}_{i\in I}$, then we define $\delta(X) := \sup_{i\in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

In the classical references on this subject (see, e.g., [11,32]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ). We have chosen this definition due to its deep geometric meaning [32].

Trivially, any bounded metric space X is $((\operatorname{diam} X)/2)$-hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. If a complete Riemannian manifold is simply connected and its sectional curvatures satisfy $K \le c$ for some negative constant c, then it is hyperbolic. See the classical references [3,11,32] in order to find further examples and results. We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [26]).

For a finite graph with n vertices it is possible to compute $\delta(G)$ in time $O(n^{3.69})$ [29]. Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows us to determine if it is hyperbolic [53]. However, determining whether or not a general infinite graph is hyperbolic is usually very difficult. Therefore, it is interesting to study the invariance of the hyperbolicity of graphs under appropriate transformations and the hyperbolicity of particular classes of graphs. The invariance of the hyperbolicity under some natural transformations on graphs have been studied in papers, for instance, removing edges of a graph is studied in [8,18]. Moreover, the hyperbolicity of some product graphs have been characterized: in [14,16,20,49] the authors characterize in a simple way the hyperbolicity of strong product of graphs, lexicographic product of graphs, graph join and corona, and Cartesian product of graphs, respectively. Some other authors have obtained results on hyperbolicity for particular classes of graphs: chordal graphs, vertex-symetric graphs, bipartite and intersection graphs, bridged graphs and expanders [12–14,41,45,47,71]. An *isomorphism* of graphs G and H is a bijection between the vertex sets of G and H

$$f : V(G) \to V(H)$$

such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices. There are previous works relating minor graphs with tree-length and tree-width, which are parameters closely related to hyperbolicity (see [9,33,55,56]).

Note that, if we consider a graph G whose edges have lengths equal to one and a graph G_k obtained from G stretching out their edges until length k, then $\delta(G_k) = k\delta(G)$. Therefore, all the results in this survey can be generalized when the edges of the graph have a length equal to k.

In recent years, much research has been focussed on studying mathematical properties of the hyperbolicity constant in graphs. In this paper, we have surveyed and discussed the principal known results obtained about this parameter. There are relations between the different parameters of the theory of graphs and the hyperbolicity constant in graphs.

In the sense theoretical on the hyperbolicity in graphs (satisfies the Rips condition with constant δ) there are many investigations, but there are several prospects and progress to carry out in the practical and computational side.

The paper is organized into four principal parts:

In Section 2, we give a brief introduction to hyperbolic spaces in the Gromov sense and we consider some previous results regarding the hyperbolicity constant. The main objective of this section is to enunciate the results that give us the procedure for the discretization of the hyperbolicity constant.

In Section 3, we present some results that show us some relations between hyperbolicity constant and other parameters in graphs, such as the diameter, the girth, the circumference, the independence number, the domination number, the maximum degree and the effective diameter.

In Section 4, we show the hyperbolicity constant for important classes of graphs. We show some relationships and properties of the hyperbolicity constant with respect to complement of graphs, product graphs (Cartesian product, lexicographic product, direct product, Cartesian sum, graph join, strong product), median graph and chordal graph.

In Section 5, we obtain quantitative information about the distortion of the hyperbolicity constant of the graph G/e obtained from the graph G by contracting an arbitrary edge e from it for simple graphs. Besides, we obtain the invariance of the hyperbolicity on many minor graphs. We also obtain information about the hyperbolicity constant of the line graph $\mathcal{L}(G)$ in terms of properties of the graph G.

2. Basic properties of the hyperbolicity in graphs

We start with some main results in this theory. As usual, a cycle in a graph G is a simple closed curve in G, i.e., a path with different vertices, except the last one, which is equal to the first vertex. We denote by $J(G)$ the union of the set $V(G)$ and the midpoints of the edges of G. Consider the set $\mathbb{T}_1$ of geodesics triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

Now, we will present the main result of this section, which states that, in order to check whether a graph is hyperbolic or to compute the hyperbolicity constant of a graph, it suffices to consider geodesics triangles such that the three points determining those triangles are vertices or midpoints of edges of the graph. Moreover, we show that the hyperbolicity constant is a multiple of $\frac{1}{4}$. These results are important because they considerably reduce the number of geodesics triangles $T = \{x, y, z\}$ to be checked and we also reduce the number of point $p \in [xy]$ for which we need to know the value of $d(p, [yz] \cup [zx])$. In order to prove these results below we need the following lemma.

Lemma 2.1. [60, Lemma 2.1] *Let us consider a geodesic metric space X. If every geodesic triangle in X which is a simple closed curve, is δ-thin, then X is δ-hyperbolic.*

Lemma 2.1 has the following direct consequence.

Corollary 2.1. *In any graph G, $\delta(G) = \sup\{\delta(T) : T$ is a geodesic triangle which is a cycle$\}$.*

Theorem 2.1. [6, Theorem 2.5] *For every graph G, we have $\delta_1(G) = \delta(G)$.*

Theorem 2.2. [6, Theorem 2.6] *For every hyperbolic graph G, $\delta(G)$ is a multiple of $\frac{1}{4}$.*

The following result is a consequence of the previous results. It states that in any hyperbolic graph there always exists a geodesic triangle for which the hyperbolicity constant is attained.

Theorem 2.3. [6, Theorem 2.7] *For any hyperbolic graph G, there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$.

Lemma 2.2. [58, Lemma 5] *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$.*

A *loop* is an edge that connects a vertex to itself and a *multiple edge* is the set of all edges (at least two) which are incident to the same two vertices. In this section we will show, that in order to study Gromov hyperbolicity, it suffices to consider graphs without loops and multiple edges.

Definition 2.1. *We say that a vertex v of a graph G is a* cut-vertex *if $G \setminus \{v\}$ is not connected. A graph is* two-connected *if it does not contain cut-vertices. Given any edge in G, let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs $\{G_n\}_n$ the* canonical T-decomposition *of G.*

Remark 2.1. *Note that every G_n in the canonical tree-decomposition of G is an isometric subgraph of G.*

Lemma 2.3. [8, Theorem 3] *Let G be any graph with canonical T-decomposition $\{G_n\}_n$. Then*

$$\delta(G) = \sup_n \delta(G_n).$$

Given a graph G, we define $A(G)$ as the graph G without its loops, and $B(G)$ as the graph G without its multiple edges, obtained by replacing each multiple edge by a single edge with the minimum length of the edges corresponding to that multiple edge.

Theorem 2.4. [8, Theorem 6] *If G is a graph with some loops, then*

$$\delta(G) = \max\left\{\delta(A(G)),\, \frac{1}{4}\right\}.$$

Theorem 2.5. [8, Theorem 8] *If G is a graph with some multiple edges, then*

$$\delta(G) = \max\left\{\delta(B(G)),\, \frac{1}{2}\right\} = \max\left\{\delta(B(A(G))),\, \frac{1}{2}\right\}.$$

3. Relations of the hyperbolicity constant and other parameters of graphs

The next result relates δ with an important parameter of a graph: the diameter. It is a simple but useful result.

Theorem 3.1. [58, Theorem 8] *In any graph G the inequality $\delta(G) \leq \frac{1}{2}\operatorname{diam} G$ holds, and furthermore, it is sharp.*

Given any graph G we define, as usual, its *girth* $g(G)$ as the infimum of the lengths of the cycles in G.

Theorem 3.2. [48, Theorem 17] *For any graph G we have $\delta(G) \geq \frac{g(G)}{4}$ and the inequality is optimal.*

Let us define the *circumference* $c(G)$ of a graph G as the supremum of the lengths of its cycles.

Theorem 3.3. [21, Proposition 2.12] *For any graph G, we have $\delta(G) \leq \frac{1}{4}c(G)$, and this inequality is sharp.*

We say that a subset $A \subset V(G)$ is an *independent set* if $[v, w] \notin E(G)$ for every $v, w \in A$. We denote by $\beta(G)$ the *independence number* of G, i.e. the cardinal of the largest independent set in G.

Theorem 3.4. [57, Theorem 2.2] *For every graph G with n vertices, we have*

$$\delta(G) \leq \min\left\{\beta(G),\, \frac{n-\beta(G)+2}{2}\right\}.$$

A set $S \subset V$ of a graph G, is a dominating set if every vertex not in S is adjacent to a vertex in S. The domination number of G, denoted by $\gamma(G)$ the *domination number* of G, is the minimum cardinality of a dominating set.

Theorem 3.5. [57, Theorem 2.8] *For every graph G, we have $\delta(G) \leq \frac{3\gamma(G)}{2}$.*

Theorem 3.6. [57, Theorem 2.9] *Let g be a graph with n vertices. If there exists a circumference C in G such that $V(G)\backslash V(C)$ is a dominating set, then*

$$\delta(G) \leq \frac{n-\gamma(G)}{4}.$$

Theorem 3.7. [57, Theorem 3.2] *Let G be any graph with n vertices and maximum degree $\Delta = n-1$ which is not a tree. Then*

$$\frac{3}{4} \leq \delta(G) \leq \frac{3}{2},$$

and both inequalities are sharp.

Theorem 3.8. [57, Theorem 3.3] *Let G be any graph with m edges and maximum degree Δ. Then*

$$\delta(G) \leq \frac{m+2-\Delta}{4}.$$

Furthermore, if $\Delta = 2$, then the inequality is attained if and only if G is isomorphic to C_m; if $\Delta = 3$, then the inequality is attained if and only if G is isomorphic to C_{m-1} with an edge attached joining two vertices of C_{m-1} at distance (in C_{m-1}) 2 or 3.

Theorem 3.9. [57, Theorem 3.4] *Let G be any graph with n vertices and minimum degree d_0. Then*

$$\delta(G) \leq \max\left\{\frac{3}{2},\, \frac{n+2-d_0}{4}\right\},$$

and the inequality is sharp.

Definition 3.1. *Given a graph G and its canonical T-decomposition $\{G_n\}$, we define the* effective diameter *as*

$$\operatorname{diameff} V(G) := \sup_n \operatorname{diam} V(G_n), \qquad \operatorname{diameff}(G) := \sup_n \operatorname{diam}(G_n).$$

Lemma 2.3 and Theorem 3.1 have the following consequence.

Lemma 3.1. [5, Lemma 4.4] *Let G be any graph. Then*

$$\delta(G) \le \frac{1}{2}\operatorname{diameff}(G).$$

Theorem 3.10. [5, Theorem 4.14] *Let G be any graph. Then $\delta(G) = 1$ if and only if* $\operatorname{diameff}(G) = 2$.

Remark 3.1. *It is not possible to bound* $\operatorname{diameff} V(G)$ *or* $\operatorname{diameff}(G)$ *if* $\delta(G) \ge \frac{3}{2}$:

Let G be the Cayley graph of the group $\mathbb{Z}\times\mathbb{Z}_2$ (G has the shape of an infinite railway). We have $\delta(G) = \frac{3}{2}$ and the canonical T-decomposition of G has just a graph $G_1 = G$; hence, $\operatorname{diameff} V(G) = \operatorname{diam} V(G_1) = \infty$ *and* $\operatorname{diameff}(G) = \infty$.

For each $n > 6$ consider the cycle graph C_n, and fix vertices $v_1 \in V(G)$ and $v_2 \in V(C_n)$. The graph G_n obtained from G and C_n by identifying v_1 and v_2 has canonical T-decomposition $\{G, C_n\}$ and $\operatorname{diameff} V(G_n) = \operatorname{diameff} V(G) = \infty$ *and* $\operatorname{diameff}(G_n) = \infty$. *Furthermore, Lemma 2.3 gives*

$$\delta(G_n) = \max\left\{\delta(G),\, \delta(C_n)\right\} = \max\left\{\frac{3}{2},\, \frac{n}{4}\right\} = \frac{n}{4}.$$

4. Hyperbolicity constant for important classes of graphs

We start this section with the precise values of the hyperbolicity constant of several important graphs.

Theorem 4.1. [58, Theorem 11] *The following graphs have these precise values of δ described in each case:*

- *The path graphs verify $\delta(P_n) = 0$ for every $n \ge 1$.*
- *The cycle graphs verify $\delta(C_n) = n/4$ for every $n \ge 3$.*
- *The complete graphs verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ for every $n \ge 4$.*
- *The complete bipartite graphs verify $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = 1$ for every $m, n \ge 2$.*
- *The Petersen graph P verifies $\delta(P) = 3/2$.*
- *The wheel graph with n vertices W_n verifies $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ for every $7 \le n \le 10$, and $\delta(W_n) = 5/4$ for $n = 6$ and for every $n \ge 11$.*

Furthermore, the graphs C_n and K_n for every $n \ge 3$, $K_{m,n}$ for every $m, n \ge 2$, the Petersen graph and W_n for every $4 \le n \le 10$, verify $\delta(G) = \frac{1}{2}$ $\operatorname{diam} G$.

Theorem 4.2. [48, Theorem 11] *Let G be any graph.*

- *$\delta(G) < 1/4$ if and only if G is a tree.*
- *$\delta(G) < 1/2$ if and only if $A(G)$ is a tree.*
- *$\delta(G) < 3/4$ if and only if $B(A(G))$ is a tree.*
- *$\delta(G) < 1$ if and only if every cycle g in G has length $L(g) \le 3$.*

Furthermore, if $\delta(G) < 1$, then $\delta(G) \in \{0, 1/4, 1/2, 3/4\}$.

Theorem 4.3. [5, Theorem 3.8] *Let G be a graph. Then $\delta(G) = 1$ if and only if the following conditions hold:*

(1) *There exists a cycle isomorphic to C_4.*

(2) *For every cycle γ such that $L(\gamma) \ge 5$ and for every vertex $w \in \gamma$, it is satisfied* $\deg_\gamma(w) \ge 3$.

Proposition 4.1. [5, Proposition 3.9] *Let G be a graph. Assume that the following conditions hold:*

(1) *There exist a cycle g in G such that $L(g) \ge 5$ and a vertex $w \in g$ satisfying* $\deg_g(w) = 2$.

(2) *For every cycle γ we have* $\operatorname{diam}(\gamma) \le \frac{5}{2}$.

Then we have $\delta(G) = \frac{5}{4}$.

4.1 Hyperbolicity and complement of graphs

Given any simple finite or infinite graph G, we denote by $\overline{G}$ the complement of G. We start with some examples. The following graphs have these precise values of δ:

- The complement of the path graphs verify $\delta(\,\overline{P_n}\,) = 5/4$ for every $n \geq 5$.
- The complement of the cycle graphs verify $\delta(\,\overline{C_n}\,) = 5/4$ for every $n \geq 5$.
- The complement of the star graphs verify $\delta(\,\overline{S_n}\,) = 1$ for every $n \geq 5$.

Theorem 4.4. [7, Theorem 2.2] *If* $\operatorname{diam}(V(G)) \geq 3$, *then* $\delta(\,\overline{G}\,) \leq 2$, *and this inequality is sharp.*

Theorem 4.5. [7, Theorem 3.1] *If G is any graph with $\delta(G) > \frac{3}{2}$, then $1 \leq \delta(\overline{G}) \leq 2$. Furthermore, if $\delta(G) > 2$, then $1 \leq \delta(\overline{G}) \leq \frac{3}{2}$. In particular, if $\delta(G) \geq 2$, then $\delta(\overline{G}) \leq \delta(G)$; if $\delta(G) > 2$, then $\delta(\overline{G}) < \delta(G)$.*

Proposition 4.2. [7, Proposition 3.4] *If G is any graph with $n \geq 4$ vertices and $\deg v \geq n-2$ for every vertex $v \in V(G)$, then $\delta(\,\overline{G}\,) < \delta(G)$.*

4.2 Hyperbolicity and product graphs

Definition 4.1. *We define the Cartesian product $G_1 \square G_2$ as the graph with vertices $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and $[(u_1, u_2), (v_1, v_2)] \in E(G_1 \square G_2)$ if and only if we have either $u_1 = v_1 \in V(G_1)$ and $[u_2, v_2] \in E(G_2)$ or $u_2 = v_2 \in V(G_2)$ and $[u_1, v_1] \in E(G_1)$.*

Theorem 4.6. [49, Theorem 13] *For every graphs G_1, G_2 we have*

$$\delta(G_1 \square G_2) \leq \min\big\{ \max\{1/2 + \operatorname{diam}_{G_2} V(G_2), \delta(G_1) + \operatorname{diam}'_{G_2} G_2\}, \max\{1/2 + \operatorname{diam}_{G_1} V(G_1), \delta(G_2) + \operatorname{diam}'_{G_1} G_1\}\big\},$$

and the inequality is sharp.

We also have the following lower bounds for $\delta(G_1 \square G_2)$.

Theorem 4.7. [49, Theorem 18] *For every graph G_1, G_2 we have*

(a) $\delta(G_1 \square G_2) \geq \max\{\delta(G_1), \delta(G_2)\}$,

(b) $\delta(G_1 \square G_2) \geq \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\}$,

(c) $\delta(G_1 \square G_2) \geq \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\} + 1/2$, *if* $\operatorname{diam}_{G_1} V(G_1) \neq \operatorname{diam}_{G_2} V(G_2)$,

(d) $\delta(G_1 \square G_2) \geq \frac{1}{2} \min\{\delta(G_1) + \operatorname{diam}_{G_2} V(G_2), \delta(G_2) + \operatorname{diam}_{G_1} V(G_1)\}$.

Furthermore, inequalities in (b) and (c) are sharp, as the first and second item in Theorem 4.9 show.

Theorems 4.6 and 4.7 have the following consequence.

Theorem 4.8. [49, Theorem 21] *Let G_1 and G_2 be two graphs, then $G_1 \square G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded.*

Theorem 4.9. [49, Theorem 23] *The following graphs have the precise values of δ:*

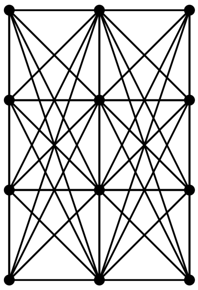
- $\delta(P_n \square P_n) = n - 1$, *for every* $n \geq 2$.
- $\delta(P_m \square P_n) = \min\{m, n\} - 1/2$, *for every* $m, n \geq 2$ *with* $m \neq n$.
- $\delta(Q_n) = n/2$, *for every* $n \geq 2$.
- $\delta(C_m \square C_n) = (m + n)/4$, *for every* $m, n \geq 3$.
- $\delta(T_1 \square T_2) = \delta(P_{1+\operatorname{diam} T_1} \square P_{1+\operatorname{diam} T_2})$, *for every tree* T_1, T_2, *i.e.,*

$$\delta(T_1 \square T_2) = \begin{cases} \operatorname{diam}_{T_1} T_1 & \text{if } \operatorname{diam}_{T_1} T_1 = \operatorname{diam}_{T_2} T_2, \\ \min\{\operatorname{diam}_{T_1} T_1, \operatorname{diam}_{T_2} T_2\} + 1/2 & \text{if } \operatorname{diam}_{T_1} T_1 \neq \operatorname{diam}_{T_2} T_2. \end{cases}$$

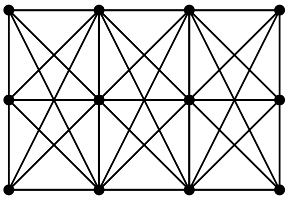
We will use the of lexicographic product definition given in [37].

Definition 4.2. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The lexicographic product $G_1 \circ G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \circ G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$, or $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$.*

Note that the lexicographic product of two graphs is not always commutative. We use the notation (x, y) for the points of the graph $G_1 \circ G_2$ with $x \in V(G_1)$ or $y \in V(G_2)$. Otherwise, this notation can be ambiguous.



$P_3 \circ P_4$



$P_4 \circ P_3$

Figure 1: Non commutative lexicographic product of two graphs ($P_3 \circ P_4 \not\simeq P_4 \circ P_3$).

By *trivial graph* we mean a graph having just a single vertex, and we denote it by E_1. If G_1 and G_2 are isomorphic, then we write $G_1 \simeq G_2$.

The next theorem shows an important qualitative result: if G_1 is not hyperbolic then $G_1 \circ G_2$ is not hyperbolic.

Theorem 4.10. [16, Theorem 3.1] *Let G_1 and G_2 two graphs, then $\delta(G_1) \leq \delta(G_1 \circ G_2)$.*

Example 4.1. *Let P_n be the path graph with $n \geq 2$. Then*

$$\delta(P_n \circ P_2) = \begin{cases} 1 & \text{if } n = 2, \\ 5/4 & \text{if } n = 3, \\ 3/2 & \text{if } n \geq 4. \end{cases}$$

Example 4.2. *Let C_n be the cycle graph with $n \geq 3$. Then*

$$\delta(C_n \circ P_2) = \begin{cases} 1 & \text{if } n = 3, \\ 5/4 & \text{if } n = 4, \\ n/4 & \text{if } n \geq 5. \end{cases}$$

Theorem 4.11. [16, Theorem 3.3] *Let G_1 be a non-trivial graph and G_2 any graph. Then*

$$\delta(G_1 \circ G_2) = \max\{\delta(\Gamma_1 \circ \Gamma_2) : \Gamma_i \text{ is isometric to } G_i \text{ for } i = 1, 2 \text{ and } \Gamma_1 \text{ non-trivial}\}.$$

Theorem 4.12. [16, Theorem 3.7] *Let G_1 be any non-trivial graph and G_2 any graph. Then we have $\delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$.*

Theorems 4.10 and 4.12 have the following consequence.

Theorem 4.13. [16, Theorem 3.10] *Let G_1 be any non-trivial graph and G_2 any graph. The lexicographic product $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

We will use the definition of direct product of graphs given in [36].

Definition 4.3. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The* direct product *$G_1 \times G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.*

Proposition 4.3. [15, Proposition 2.8] *Let G_1 and G_2 be two unbounded graphs. Then $G_1 \times G_2$ is not hyperbolic.*

Theorem 4.14. [15, Theorem 2.11] *Let G_1 be a graph and G_2 be a non-trivial bounded graph with some odd cycle. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Theorem 4.15. [15, Theorem 3.6] *If G_1 and G_2 are bipartite graphs with $k_1 := \operatorname{diam} V(G_1)$ and $k_2 := \operatorname{diam} V(G_2)$ such that $k_1 \geq k_2 \geq 1$, then*

$$\max\Big\{\min\Big\{\frac{k_1-1}{2}, k_2-1\Big\}, \delta(G_1), \delta(G_2)\Big\} \leq \delta(G_1 \times G_2) \leq \frac{k_1}{2}.$$

Furthermore, if $k_1 \leq 2k_2 - 2$ and k_1 is even, then $\delta(G_1 \times G_2) = k_1/2$.

Theorem 4.16. [15, Theorem 3.7] *For every odd number $m \geq 3$ and every $n \geq 2$,*

$$\delta(C_m \times P_n) = \begin{cases} m/2 & \text{if } n-1 \leq m, \\ (n-1)/2 & \text{if } m < n-1 < 2m, \\ m - 1/2 & \text{if } n-1 \geq 2m. \end{cases}$$

We will use the definition of Cartesian sum of graphs given in [46].

Definition 4.4. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The* Cartesian sum *$G_1 \oplus G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \oplus G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$ or $[v_1, v_2] \in E(G_2)$.*

Remark 4.1. *The Cartesian, strong and lexicographic products of two graphs are subgraphs of the Cartesian sum product of two graphs, i.e., $G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \circ G_2 \subseteq G_1 \oplus G_2$.*

Theorem 4.17. [17, Theorem 3.5] *For every non-trivial graph G_1, G_2, we have*

$$\delta(G_1 \oplus G_2) = \max\{\delta(\Gamma_1 \oplus \Gamma_2) : \Gamma_i \text{ is an isometric subgraph of } G_i \text{ and } \Gamma_i \text{ is non-trivial for } i = 1, 2\}.$$

The following result characterizes the hyperbolic Cartesian sums.

Theorem 4.18. [17, Theorem 3.9] *Let G_1 and G_2 be any graphs.*

(1) *If G_1 is a trivial graph, then the Cartesian sum $G_1 \oplus G_2$ is hyperbolic if and only if G_2 is hyperbolic. Furthermore,*

$$\delta(G_1 \oplus G_2) = \delta(G_2).$$

(2) *If G_2 is a trivial graph, then the Cartesian sum $G_1 \oplus G_2$ is hyperbolic if and only if G_1 is hyperbolic. Furthermore,*

$$\delta(G_1 \oplus G_2) = \delta(G_1).$$

(3) *For every non-trivial graph G_1, G_2 the Cartesian sum $G_1 \oplus G_2$ is hyperbolic with*

$$1 \leq \delta(G_1 \oplus G_2) \leq 3/2.$$

Furthermore, the hyperbolicity constant $\delta(G_1 \oplus G_2)$ belongs to $\{1, 5/4, 3/2\}$.

Theorem 4.19. [17, Theorem 3.12] *Let G_1, G_2 be any graphs. If $\operatorname{diam} V(G_i) \geq 3$ for $i \in \{1, 2\}$, then $\delta(G_1 \oplus G_2) = 3/2$.*

Theorem 4.20. [17, Theorem 3.22] *Let G_1, G_2 be any trees. Then*

$$\delta(G_1 \oplus G_2) = \begin{cases} 0 & \text{if } G_1 \simeq E_1 & \text{or} & G_2 \simeq E_1, \\ 1 & \text{if } 1 \leq \operatorname{diam} G_1 \leq 2 & \text{and} & 1 \leq \operatorname{diam} G_2 \leq 2, \\ 5/4 & \text{if } 1 \leq \operatorname{diam} G_1 \leq 2 & \text{and} & \operatorname{diam} G_2 \geq 3, \\ 5/4 & \text{if } \operatorname{diam} G_1 \geq 3 & \text{and} & 1 \leq \operatorname{diam} G_2 \leq 2, \\ 3/2 & \text{if } \operatorname{diam} G_1 \geq 3 & \text{and} & \operatorname{diam} G_2 \geq 3. \end{cases}$$

Theorem 4.21. [17, Theorem 4.5] *Let G_1, G_2 be any graphs. If $\operatorname{diam} V(G_i) \geq 3$ for $i \in \{1, 2\}$, then*

$$\frac{3}{2} \leq \delta(\overline{G_1 \oplus G_2}) \leq 2.$$

Definition 4.5. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs with $V(G) \cap V(H) = \emptyset$. The* graph join *$G + H$ of G and H has $V(G+H) = V(G) \cup V(H)$ and two different vertices u and v of $G+H$ are adjacent if $u \in V(G)$and $v \in V(H)$, or $[u, v] \in E(G)$ or $[u, v] \in E(H)$.*

Corollary 4.1. [20, Corollary 3.2] *For any graphs G, H, the graph join $G+H$ is hyperbolic with $\delta(G+H) \leq 3/2$, and the inequality is sharp.*

Theorem 4.22. [20, Theorem 3.9] *Let G, H be two graphs.*

(1) $\delta(G+H)=0$ *if and only if G and H are empty graphs and one of them is isomorphic to E_1.*

(2) $\delta(G+H)=3/4$ *if and only if $G \simeq E_1$ and $\Delta_H = 1$, or $H \simeq E_1$ and $\Delta_G = 1$.*

If G is a graph with connected components $\{G_j\}$, we define

$$\operatorname{diam}^* G := \sup_j \{\operatorname{diam} G_j\}.$$

Theorem 4.23. [20, Theorem 3.22] *Let G, H be any two graphs. Then*

$$\delta(G+H) = \begin{cases} 0 & \text{if } G \simeq E_n \text{ and } H \simeq E_m \text{ with } n=1 \text{ or } m=1, \\ 3/4 & \text{if } G \simeq E_1 \text{ and } \Delta_H = 1, \text{ or } H \simeq E_1 \text{ and } \Delta_G = 1, \\ 1 & \text{if } G \simeq E_1 \text{ and } 1 < \operatorname{diam}^* H \leq 2; \text{ or } H \simeq E_1 \text{ and } 1 < \operatorname{diam}^* G \leq 2; \text{ or} \\ & |\, V(G)\,|, |\, V(H)\,| \geq 2, \text{ and } \operatorname{diam} G \leq 2 \text{ or } \operatorname{diam}^* G = 0, \text{ and } \operatorname{diam} H \leq 2 \text{ or } \operatorname{diam}^* H = 0; \\ 3/2 & \text{if } G \in \mathcal{F} \text{ or } H \in \mathcal{F}, \\ 5/4 & \text{otherwise}. \end{cases}$$

We will use the definition of strong product of graphs given by Sabidussi in [61].

Definition 4.6. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ two graphs. The strong product $G_1 \boxtimes G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \boxtimes G_2$ are adjacent if either $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$, or $[u_1, u_2] \in E(G_1)$ and $v_1 = v_2$, or $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.*

Corollary 4.2. [14, Corollary 12] *For all graphs G_1, G_2, we have*

$$\delta(G_1 \boxtimes G_2) \leq \frac{\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1}{2},$$

and the inequality is sharp.

Theorem 4.24. [14, Theorem 15] *For all graphs G_1, G_2 we have:*

(a) $\delta(G_1 \boxtimes G_2) \geq \max\{\delta(G_1), \delta(G_2)\}$,

(b) $\delta(G_1 \boxtimes G_2) \geq \frac{1}{2} \min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$,

(c) $\delta(G_1 \boxtimes G_2) \geq \frac{1}{2}\big(\operatorname{diam} V(G_1) + 1\big)$, *if* $0 < \operatorname{diam} V(G_1) < \operatorname{diam} V(G_2)$,

(d) $\delta(G_1 \boxtimes G_2) \geq \frac{1}{4} \min\{\operatorname{diam} V(G_1) + 2\delta(G_2), \operatorname{diam} V(G_2) + 2\delta(G_1)\}$.

The following one is a qualitative result about the hyperbolicity of $G_1 \boxtimes G_2$.

Theorem 4.25. [14, Theorem 17] *If G_1 and G_2 are infinite graphs, then $G_1 \boxtimes G_2$ is not hyperbolic.*

Now we state the main result about the hyperbolicity of $G_1 \boxtimes G_2$.

Theorem 4.26. [14, Theorem 23] *For all G_1, G_2 we have that $G_1 \boxtimes G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded.*

Theorem 4.27. [14, Theorem 34] *Let C_n, C_m be two cycle graphs with $3 \leq n \leq m$. Then*

$$\delta(C_n \boxtimes C_m) = \begin{cases} \lfloor m/2 \rfloor /2 + 1/2 & \text{if } \lfloor m/2 \rfloor < 2\lfloor n/2 \rfloor, \\ \lfloor m/2 \rfloor /2 + 1/4 & \text{if } \lfloor m/2 \rfloor = 2\lfloor n/2 \rfloor, \\ m/4 & \text{if } \lfloor m/2 \rfloor > 2\lfloor n/2 \rfloor. \end{cases}$$

Theorem 4.28. [14, Theorem 35] *For every $m \geq 2$, $n \geq 3$,*

$$\delta(C_n \boxtimes P_m) = \begin{cases} \lfloor n/2 \rfloor + 1/2 & \text{if } \lfloor n/2 \rfloor < (m-1)/2, \\ \lfloor n/2 \rfloor + 1/4 & \text{if } \lfloor n/2 \rfloor = (m-1)/2, \\ m/2 & \text{if } (m-1)/2 < \lfloor n/2 \rfloor \leq (m-1), \\ \big(\lfloor n/2 \rfloor + 1\big)/2 & \text{if } m-1 < \lfloor n/2 \rfloor < 2(m-1), \\ \lfloor n/2 \rfloor/2 + 1/4 & \text{if } \lfloor n/2 \rfloor = 2(m-1), \\ n/4 & \text{if } \lfloor n/2 \rfloor > 2(m-1). \end{cases}$$

4.3 Hyperbolicity in median graphs

In any graph, for any two vertices a and b, we define the *interval* of vertices that lie on shortest paths $I(a,b) := \{v/\, d(a,b) = d(a,v)+d(v,b)\}$. A *median graph* is defined by the property that, for any three vertices a, b, c, the intervals $I(a,b), I(a,c), I(b,c)$ intersect in a single point.

Proposition 4.4. [66, Proposition 2.1] *The following equalities hold:*

(1) *If G is any tree, then $\delta(G) = 0$.*

(2) *If G is the Cayley graph of the group $\mathbb{Z}^n$, then $\delta(G) = 0$ for $n = 1$ and $\delta(G) = \infty$ for $n > 1$.*

(3) *If G is the n-cube graph $Q_n = K_2 \times \cdots \times K_2$, then $\delta(G) = n/2$.*

Definition 4.7. *Given a graph G, let us consider the set $\mathbb{T}_2$ of geodesics triangles T in G which are cycles and such that the three vertices of the triangle T are vertices of G. We define the constant $\delta_2(G)$ as the supremum of $\delta(T)$ for every $T \in \mathbb{T}_2$.*

Theorem 4.29. [66, Theorem 2.4] *For every graph G, we have*

$$\delta_2(G) \leq \delta(G) \leq 4\delta_2(G) + \frac{1}{2}\,.$$

Theorem 4.30. [66, Theorem 2.5] *Let G be any median graph. Then G is hyperbolic if and only if the bigons of G are uniformly thin. In fact, if every geodesic bigon in G is δ-thin, then*

$$\delta_2(G) \leq 3\delta, \qquad \delta(G) \leq 12\delta + \frac{1}{2}\,.$$

The following property about median graphs and tree-decompositions holds.

Proposition 4.5. [66, Proposition 2.7] *The following statements are equivalent for any graph G:*

(1) *G is a median graph.*

(2) *G_n is a median graph for every n for some tree-decomposition $\{G_n\}_n$ of G.*

(3) *G_n is a median graph for every n for every tree-decomposition $\{G_n\}_n$ of G.*

5. Study on the invariance of the hyperbolicity in graphs under transformations

5.1 Hyperbolicity constant of line graphs

Although the terminology of a line graph was used in [7] for the first time, line graphs were initially introduced in the paper [39]. If G is a graph, we denote by $\mathcal{L}(G)$ its line graph.

Theorem 5.1. [21, Theorem 2.2] *There exists a $(1/2)$-full $(1,1)$-quasi-isometry from G on its line graph $\mathcal{L}(G)$ and, consequently, G is hyperbolic if and only if $\mathcal{L}(G)$ is hyperbolic.*
Furthermore, if G (respectively, $\mathcal{L}(G)$) is δ - hyperbolic, then $\mathcal{L}(G)$(respectively, G) is δ'- hyperbolic, where δ' is a constant which depends on δ.

Proposition 5.1. [21, Proposition 3.3] *Let T be any tree with maximum degree Δ. Then,*

$$\delta(\mathcal{L}(T)) = \begin{cases} 1 & \text{if } \Delta \geq 4, \\ 3/4 & \text{if } \Delta = 3, \\ 0 & \text{if } \Delta \leq 2. \end{cases}$$

Theorem 5.2. [21, Theorem 3.13] *For any graph G, we have*

$$\frac{1}{4}g(G) \leq \delta(\mathcal{L}(G)) \leq \frac{1}{4}c(G) + 2.$$

5.2 Gromov hyperbolicity of minor graphs

If G is a graph and $e := [a, b] \in E(G)$, we denote by G/e the graph obtained by contracting the edge e from it (we remove e from G while simultaneously we merge a and b).

We will denote by v_e the vertex in G/e obtained by identifying a and b. Note that any vertex $v \in V(G) \setminus \{a, b\}$ can be seen as a vertex in $V(G/e)$. Also we can write any edge in $E(G/e)$ in terms of its endpoints, but we write v_e instead of a or b. If $[v, a]$ and $[v, b]$ are edges of G for some $v \in V(G)$, then we need to replace both edges by a single edge $[v, v_e] \in G/e$ since we work with simple graphs, see Figure 2.

We define the map $h : G \to G/e$ in the following way: if x belongs to the edge e, then $h(x) := v_e$; if $x \in G$ does not belong to e, then $h(x)$ is the "natural inclusion map". Clearly h is onto, i.e., $h(G) = G/e$. Besides, h is an injective map in the union of edges without endpoints in $\{a, b\}$.

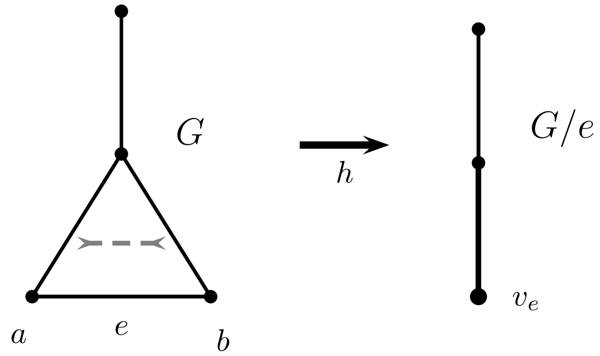


Figure 2: The map h.

Theorem 5.3 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic. Furthermore, if Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on α, β, δ.*

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ-hyperbolic, then Y is δ'-hyperbolic, where δ' is a constant which just depends on $\alpha, \beta, \delta, \varepsilon$.

Remark 5.1. *The definition of $\delta(G)$ when G is a non-connected graph gives that Theorem 5.3 also holds for non-connected graphs.*

Using the invariance of hyperbolicity (Theorem 5.3), we can obtain the following very important qualitative result.

Theorem 5.4. [19, Theorem 2.5] *Let G be a graph and $e \in E(G)$. Then G is hyperbolic if and only if G/e is hyperbolic. Furthermore, if G (respectively, G/e) is δ-hyperbolic, then G/e (respectively, G) is δ'-hyperbolic, where δ' is a constant which just depends on δ.*

Some previous results allow to obtain a quantitative version of Theorem 5.4.

Theorem 5.5. [19, Theorem 2.13] *Let G be a graph and $e \in E(G)$. Then*

$$\frac{1}{3}\,\delta(G/e) \leq \delta(G) \leq \frac{16}{3}\,\delta(G/e) + 1. \tag{1}$$

We say that and edge $e \in E(G)$ is a *cut-edge* if $G \setminus e$ is not connected.

Proposition 5.2. [19, Proposition 2.17] *Let G be a graph and e a cut-edge in G. Then*

$$\delta(G/e) = \delta(G) = \delta(G \setminus e).$$

Consider a subset $\{e_j\}_{j\in J} \subset E(G)$ with $e_j = [a_j, b_j]$ for any $j \in J$. We say that $\{e_j\}_{j\in J}$ is a *proper-removal subset* if $\Lambda(G, \{e_j\}_{j\in J}) < \infty$, where

$$\Lambda\big(G, \{e_j\}_{j\in J}\big) := \sup\Big\{ d_{G\setminus\{e_j\}_{j\in J}}(a_k, b_k) \,\Big|\, k \in J \text{ with } a_k, b_k \text{ in the same connected component of } G \setminus \{e_j\}_{j\in J} \Big\}.$$

Proposition 5.3. [19, Proposition 3.4] *Let G be a graph and $\{e_j\}_{j\in J}$ a proper-removal subset of $E(G)$. Then $G \setminus \{e_j\}_{j\in J}$ is hyperbolic if and only if G is hyperbolic.*

Consider a subset $\{e_j\}_{j\in J} \subset E(G)$ with connected components $\{K_i\}_{i\in I}$. We say that $\{e_j\}_{j\in J}$ is a *proper-contraction subset* if $\sup_{i\in I} \operatorname{diam}_G K_i < \infty$.

Proposition 5.4. [19, Proposition 3.5] *Let G be a graph and $\{e_j\}_{j\in J}$ a proper-contraction subset of $E(G)$. Then $G/\{e_j\}_{j\in J}$ is hyperbolic if and only if G is hyperbolic.*

Finally, since the hyperbolicity constant of any isolated vertex is 0, Propositions 5.3 and 5.4 give the following qualitative result.

Theorem 5.6. [19, Theorem 3.6] *Let G be a graph, G_1 a minor graph of G obtained by contracting a proper-contraction subset of $E(G)$, G_2 a minor graph of G_1 obtained by deleting a proper-removal subset of $E(G_1)$, and G' a minor graph of G_2 (and of G) obtained by deleting any amount of isolated vertices. Then G is hyperbolic if and only if G' is hyperbolic.*

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