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Rado numbers of regular nonhomogeneous equations

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Abstract

We consider Rado numbers of the regular equations $\mathcal{E}_k(b)$ of the form

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b,$$

where $b \in \mathbb{Z}$ and $c_i \in \mathbb{Z}^+$ for all i. We give universal upper bounds and qualified lower bounds for t-color Rado numbers $r(\mathcal{E}_k(b);t)$ in terms of $r(\mathcal{E}_k(0);t)$. The qualification is based on a new concept we name the *excellence condition*. We also give examples where the exact values of Rado numbers are obtained from these results.

Keywords: Ramsey theory; experimental mathematics.

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1. Introduction

Issai Schur, in the paper [10] pulished in 1916, showed that for any t colors, $t \geq 1$, there is a least positive integer $s(t)$ such that for any t-coloring on the interval $[1, s(t)]$, there must be a monochromatic solution to $x + y = z$ where x, y and z are positions on the interval. This result is part of Ramsey Theory. The numbers $s(t)$ are called *Schur numbers*. For example $s(2) = 5$ and the longest possible interval that avoids the monochromatic solution to $x + y = z$ is $[1, 2, 2, 1]$ (1 represents red color and 2 represents blue color, for example). For 3 colors, $s(3) = 14$ and one of the longest interval that avoids the monchromatic solution to $x + y = z$ is $[1, 2, 2, 1, 3, 3, 3, 3, 3, 1, 2, 2, 1]$. It is also known that $s(4) = 45$ and $s(5) = 161$.

We give definitions that relate to the examples of Schur.

Definition 1.1. *We say an equation $\mathcal{E}$ is t-regular if a number analogous to $s(t)$ exists for a given t and regular if these numbers exist for all t, $t \geq 1$.*

We also see that $[1, 2, 2, 1]$ and $[1, 2, 2, 1, 3, 3, 3, 3, 3, 1, 2, 2, 1]$ are "good colorings" for 2-colorings and 3-colorings to $x+y = z$.

Definition 1.2. *A coloring χ on an interval I is good if it contains no monochromatic solution to equation $\mathcal{E}$.*

Later on, Richard Rado, a Ph.D. student of Schur, generalized Schur's work to a linear homogeneous equation

$$\sum_{i=1}^{k} c_i x_i = 0$$

and found the condition for regularity of these equations; see the references [4, 5].

Theorem 1.1 (Radó's Single Equation Theorem)**.** *Let $k \geq 2$. Let $c_i \in \mathbb{Z} - \{0\}, 1 \leq i \leq k$, be constants. Then*

$$\sum_{i=1}^{k} c_i x_i = 0$$

is regular if and only if there exists a nonempty set $D \subseteq \{c_i,\ 1 \leq i \leq k\}$ such that $\sum_{d \in D} d = 0$.

As with Schur numbers, for a linear equation $\mathcal{E}$, we denote by $r(\mathcal{E};t)$ the minimal integers, if it exists, such that any t-coloring of $[1, r(\mathcal{E};t)]$ must admit a monochromatic solution to $\mathcal{E}$. The numbers $r(\mathcal{E};t)$ are called *t-color Rado numbers for equation $\mathcal{E}$*.

An analog to Rado's Theorem which gives the regularity condition for a linear non-homogeneous equation is given below.

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Theorem 1.2. *Let $k \geq 2$ and let $b, c_1, c_2, \ldots, c_k$ be nonzero integers. Let $\mathcal{E}(b)$ be the equation*

$$\sum_{i=1}^{k} c_i x_i = b,$$

and let $s = \sum_{i=1}^{k} c_i$. Then $\mathcal{E}(b)$ is regular if and only if one of the following conditions holds:

(i) $\dfrac{b}{s} \in \mathbb{Z}^+$;

(ii) $\dfrac{b}{s}$ *is a negative integer and $\mathcal{E}(0)$ is regular.*

We note that it is possible that an equation does not have a monochromatic solution for a coloring on $\mathbb{Z}^+$. For example, the coloring $[1, 2, 1, 2, 1, 2, \ldots]$ avoids the monochromatic solution to the equation $x + y = 2b + 1$ for any $b \geq 0$. Also some equations are t-regular but not regular. For example, $3x + y - z = 2$ is 2-regular with $r(\mathcal{E}; 2) = 8$ but not regular according to Theorem 1.2.

In this paper, we partially quantify Theorem 1.2 by giving Rado numbers to equations $\mathcal{E}(\tilde{b})$ of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_{k-1} x_{k-1} = x_k + \tilde{b}, \tag{1}$$

where $c_i \in \mathbb{Z}^+$ for all i and $\tilde{b}$ satisfies the condition (i) or (ii) of Theorem 1.2. The Rado numbers of (1) will be written in terms of the Rado numbers of the corresponding homogeneous equation, $\mathcal{E}(0)$.

In order to distinguish the Rado numbers of the homogeneous equation from those of the non-homogeneous one, we denote by

$$R_C(t) = R_{[c_1, c_2, \ldots, c_{k-1}]}(t) := r(\mathcal{E}(0); t)$$

the Rado number of the homogeneous equation, $\mathcal{E}(0)$, with t colors.

For convenient, we restate (1) formally and will use these notations for the rest of the paper.

Important Notations. For $\tilde{b}$ an integer, we let $\mathcal{E}_k(\tilde{b})$ represent the equation

$$\sum_{i=1}^{k-1} c_i x_i = x_k + \tilde{b}, \quad c_i \in \mathbb{Z}^+ \text{ for all } i$$

and let $\mathcal{F}_k(\tilde{b})$ represent the equation

$$\sum_{i=1}^{k-1} x_i = x_k + \tilde{b}.$$

We also reserve s: we always take $s = \sum_{i=1}^{k-1} c_i - 1$. Hence for $\mathcal{F}_k(\tilde{b})$, $s = k - 1 - 1 = k - 2$.

2. Main results; case $\tilde{b} < 0$

We consider the Rado numbers of $\mathcal{E}_k(\tilde{b})$ where the constant $\tilde{b}$ is negative. Theorem 2.1 gives the upper bounds and Theorem 2.2 gives qualified lower bounds.

Theorem 2.1. *Consider equation $\mathcal{E}_k(-b)$ where $b > 0$. If $s|b$ and $\mathcal{E}_k(0)$ is t-regular then*

$$r(\mathcal{E}_k(-b); t) \leq \left(\frac{b}{s} + 1\right) \cdot R_C(t) - \frac{b}{s}.$$

Proof. Assume $s|b$ and $\mathcal{E}_k(0)$ is t-regular. Let $r = (\frac{b}{s} + 1) \cdot R_C(t) - \frac{b}{s}$.

The proof is based on the fact that the dilation and appropriate translation preserves the equation. Define an injective map f from $[1, R_C(t)]$ to $[1, r]$ by

$$f(w) = \left(\frac{b}{s} + 1\right) \cdot w - \frac{b}{s}.$$

Notice that the k-tuple $(w_1, w_2, \ldots, w_{k-1}, \sum_{i=1}^{k-1} c_i w_i)$ of $\mathcal{E}_k(0)$ is made to correspond to the k-tuple

$$\left(f(w_1), f(w_2), \ldots, f(w_{k-1}), f\left(\sum_{i=1}^{k-1} c_i w_i\right)\right)$$

of $\mathcal{E}_k(-b)$. Now given any coloring α on $[1, r]$, we define the coloring χ on the interval $[1, R_C(t)]$ by

$$\chi(w) := \alpha(f(w)), \quad w = 1, 2, \ldots, R_C(t).$$

From the definition of the Rado number, any coloring on $[1, R_C(t)]$ must contain a monochromatic tuple to $\mathcal{E}_k(0)$. Hence there is also a monochromatic tuple on $[1, r]$ to $\mathcal{E}_k(-b)$. $\square$

Next we define a sufficient condition for the qualified lower bounds.

Definition 2.1 (excellence condition)**.** *The coloring on an interval* $[1,n]$ *satisfies the excellence condition if it does not contain any monochromatic solution to*

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k,$$

for each $j, \quad 0 \le j \le s$.

Theorem 2.2. *Consider equation* $\mathcal{E}_k(-b)$ *where* $b > 0$. *If* $s|b$ *and there is a* t*-coloring on the interval* $[1,n]$ *which satisfies the excellence condition then*

$$r(\mathcal{E}_k(-b);t) \ge \left(\frac{b}{s}+1\right)\cdot n + 1.$$

Proof. Assume $s|b$ and let χ be the coloring on $[1,n]$ that satisfies the excellence condition to the equation

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k, \quad 0 \le j \le s.$$

Let $r = (\frac{b}{s}+1)\cdot n + 1$. We show that there is a good coloring to $\mathcal{E}_k(-b)$ on the interval $[1, r-1] = [1, (\frac{b}{s}+1)\cdot n]$. We define the coloring α on $[1, (\frac{b}{s}+1)\cdot n]$ by

$$\alpha(i) = \chi\left(\left\lceil \frac{i}{\frac{b}{s}+1} \right\rceil\right).$$

Basically, we create the coloring by repeating each point of the original coloring on the $[1,n]$ interval $\frac{b}{s}+1$ times. We now prove the statement by contradiction:

Assume there is a monochromatic k-tuple on $[1, (\frac{b}{s}+1)\cdot n]$ to $\mathcal{E}_k(-b)$ written in the form

$$\left(d_1\left(\frac{b}{s}+1\right) - e_1, d_2\left(\frac{b}{s}+1\right) - e_2, \ldots, d_{k-1}\left(\frac{b}{s}+1\right) - e_{k-1}, \left(\frac{b}{s}+1\right)\cdot\sum_{i=1}^{k-1} c_id_i - \sum_{i=1}^{k-1} c_ie_i + b\right),$$

where $1 \le d_i \le n$ for all i and $0 \le e_i \le b/s$.

Notice that $\alpha(d_i(\frac{b}{s}+1) - e_i) = \chi(d_i)$. However, by this mapping, we have the monochromatic k-tuple in χ as

$$\left(d_1, d_2, \ldots, d_{k-1}, \sum_{i=1}^{k-1} c_id_i + \left\lceil \frac{b - \sum_{i=1}^{k-1} c_ie_i}{\dfrac{b}{s}+1} \right\rceil\right).$$

But this is a monochromatic solution to

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k,$$

for some $j, \quad 0 \le j \le \left\lceil \frac{sb}{b+s} \right\rceil$ which contradicts the excellence condition of χ we assumed it to have. $\square$

We note that the upper bounds and lower bounds meet if there is a good coloring of length $n = R_C(t) - 1$ that satisfies the excellence condition. These turn out to be the cases for the following two corollaries.

Corollary 2.1. *Consider the equation* $\mathcal{F}_k(-b)$, *with* $k \ge 2, \quad b > 0$ *and* $(k-2)|b$. *Then*

$$r(\mathcal{F}_k(-b);2) = (k+1)(b+k-2) + 1.$$

Proof. It is known from from the paper [1] that

$$r(x_1 + x_2 + \cdots + x_{k-1} = x_k; 2) = k^2 - k - 1, \qquad \text{for } k \ge 2.$$

The coloring $\chi = [1^{k-2}, 2^{(k-1)(k-2)}, 1^{k-2}]$ satisfies the excellence condition for each k. The result follows from Theorems 2.1 and 2.2. $\square$

This result agrees with Theorems 9.14 and 9.26 of the reference [3] and the main result of the paper [9] which applies to any 2-coloring but for a more general b (not only $(k-2)|b$). The next result, we apply our theorems to 3-coloring.

Corollary 2.2. *For* $m > 0$,

$$\begin{aligned}
r(x + y - z = -m; 3) &= 13m + 14,\\
r(x + y + z - w = -2m; 3) &= 42m + 43,\\
r(x_1 + x_2 + x_3 + x_4 - x_5 = -3m; 3) &= 93m + 94,\\
r(x_1 + x_2 + x_3 + x_4 + x_5 - x_6 = -4m; 3) &= 172m + 173.
\end{aligned}$$

The first result was also mentioned in the papers [6, 7]. The rest are new. The good colorings (that also satisfy the excellence condition) of the first two equations can be found by the accompanying program `Schaal` (The coloring at the end of the first paragraph of section 1 works too). The good colorings (that also satisfy the excellence condition) of the equations $x_1 + x_2 + x_3 + x_4 = x_5$ and $x_1 + x_2 + x_3 + x_4 + x_5 = x_6$ were given in the reference [8].

3. Main results; case $\tilde{b} > 0$

We consider the Rado numbers of $\mathcal{E}_k(\tilde{b})$ where the constant $\tilde{b}$ is positive. Theorem 3.1 gives the upper bounds and Theorem 3.2 gives qualified lower bounds.

Theorem 3.1. *Consider equation $\mathcal{E}_k(b)$ where $b > 0$. If $s|b$ and $\mathcal{E}_k(0)$ is t-regular then*

$$r(\mathcal{E}_k(b); t) \leq \frac{b}{s} - \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil + 1.$$

Proof. Assume $s|b$ and $\mathcal{E}_k(0)$ is t-regular. Let

$$r = \frac{b}{s} - \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil + 1.$$

We write b as $b = s\,(R_C(t) \cdot m - q)$ where

$$m = \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil$$

and $0 \leq q \leq R_C(t) - 1$. Then $r = (R_C(t) - 1) \cdot m - q + 1$.

We show that there is no good coloring on the interval $[1, r]$.

Case 1: $m = 1$.
Then $r = R_C(t) - q = b/s$. We have a trivial monochromatic solution to $\mathcal{E}_k(b)$ via $x_1 = x_2 = x_3 = \cdots = x_k = r$.

Case 2: $m > 1$.
Define an injective map f from $[1, R_C(t)]$ to $[1, r]$ by

$$f(w) = (R_C(t) - w) \cdot m - q + w.$$

Notice that a k-tuple for $\mathcal{E}_k(0)$ is made to correspond to the k-tuple of $\mathcal{E}_k(b)$.

Now, given any coloring of α on $[1, r]$, we define the coloring χ on the interval $[1, R_C(t)]$ by

$$\chi(w) = \alpha(f(w)), \quad w = 1, 2, \ldots, R_C(t).$$

From the definition of the Rado number, any coloring on $[1, R_C(t)]$ must contain a monochromatic tuple to $\mathcal{E}_k(0)$. Hence there is also a monochromatic tuple on $[1, r]$ to $\mathcal{E}_k(b)$. In both cases, there is no good coloring on $[1, r]$. □

The lower bounds can be stated in similar way to Theorem 2.2.

Theorem 3.2. *Consider equation $\mathcal{E}_k(b)$ where $b > 0$. If $s|b$ and there is a t-coloring on the interval $[1, n]$ which satisfies the excellence condition then*

$$r(\mathcal{E}_k(b); t) \geq \frac{b}{s} - \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil + 1.$$

Proof. We invoke the result of Theorem 2.2. Since $s|b$, we can write b in the form $b = s\,[(n+1)m - q]$ where

$$m = \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil$$

and $0 \leq q \leq n$. Then

$$r = \frac{b}{s} - \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil + 1 = (n+1)m - q - m + 1 = nm - q + 1.$$

We show that there is a good coloring on the interval $[1, r-1] = [1, nm - q]$ to $\mathcal{E}_k(b)$.

First we rewrite $\mathcal{E}_k(b)$ as

$$c_1 x_1 + c_2 x_2 + \cdots + c_{k-1} x_{k-1} = x_k + s\,[(n+1)m - q]\,.$$

We then rewrite this equation again as

$$c_1\,[(n+1)m - q - x_1] + c_2\,[(n+1)m - q - x_2] + \cdots + c_{k-1}\,[(n+1)m - q - x_{k-1}] = (n+1)m - q - x_k.$$

Next we add $-s(m-1)$ on both sides of the equation,

$$c_1\left[nm-q+1-x_1\right]+c_2\left[nm-q+1-x_2\right]+\cdots+c_{k-1}\left[nm-q+1-x_{k-1}\right]=\left[nm-q+1-x_k\right]-s(m-1).$$

We let $x_i'=nm-q+1-x_i$ for each i. The reader sees that x_i' is x_i after reversing the interval $[1,nm-q]$. The equation after substitution is

$$c_1x_1'+c_2x_2'+\cdots+c_{k-1}x_{k-1}'=x_k'-s(m-1). \tag{2}$$

The next step is clear. We invoke the result from Theorem 2.2 that there is a good coloring α on the interval $[1,mn]$ to (2). We can then make a good coloring to $\mathcal{E}_k(b)$ from this interval by taking the elements 1 to $mn-q$ of α and reverse the interval. □

Below are some applications of Theorems 3.1 and 3.2. The proofs of Corollary 3.1 and 3.2 are similar to those of Corollary 2.1 and 2.2.

Corollary 3.1. *Consider equation $\mathcal{F}_k(b)$ with $k\geq 2,\;\; b\geq 1$ and $(k-2)|b$. Then*

$$r(\mathcal{F}_k(b);2)=\frac{b}{k-2}-\left\lceil\frac{b}{(k-2)(k^2-k-1)}\right\rceil+1.$$

Corollary 3.2. *For $m\geq 1$,*

$$\begin{aligned}
r(x+y-z=m;3)&=m-\left\lceil\frac{m}{14}\right\rceil+1,\\
r(x+y+z-w=2m;3)&=m-\left\lceil\frac{m}{43}\right\rceil+1,\\
r(x_1+x_2+x_3+x_4-x_5=3m;3)&=m-\left\lceil\frac{m}{94}\right\rceil+1,\\
r(x_1+x_2+x_3+x_4+x_5-x_6=4m;3)&=m-\left\lceil\frac{m}{173}\right\rceil+1.
\end{aligned}$$

This first result was a part of Theorem 9.15 of the paper [3]. Although it was wrongly claimed that

$$r(x+y-z=b;3)=b-\left\lceil\frac{b-1}{14}\right\rceil.$$

The rest of the results are new.

For the situation when the equation $\mathcal{E}_k(0)$ is not t-regular, the trivial bounds of the Rado numbers to $\mathcal{E}_k(b)$ where $b>0$ and $s|b$ are

$$\left\lceil\frac{b+1}{s+1}\right\rceil\leq r(\mathcal{E}_k(b);t)\leq\frac{b}{s},\quad\text{for any } t\geq 1.$$

The monochromatic solution for the upper bound arises from the tuple $(\frac{b}{s},\frac{b}{s},\dots,\frac{b}{s})$.

4. Final remarks

So far, our results were obtained by checking the excellence condition of each good coloring. We made two general conjectures based on the niceness of the excellence condition that we had experienced with. The second conjecture will be just a corollary if the first conjecture is true.

Conjecture 4.1. *For $t=2$ or 3. Consider the t-regular equation $\mathcal{E}_k(0)$ with the Rado number $R_C(t)$. There always exists a coloring of length $n=R_C(t)-1$ to $\mathcal{E}_k(0)$ that satisfies the excellence condition.*

Conjecture 4.2. *For $t=2$ or 3. Consider the equation $\mathcal{E}_k(\tilde{b})$. If $s|\tilde{b}$ and $\mathcal{E}_k(0)$ is t-regular then*

$$r(\mathcal{E}_k(\tilde{b});t)=\begin{cases}\dfrac{\tilde{b}}{s}-\left\lceil\dfrac{\tilde{b}}{s\cdot R_C(t)}\right\rceil+1, & \text{for } \tilde{b}>0,\\[2ex] \left(-\dfrac{\tilde{b}}{s}+1\right)R_C(t)+\dfrac{\tilde{b}}{s}, & \text{for } \tilde{b}<0.\end{cases}$$

For t-colorings where $t\geq 4$, our Maple program is too slow to give any tangible observations. A faster program could used to verify whether this conjecture still holds.

Lastly, the reader might wonder about the other type of equations that we did not consider, i.e.

$$\sum_{i=1}^{k-1}c_ix_i=c_kx_k+b,\quad\text{where } c_i\in\mathbb{Z}^+,\quad\text{for } 1\leq i\leq k \text{ and } c_k\geq 2.$$

It turns out that the Rado numbers of these equations exhibit more complicated patterns from those discovered in this paper.

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