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Average length of the longest increasing subsequences in random involutions avoiding 231 and a layered pattern

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Abstract

The problems of deciding the length of the longest increasing subsequence for a permutation or an involution, the expectation and the limiting distribution, surveyed in Stanley's ICM (International Congress of Mathematicians) Plenary address, have been widely studied by combinatorialists, analysts and probabilists. Partially motivated by the intriguing phenomenon stated by Simion and Schmidt [*European J. Combin.* **6** (1985) 383–406] that 231-avoiding permutations are exactly the set of layered permutations, in this paper, we investigate the limiting behavior of the average length of the longest increasing subsequences in random involutions avoiding 231 and another pattern which is a layered permutation. We obtain an explicit formula of the generating function and apply it to exemplify a set of interesting examples, which extend recent results of the first author with Yıldırım [*Turkish J. Math.* **43** (2019) 2183–2192], where the longest increasing subsequences in involutions avoiding a pair of patterns of length 3 are studied.

Keywords: longest increasing subsequence problem; pattern-avoidance; pattern-avoiding involutions; generating functions.

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1. Introduction

Let $\mathcal{S}_n$ be the set of all permutations of length n on the set $[n]=\{1,2,\cdots,n\}$. An *involution* $\pi=\pi_1\cdots\pi_n$ of length n is a permutation in $\mathcal{S}_n$ such that $\pi^2=1$, that is, $\pi_{\pi_i}=i$ for all $i\in[n]$. For $\tau=\tau_1\cdots\tau_k\in\mathcal{S}_k$ and $\pi=\pi_1\cdots\pi_n\in\mathcal{S}_n$, we say that τ *occurs* as a *pattern* in π if there exists a subset of indices $1\le i_1<i_2<\cdots<i_k\le n$ such that $\pi_{i_s}<\pi_{i_t}$ if and only if $\tau_s<\tau_t$ for all $1\le s,t\le k$. For example, the permutation 231 occurs as a pattern in 24315 because it has the subsequences $24-1-$ or $2-31-$. If τ doesn't occur as a pattern in π, then π is called a τ-*avoiding* permutation. We denote by $\mathcal{S}_n(\tau)$ (resp. $\mathcal{I}_n(\tau)$) the set of all τ-avoiding permutations (resp. involutions) of length n. More generally, for a set T of patterns, we use the notation $\mathcal{S}_n(T)=\cap_{\tau\in T}\mathcal{S}_n(\tau)$ and $\mathcal{I}_n(T)=\cap_{\tau\in T}\mathcal{I}_n(\tau)$. For more on the subject, see [4,7,8,10].

For $\pi\in S_n$, we use $L_n(\pi)$ to denote the length of a *longest increasing subsequence* in π, that is,

$$L_n(\pi)=\max\{k\in[n]:12\cdots k \text{ occurs as a pattern in } \pi\}.$$

The problem of determining the asymptotic behavior of L_n for uniformly random permutations and involutions has a long and interesting history. The asymptotic behavior of the expected value of $L_n(\pi)$ was obtained by VershikâĂŞKerov [20] and LoganâĂŞShepp [9]. The entire limiting distribution of $L_n(\pi)$ was determined by Baik, Deift, and Johansson [2], which opens up unexpected connections between increasing subsequences and random matrices. In [17], the result of [2] is listed as one of three major breakthroughs in recent algebraic combinatorics. Theory and techniques of increasing and decreasing subsequences of permutations are reviewed in Stanley's plenary address [18] at the International Congress of Mathematicians, Madrid. See also [1,3,5,6,13,15] and references therein. In specific, it is known that $\mathbf{E}(L_n)\sim 2\sqrt{n}$ [9,20] (see also [1]) and $n^{-1/6}(L_n-\mathbf{E}(L_n))$ converges in distribution to the Tracy-Widom distribution as $n\to\infty$ [2]. The Tracy-Widom distribution first appeared in the context of the random matrix theory as the limiting distribution for the rescaled largest eigenvalue for the Gaussian unitary ensemble [19].

In this paper, we study asymptotic behaviors of the longest increasing subsequence problem for some pattern-avoiding involution classes. We let $\mathbf{E}^\tau(L_n)$ be the expectation of the length of a longest increasing subsequence in $\pi\in\mathcal{I}_n(\tau)$, and more generally, $\mathbf{E}^{\tau_1,\tau_2}(L_n)$ is the expectation of the length of a longest increasing subsequence in $\pi\in\mathcal{I}_n(\tau_1,\tau_2)$ (i.e. $\mathcal{I}_n(\{\tau_1,\tau_2\})$). Motivated by Simion and Schmidt's interesting result that $\mathcal{I}_n(231)$ and $\mathcal{I}_n(312)$ are "not only equal in number, but they are equal as sets" [16], which complements previous work by Rotem [14], we shall fix τ_1 to be 231 and let τ_2 be a layered pattern

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defined in the following section. We determine the exact and asymptotic formulas for the average length of the longest increasing subsequences for such permutation classes. To some extent, our major results in this current paper extends recent results of the first author with Yıldırım [11, 12], where longest increasing subsequences in involutions avoiding a pair of patterns of length 3 are studied.

2. Main results

For a finite sequence $S = s_1 s_2 \cdots s_p$ and an integer k, we let $S+k$ denote the sequence $(s_1+k)(s_2+k)\cdots(s_p+k)$. We denote the permutation $k(k-1)\cdots 1$ by $\langle k\rangle$.

We call a permutation *layered* whenever it has the form

$$[\ell_1,\ldots,\ell_m] = (\langle\ell_1\rangle)(\langle\ell_2\rangle+\ell_1)\cdots(\langle\ell_m\rangle+\ell_1+\cdots+\ell_{m-1})$$

for some sequence $\ell_1,\ldots,\ell_m \geq 1$. Thus, $(12\cdots n) = [1,1,\ldots,1]$ and $(n\cdots 21) = [n]$. It is not difficult to decide that the number of layered permutations of length n is 2^{n-1}. The 4 layered permutation of 3 letters are 123, 132, 213 and 321; and the 8 layered permutation of 4 letters are 1234, 1243, 1324, 1432, 2134, 2143, 3214 and 4321.

As explained in the remarks following the proofs of Propositions 6 and 12 [16], the sets $\mathcal{I}_n(231)$ and $\mathcal{I}_n(312)$ are connected with layered permutations, namely, the set $\mathcal{I}_n(231) = \mathcal{I}_n(312)$ is exactly the set of layered permutations of length n, for all $n \geq 0$.

Fix $\tau = [\ell_1,\ldots,\ell_m] \in \mathcal{I}_k(231)$ to be a pattern. Let

$$A_\tau(x,q) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{I}_n(231,\tau)} q^{L_n(\pi)}x^n$$

be the bivariate generating function for the number of involutions in $\mathcal{I}_n(231,\tau)$ according to L_n and n. We discover an explicit formula for $A_\tau(x,q)$ and use it to derive the expected value of $L_n(\pi)$ for $\pi \in \mathcal{I}_n(231,\tau)$.

Theorem 2.1. *Let $A_\tau(x,q) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{I}_n(231,\tau)} q^{L_n(\pi)}x^n$. For all $m \geq 1$, we have*

$$A_{[\ell_1,\ldots,\ell_m]}(x,q) = \sum_{j=1}^{m}\frac{q^{j-1}x^{\ell_1+\ell_2+\cdots+\ell_{j-1}}}{(1-x)^{j-1}\prod_{i=1}^{j}(1-q(x+x^2+\cdots+x^{\ell_i-1}))}.$$

Proof. It is important to note that any involution π in $\mathcal{I}_n(231)$ can be presented as $\pi = (\langle j\rangle)(\pi'+j)$, where $\pi' \in \mathcal{I}_{n-j}(231)$. We discuss it according to $j < l_1$ or $j \geq l_1$. Thus we have

$$A_\tau(x,q) = 1 + q(x+x^2+\cdots+x^{\ell_1-1})A_\tau(x,q) + \frac{qx^{\ell_1}}{1-x}A_{\tau'}(x,q),$$

where $\tau' = [\ell_2,\ldots,\ell_m]$. Here, 1 counts the empty involution, $q(x+x^2+\cdots+x^{\ell_1-1})A_\tau(x,q)$ counts all involutions in the case $1 \leq j \leq \ell_1 - 1$, and

$$\frac{qx^{\ell_1}}{1-x}A_{\tau'}(x,q)$$

counts all involutions in the case $j \geq \ell_1$. This means

$$A_\tau(x,q) = \frac{1}{1-q(x+x^2+\cdots+x^{\ell_1-1})} + \frac{qx^{\ell_1}}{(1-x)(1-q(x+x^2+\cdots+x^{\ell_1-1}))}A_{\tau'}(x,q).$$

By induction on m, we get

$$A_{[\ell_1,\ldots,\ell_m]}(x,q) = \sum_{j=1}^{m}\frac{q^{j-1}x^{\ell_1+\ell_2+\cdots+\ell_{j-1}}}{(1-x)^{j-1}\prod_{i=1}^{j}(1-q(x+x^2+\cdots+x^{\ell_i-1}))}.$$

□

Based on Theorem 2.1 and the definition of $A_\tau(x,q)$, we may find $\mathbf{E}^{231,\tau}(L_n)$ by the following formula.

$$\mathbf{E}^{231,\tau}(L_n) = \frac{[x^n]\frac{\partial}{\partial q}A_\tau(x,q)\mid_{q=1}}{[x^n]A_\tau(x,1)}, \tag{1}$$

where $[x^n]h(x)$ represents the coefficient of x^n in the series $h(x)$ in powers of x [21, p. 7].

The first two examples are the following. As in usual convention, "$f(n) \sim g(n)$" means that

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1.$$

Corollary 2.1. *For all $k \geq 3$, we have*

$$\mathbf{E}^{231,12\cdots k}(L_n) \sim k-1.$$

Proof. By (1), we get

$$\mathbf{E}^{231,12\cdots k}(L_n) = \frac{[x^n]\frac{\partial}{\partial q}A_{[1,\ldots,1]}(x,q)\mid_{q=1}}{[x^n]A_{[1,\ldots,1]}(x,1)} = \frac{\sum_{i=1}^{k-1} i\binom{n-1}{i-1}}{\sum_{i=1}^{k-1}\binom{n-1}{i-1}} \sim k-1.$$

□

Corollary 2.2. *For all $k \geq 3$, we have*

$$\mathbf{E}^{231,2134\cdots k}(L_n) \sim \frac{n}{k-1}.$$

Proof. By (1), we have

$$\mathbf{E}^{231,2134\cdots k}(L_n) = \frac{[x^n]\frac{\partial}{\partial q}A_{[2,1,\ldots,1]}(x,q)\mid_{q=1}}{[x^n]A_{[2,1,\ldots,1]}(x,1)} = \frac{n+\sum_{i=2}^{k-1}((i-1)\binom{n}{i}-(i-2)\binom{n-1}{i})}{\sum_{i=1}^{k-1}\binom{n-1}{i-1}} \sim \frac{n}{k-1}.$$

□

As in the above corollaries we can obtain the asymptotic behaviors of $E^{231,\tau}(L_n)$ for other general patterns $\tau \in S_k$. For instance, $\tau = 132456\ldots k$, or $\tau = 321456\ldots k$. For applications of Theorem 2.1 on layered patterns of length $n \leq 5$, we state the results below but skipping the details of the calculations.

By Theorem 2.1 with $\tau \in \mathcal{I}_k(231)$ where $k = 3, 4$, we obtain the asymptotic behaviors of $\mathbf{E}^{231,\tau}(L_n)$.

$$\mathbf{E}^{231,123}(L_n) \sim 2, \qquad \mathbf{E}^{231,321}(L_n) \sim \frac{(1+\sqrt{5})n}{2\sqrt{5}}, \qquad \mathbf{E}^{231,132}(L_n) \sim \mathbf{E}^{231,213}(L_n) \sim \frac{n}{2},$$

and

$$\begin{aligned}
&\mathbf{E}^{231,1234}(L_n) \sim 3, && \mathbf{E}^{231,2143}(L_n) \sim \frac{2n}{3},\\
&\mathbf{E}^{231,1432}(L_n) \sim \mathbf{E}^{231,3214}(L_n) \sim \frac{(1+\sqrt{5})n}{2\sqrt{5}}, && \mathbf{E}^{231,4321}(L_n) \sim \frac{(1-a^3)n}{2-4a^3},\\
&\mathbf{E}^{231,1243}(L_n) \sim \mathbf{E}^{231,1324}(L_n) \sim \mathbf{E}^{231,2134}(L_n) \sim \frac{n}{3},
\end{aligned}$$

where a is the smallest positive root of the polynomial x^3+x^2+x-1. Moreover, applying Theorem 2.1 with $\tau \in \mathcal{I}_5(231)$ gives

$$\begin{aligned}
&\mathbf{E}^{231,12345}(L_n) \sim 4,\\
&\mathbf{E}^{231,13254}(L_n) \sim \mathbf{E}^{231,21354}(L_n) \sim \mathbf{E}^{231,21435}(L_n) \sim \frac{n}{2},\\
&\mathbf{E}^{231,12354}(L_n) \sim \mathbf{E}^{231,12435}(L_n) \sim \mathbf{E}^{231,13245}(L_n) \sim \mathbf{E}^{231,21345}(L_n) \sim \frac{n}{4},\\
&\mathbf{E}^{231,12543}(L_n) \sim \mathbf{E}^{231,14325}(L_n) \sim \mathbf{E}^{231,21543}(L_n) \sim \mathbf{E}^{231,32145}(L_n) \sim \mathbf{E}^{231,32154}(L_n) \sim \frac{(1+\sqrt{5})n}{2\sqrt{5}},\\
&\mathbf{E}^{231,15432}(L_n) \sim \mathbf{E}^{231,43215}(L_n) \sim \frac{(1-a^3)n}{2-4a^3},\\
&\mathbf{E}^{231,54321}(L_n) \sim \frac{(1-b^4)n}{2-5b^4},
\end{aligned}$$

where a and b are the smallest positive roots of the polynomials x^3+x^2+x-1 and $x^4+x^3+x^2+x-1$, respectively.

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