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On complementary equienergetic strongly regular graphs

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Abstract

The energy of a graph is the sum of absolute values of the eigenvalues of its adjacency matrix. Two graphs are said to be equienergetic if they have same energy. A graph is said to be complementary equienergetic if it is equienergetic with its complement. In this paper, we characterize the strongly regular graphs, which are complementary equienergetic. In addition, by means of Cartesian and strong graph products, we construct equienergetic graphs using complementary equienergetic graphs.

Keywords: energy of a graph; equienergetic graphs; complementary equienergetic graphs; strongly regular graphs.

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1. Introduction

The energy of a graph is the sum of absolute values of the eigenvalues of its $(0, 1)$-adjacency matrix [10, 17]. It is closely related to the total π-electron energy of a molecule, calculated with the Hückel molecular orbital method [11, 12]. More results on graph energy can be found in [17, 19]. Two graphs are said to be equienergetic if they have same energies. Numerous results on equienergetic graphs have been reported [1, 3–5, 8, 15, 16, 18, 21, 22, 24–26]. Recently, Ramane et al. [20] designed graphs that are equienergetic with their complements. Such graphs are referred to as complementary equienergetic [2]. Ali et al. [2] determined all complementary equienergetic graphs with at most 10 vertices. They also found all graphs of order at most ten, whose line graphs are complementary equienergetic. In this paper we characterize the complementary equienergetic strongly regular graphs. In addition, by means of Cartesian and strong graph products, we construct equienergetic graphs using complementary equienergetic species.

2. Preliminaries

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. A graph is r-regular if all its vertices have the same degree equal to r. The complement of a graph G is the graph $\overline{G}$ with vertex set $V(\overline{G}) = V(G)$ in which two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph G is self–complementary if it is isomorphic to its complement, $G \cong \overline{G}$. As usual, we denote by K_n and C_n the complete graph and the cycle on n vertices, and by $K_{p,q}$ be the complete bipartite graph on $n = p + q$ vertices [13].

The adjacency matrix of a graph G is the $n \times n$ matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ labeled as $x_1 \geq x_2 \geq \cdots \geq x_n$ are said to be the eigenvalues of the graph G, forming the spectrum of G [7]. If $x_1, x_2, \ldots, x_k$ are the distinct eigenvalues of G with respective multiplicities $m_1, m_2, \ldots, m_k$, then the spectrum of G is

$$\text{Spec}(G) = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix},$$

where $m_1 + m_2 + \cdots + m_k = n$. Two graphs G_1 and G_2 are said to be cospectral if $\text{Spec}(G_1) = \text{Spec}(G_2)$.

The energy of the graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of the eigenvalues of G [10], that is,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |x_i| .$$

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Two graphs G_1 and G_2 of the same order are said to be equienergetic if $\mathcal{E}(G_1) = \mathcal{E}(G_2)$. A non-self–complementary graph G is said to be complementary equienergetic if $\mathcal{E}(G) = \mathcal{E}(\overline{G})$. In other words, two non-self–complementary equienergetic graphs G_1 and G_2 satisfying $G_1 \cong \overline{G_2}$ are complementary equienergetic.

Lemma 2.1. [7] *Let G be an r-regular graph of order n with the eigenvalues $r, x_2, \dots, x_n$. Then the eigenvalues of $\overline{G}$ are $n-r-1, -x_2-1, \dots, -x_n-1$.*

3. Strongly regular complementary equienergetic graphs

A strongly regular graph with parameters (n, r, λ, μ) is an r-regular graph ($0 < r < n-1$) on n vertices in which any two adjacent vertices have exactly λ common neighbors and any two non-adjacent vertices have exactly μ common neighbors. If G is a strongly regular graph with parameters (n, r, λ, μ), then its complement $\overline{G}$ is also strongly regular graph with parameters $(n, n-r-1, n-2r+\mu-2, n-2r+\lambda)$. A strongly regular graph has only three distinct eigenvalues [7].

Theorem 3.1. [7] *If G is a strongly regular graph with parameters (n, r, λ, μ), then*

$$\text{Spec}(G) = \begin{pmatrix} r & \frac{1}{2}(\lambda-\mu+t) & \frac{1}{2}(\lambda-\mu-t) \\ 1 & \frac{1}{2}(n-1-\frac{\Delta}{t}) & \frac{1}{2}(n-1+\frac{\Delta}{t}) \end{pmatrix}, \tag{1}$$

where $t = \sqrt{(\lambda-\mu)^2 + 4(r-\mu)}$ and $\Delta = 2r + (n-1)(\lambda-\mu)$.

Theorem 3.2. *A strongly regular graph with parameters (n, r, λ, μ) is complementary equienergetic if and only if $n-2r-1 = \frac{\Delta}{t}$, where $t = \sqrt{(\lambda-\mu)^2 + 4(r-\mu)}$ and $\Delta = 2r + (n-1)(\lambda-\mu)$.*

Proof. Let G be a strongly regular graph with parameters (n, r, λ, μ). Then by Equation (1),

$$\begin{aligned} \mathcal{E}(G) &= |r| + \frac{1}{4}\left(n-1-\frac{\Delta}{t}\right)|\lambda-\mu+t| + \frac{1}{4}\left(n-1+\frac{\Delta}{t}\right)|\lambda-\mu-t| \\ &= r + \frac{1}{4}\left(n-1-\frac{\Delta}{t}\right)(\lambda-\mu+t) + \frac{1}{4}\left(n-1+\frac{\Delta}{t}\right)(t-\lambda+\mu). \end{aligned} \tag{2}$$

By Lemma 2.1, the spectrum of $\overline{G}$ is

$$\text{Spec}(\overline{G}) = \begin{pmatrix} n-r-1 & -1-\frac{1}{2}(\lambda-\mu+t) & -1-\frac{1}{2}(\lambda-\mu-t) \\ 1 & \frac{1}{2}(n-1-\frac{\Delta}{t}) & \frac{1}{2}(n-1+\frac{\Delta}{t}) \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \mathcal{E}(\overline{G}) &= |n-r-1| + \frac{1}{2}\left(n-1-\frac{\Delta}{t}\right)\left|-1-\frac{1}{2}(\lambda-\mu+t)\right| \\ &\quad + \frac{1}{2}\left(n-1+\frac{\Delta}{t}\right)\left|-1-\frac{1}{2}(\lambda-\mu-t)\right| \\ &= n-r-1 + \frac{1}{2}\left(n-1-\frac{\Delta}{t}\right)\left(1+\frac{1}{2}(\lambda-\mu+t)\right) \\ &\quad + \frac{1}{2}\left(n-1+\frac{\Delta}{t}\right)\left(-1-\frac{1}{2}(\lambda-\mu-t)\right). \end{aligned} \tag{3}$$

Since G is complementary equienergetic, $\mathcal{E}(G) = \mathcal{E}(\overline{G})$. Therefore, by equating Equations (2) and (3) we get $n-2r-1 = \frac{\Delta}{t}$.

Conversely, suppose that $n-2r-1 = \frac{\Delta}{t}$ is true. Then we need to prove that $\mathcal{E}(G) = \mathcal{E}(\overline{G})$. Suppose that $\mathcal{E}(G) \neq \mathcal{E}(\overline{G})$, which implies that $n-2r-1 \neq \frac{\Delta}{t}$, a contradiction. Thus, $\mathcal{E}(G) = \mathcal{E}(\overline{G})$. □

Theorem 3.3. *Let G be a strongly regular graph of order n with regularity $r = (n-1)/2$, which implies that n must be odd. Then G is complementary equienergetic if and only if G and $\overline{G}$ are cospectral.*

Proof. Let G be a strongly regular graph with parameters (n, r, λ, μ), where $r = (n-1)/2$. By Equation (1) and by Lemma 2.1,

$$\text{Spec}(\overline{G}) = \begin{pmatrix} n-r-1 & -1-\frac{1}{2}(\lambda-\mu+t) & -1-\frac{1}{2}(\lambda-\mu-t) \\ 1 & \frac{1}{2}(n-1-\frac{\Delta}{t}) & \frac{1}{2}(n-1+\frac{\Delta}{t}) \end{pmatrix}, \tag{4}$$

where

$$t = \sqrt{(\lambda-\mu)^2+4(r-\mu)} \quad \text{and} \quad \Delta = 2r+(n-1)(\lambda-\mu)\,. \tag{5}$$

If G is complementary equienergetic, then by Theorem 3.2, $n-2r-1=\frac{\Delta}{t}$. Further, as $r=(n-1)/2$, so $\Delta/t=0$, implies $\Delta=0$. Therefore, by Equation (5), $\lambda-\mu=-1$. Substituting these values in Equations (1) and (4), we get

$$\mathrm{Spec}(G) = \begin{pmatrix} \frac{n-1}{2} & \frac{-1+t}{2} & \frac{-1-t}{2} \\ 1 & \frac{n-1}{2} & \frac{n-1}{2} \end{pmatrix} \qquad \text{and} \qquad \mathrm{Spec}(\overline{G}) = \begin{pmatrix} \frac{n-1}{2} & \frac{-1+t}{2} & \frac{-1-t}{2} \\ 1 & \frac{n-1}{2} & \frac{n-1}{2} \end{pmatrix}.$$

Hence G and $\overline{G}$ are co-spectral.

The converse is obvious. □

An orthogonal array $OA(m,n)$ is an $m \times n^2$ array with entries from $\{1,2,\ldots,n\}$ with the property that the columns of any $2\times n^2$ subarray consist of all n^2 possible pairs.

The block graph denoted by $X_{OA(m,n)}$ of an orthogonal array $OA(m,n)$ is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row where they have the same entry [9]. It is also called as pseudo–net graph or pseudo–Latin square graph [14]. Orthogonal arrays have applications in combinatorial designs, particularly in testing designs [6, 14].

Theorem 3.4. [9] *If $OA(m,n)$ is an orthogonal array with $m<n+1$, then its block graph $X_{OA(m,n)}$ is strongly regular with parameters*

$$\Big(n^2, m(n-1), (m-1)(m-2)+n-2, m(m-1)\Big)\,. \tag{6}$$

Theorem 3.5. *The orthogonal array block graph $G=X_{OA(m,n)}$, with $m<n+1$, is complementary equienergetic.*

Proof. By Theorem 3.4, the orthogonal array block graph $X_{OA(m,n)}$ is a strongly regular graph with parameters (6), where $m<n+1$. Recall that the number of its vertices is n^2. These parameters satisfies the condition of Theorem 3.2. Hence the result follows. □

Remark 3.1. *Some of the strongly regular graphs, considered in [20], are special cases of the block graph $G=X_{OA(m,n)}$ of an orthogonal array. These are:*

(i) the strongly regular graph with parameters $(n^2, 2n-2, n-2, 2)$, which is a block graph $X_{OA(2,n)}$ (isomorphic to the line graph of $K_{n,n}$);

(ii) the strongly regular graph with parameters $(n^2, 3n-3, n, 6)$, which is a block graph $X_{OA(3,n)}$;

(iii) the strongly regular graph with parameters $(n^2, n^2-n, n^2-2n, n^2-n)$, which is a block graph $X_{OA(n,n)}$ (isomorphic to $\overline{nK_n}$);

(iv) the strongly regular graph with parameters $(4n^2, 2n^2-n, n^2-n, n^2-n)$, which is a block graph $X_{OA(n,2n)}$.

All graphs mentioned in Remark 3.1 satisfy the relation $\mathcal{E}(G)=\mathcal{E}(\overline{G})$.

4. Equienergetic graphs

Equienergetic graphs constructed by using Cartesian product, tensor product and join are reported in [4, 20, 21]. Here we design non-cospectral equienergetic graphs using complementary equienergetic graphs through Cartesian and strong products.

The Cartesian product of two graphs G_1 and G_2 is the graph $G_1 \square G_2$ with vertex set $V(G_1)\times V(G_2)$, in which the vertices (u_1,u_2) and (v_1,v_2) are adjacent if either u_1 is adjacent to v_1 in G_1 and $u_2=v_2$ or $u_1=v_1$ and u_2 is adjacent to v_2 in G_2.

The strong product of two graphs G_1 and G_2 is the graph $G_1 \boxtimes G_2$ with vertex set $V(G_1)\times V(G_2)$, in which the vertices (u_1,u_2) and (v_1,v_2) are adjacent whenever u_1 and v_1 are equal or adjacent in G_1, and u_2 and v_2 are equal or adjacent in G_2.

Lemma 4.1. [7] *If $x_1,x_2,\ldots,x_n$ are the eigenvalues of G_1 and $y_1,y_2,\ldots,y_m$ are the eigenvalues of G_2, then*
(i) the eigenvalues of $G_1\square G_2$ are x_i+y_j, $i=1,2,\ldots,n$; $j=1,2,\ldots,m$;
(ii) the eigenvalues of $G_1\boxtimes G_2$ are $(x_i+1)(y_j+1)-1$, $i=1,2,\ldots,n$; $j=1,2,\ldots,m$.

The join of two graphs G_1 and G_2 is the graph G_1+G_2 obtained from G_1 and G_2 by joining every vertex of G_1 to every vertex of G_2.

Theorem 4.1. [21] *If G_i is an r_i-regular graph of order n_i, $i=1,2$, then*

$$\mathcal{E}(G_1+G_2)=\mathcal{E}(G_1)+\mathcal{E}(G_2)+\sqrt{(r_1+r_2)^2+4(n_1\,n_2-r_1\,r_2)}-(r_1+r_2)\,.$$

Theorem 4.2. *Let G be an r-regular complementary equienergetic graph of order n. Then $\mathcal{E}\left(\overline{G}\boxtimes\overline{C_4}\right)=\mathcal{E}\left(\overline{\overline{G}\boxtimes\overline{C_4}}\right)$.*

Proof. Let

$$\text{Spec}(G)=\begin{pmatrix} r & \lambda_2 & \cdots & \lambda_k \\ 1 & m_2 & \cdots & m_k \end{pmatrix}, \tag{7}$$

where $1+\sum\limits_{i=2}^{k} m_i=n$. Therefore, by Lemma 2.1,

$$\text{Spec}\left(\overline{G}\right)=\begin{pmatrix} n-r-1 & -\lambda_2-1 & \cdots & -\lambda_k-1 \\ 1 & m_2 & \cdots & m_k \end{pmatrix}$$

and

$$\text{Spec}\left(\overline{C_4}\right)=\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

By Lemma 4.1,

$$\text{Spec}\left(\overline{G}\boxtimes\overline{C_4}\right)=\begin{pmatrix} 2n-2r-1 & -1-2\lambda_2 & \cdots & -1-2\lambda_k & -1 \\ 2 & 2m_2 & \cdots & 2m_k & 2n \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \mathcal{E}\left(\overline{G}\boxtimes\overline{C_4}\right) &= 2|2n-2r-1|+\sum_{i=2}^{k} 2m_i\,|-1-2\lambda_i|+2n|-1| \\ &= 6n-4r-2+2\sum_{i=2}^{k} m_i\,|1+2\lambda_i|. \end{aligned} \tag{8}$$

Since G is complementary equienergetic, $\mathcal{E}(G)=\mathcal{E}(\overline{G})$. Therefore,

$$r+\sum_{i=2}^{k} m_i\,|\lambda_i|=n-r-1+\sum_{i=2}^{k} m_i\,|1+\lambda_i|,$$

which yields

$$\sum_{i=2}^{k} m_i\,|1+\lambda_i|=-n+2r+1+\sum_{i=2}^{k} m_i\,|\lambda_i|\,. \tag{9}$$

Replacing λ_i by $2\lambda_i$ in Equation (9) we get

$$\begin{aligned} \sum_{i=2}^{k} m_i\,|1+2\lambda_i| &= -n+2r+1+\sum_{i=2}^{k} m_i\,|2\lambda_i| \\ &= -n+2r+1+2[\mathcal{E}(G)-r]=-n+1+2\,\mathcal{E}(G)\,. \end{aligned} \tag{10}$$

Substituting Equation (10) in Equation (8) results in

$$\mathcal{E}\left(\overline{G}\boxtimes\overline{C_4}\right)=4n-4r+4\,\mathcal{E}(G)\,.$$

The graph $\overline{G}\boxtimes\overline{C_4}$ is regular on $4n$ vertices with regularity $2n-2r-1$. Therefore, applying Lemma 2.1 to the spectrum of $\overline{G}\boxtimes\overline{C_4}$, we arrive at

$$\text{Spec}\left(\overline{\overline{G}\boxtimes\overline{C_4}}\right)=\begin{pmatrix} 2n+2r & -2n+2r & 2\lambda_2 & \cdots & 2\lambda_k & 0 \\ 1 & 1 & 2m_2 & \cdots & 2m_k & 2n \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \mathcal{E}\left(\overline{\overline{G}\boxtimes\overline{C_4}}\right) &= |2n+2r|+|-2n+2r|+\sum_{i=2}^{k} 2m_i\,|2\lambda_i|+2n|0| \\ &= 4n+4(\mathcal{E}(G)-r)=4n-4r+4\,\mathcal{E}(G)\,. \end{aligned}$$

Thus, $\mathcal{E}\left(\overline{G}\boxtimes\overline{C_4}\right)=\mathcal{E}\left(\overline{\overline{G}\boxtimes\overline{C_4}}\right)$. □

Theorem 4.3. *Let G be an r-regular complementary equienergetic graph of order n. Then $\mathcal{E}\left(\overline{G \boxtimes K_p}\right) = \mathcal{E}\left(\overline{\overline{G} \boxtimes K_p}\right)$.*

Proof. Using Equation (7) and recalling that

$$\mathrm{Spec}(K_p) = \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}.$$

By means of Lemma 4.1 we get

$$\mathrm{Spec}(G \boxtimes K_p) = \begin{pmatrix} rp+p-1 & p+p\lambda_2-1 & \cdots & p+p\lambda_k-1 & -1 \\ & & & & \\ 1 & m_2 & \cdots & m_k & n(p-1) \end{pmatrix}.$$

The graph $G \boxtimes K_p$ is regular on np vertices with regularity $pr+p-1$. Therefore, applying Lemma 2.1 to its spectrum, we obtain

$$\mathrm{Spec}\left(\overline{G \boxtimes K_p}\right) = \begin{pmatrix} np-rp-p & -p-p\lambda_2 & \cdots & -p-p\lambda_k & 0 \\ & & & & \\ 1 & m_2 & \cdots & m_k & n(p-1) \end{pmatrix},$$

implying

$$\begin{aligned} \mathcal{E}\left(\overline{G \boxtimes K_p}\right) &= |np-rp-p| + \sum_{i=2}^{k} m_i \, |-p-p\lambda_i| + n(p-1)|0| \\ &= np-rp-p+p\sum_{i=2}^{k} m_i \, |1+\lambda_i| \\ &= np-rp-p+p\left(-n+2r+1+\sum_{i=2}^{k} m_i \, |\lambda_i|\right) = p\,\mathcal{E}(G)\,. \end{aligned} \tag{11}$$

Replacing G by $\overline{G}$ in Equation (11) results in

$$\mathcal{E}\left(\overline{\overline{G} \boxtimes K_p}\right) = p\,\mathcal{E}(\overline{G})\,. \tag{12}$$

Since G is complementary equienergetic, $\mathcal{E}(G) = \mathcal{E}(\overline{G})$. Therefore, by Equations (11) and (12), the result follows. □

Theorem 4.4. *Let G be an r-regular complementary equienergetic graph of order n. Then $\mathcal{E}\left(\overline{G \square \overline{K_{1,p}}}\right) = \mathcal{E}\left(\overline{\overline{G} \square \overline{K_{1,p}}}\right)$, where $p \geq r$.*

Proof. Note that $\overline{G \square \overline{K_{1,p}}}$ is same as $\overline{G} + \overline{G \square K_p}$. Let $\mathrm{Spec}(G)$ and $\mathrm{Spec}(K_p)$ be same as in Theorem 4.3. Then by Lemma 4.1,

$$\mathrm{Spec}(G \square K_p) = \begin{pmatrix} r{+}p{-}1 & r{-}1 & \lambda_2{+}p{-}1 & \cdots & \lambda_k{+}p{-}1 & \lambda_2{-}1 & \cdots & \lambda_k{-}1 \\ & & & & & & & \\ 1 & p{-}1 & m_2 & \cdots & m_k & m_2(p{-}1) & \cdots & m_k(p{-}1) \end{pmatrix}.$$

The graph $G \square K_p$ is regular on np vertices with regularity $r+p-1$. Therefore, by applying Lemma 2.1 to the spectrum of $G \square K_p$, we get

$$\mathrm{Spec}\left(\overline{G \square K_p}\right) = \begin{pmatrix} np{-}rp{-}p & -r & -\lambda_2{-}p & \cdots & -\lambda_k{-}p & -\lambda_2 & \cdots & -\lambda_k \\ & & & & & & & \\ 1 & p{-}1 & m_2 & \cdots & m_k & m_2(p{-}1) & \cdots & m_k(p{-}1) \end{pmatrix}$$

from which, we have

$$\begin{aligned} \mathcal{E}\left(\overline{G \square K_p}\right) &= |np-r-p| + (p-1)|-r| + \sum_{i=2}^{k} m_i |-\lambda_i - p| + \sum_{i=2}^{k} m_i \,(p-1)|-\lambda_i| \\ &= np-2r-p+pr+\sum_{i=2}^{k} m_i \,|\lambda_i+p| + (p-1)\sum_{i=2}^{k} m_i \,|\lambda_i|\,. \end{aligned} \tag{13}$$

Every r-regular graph satisfies $\lambda_i \geq -r$, $i = 1, 2, \ldots, n$ [7]. Therefore, $\lambda_i + p \geq -r + p \geq 0$ as $p \geq r$. Equation (13) becomes then

$$\begin{aligned} \mathcal{E}\left(\overline{G \square K_p}\right) &= np-2r-p+pr+\sum_{i=2}^{k} m_i \,(\lambda_i+p) + (p-1)\sum_{i=2}^{k} m_i \,|\lambda_i| \\ &= np-2r-p+pr-r+p(n-1)+(p-1)\big[\mathcal{E}(G)-r\big] \end{aligned}$$

$$= \quad 2(np-p-r)+(p-1)\,\mathcal{E}(G)\,.$$

By Theorem 4.1, we have

$$\begin{aligned}
\mathcal{E}\left(\overline{G\square K_{1,p}}\right) &= \mathcal{E}\left(\overline{G}+\overline{G\square K_p}\right)) = \mathcal{E}(\overline{G})+\mathcal{E}\left(\overline{G\square K_p}\right) \\
&\quad +\sqrt{\left[(n-r-1)+(np-p-r)\right]^2+4\left[n^2p-(n-r-1)(np-p-r)\right]} \\
&\quad -\left[(n-r-1)+(np-p-r)\right] \\
&= \mathcal{E}(\overline{G})+np-n-p+1+(p-1)\,\mathcal{E}(G) \\
&\quad +\sqrt{n^2p^2+2n^2p+4np+p^2+n^2+1-2np^2-2p-2n} \\
&= np-n-p+1+p\,\mathcal{E}(G) \\
&\quad +\sqrt{n^2p^2+2n^2p+4np+p^2+n^2+1-2np^2-2p-2n}\,, \qquad (14)
\end{aligned}$$

because $\mathcal{E}(G)=\mathcal{E}(\overline{G})$.

Replacing G by $\overline{G}$ in Equation (14) we get

$$\mathcal{E}\left(\overline{\overline{G}\square K_{1,p}}\right) = np-n-p+1+p\mathcal{E}(\overline{G})+\sqrt{n^2p^2+2n^2p+4np+p^2+n^2+1-2np^2-2p-2n}\,. \qquad (15)$$

Since G is complementary equienergetic, $\mathcal{E}(G)=\mathcal{E}(\overline{G})$. Therefore, by Equations (14) and (15), the result follows. □

Note that $\overline{G\square K_{1,p}}$ and $\overline{\overline{G}\square K_{1,p}}$ are non-regular graphs.

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