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Domination number and traceability of graphs

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Abstract

For $k \geq 1$, let G be a k-connected graph of order n. In this note, it is proved that if $\gamma(G^c) \geq n - k - 1$ then G is either traceable or $K_k \vee K^c_{k+2}$, where $\gamma(G^c)$ is the domination number of the complement of the graph G and $K_k \vee K^c_{k+2}$ is the join of K_k and K^c_{k+2}.

Keywords: traceability; domination number.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology, not defined here, follow those in [2]. Let G be a graph. We use G^c to denote the complement of G. We also use $\gamma(G)$, $\omega(G)$, and $\alpha(G)$ to denote the domination number, the clique number, and the independent (or stability) number of G, respectively. If $S \subseteq V(G)$, then $N(S)$ denotes the neighborhood of S, that is, the set of all vertices in G adjacent to at least one vertex in S. For a subgraph H of G and $S \subseteq V(G) - V(H)$, let $N_H(S) = N(S) \cap V(H)$. We use $G \vee H$ to denote the the join of two disjoint graphs G and H. If P is a path of G, we use $\overrightarrow{P}$ to denote the path P with a given direction. For two vertices x, y in P, we use $\overrightarrow{P}[x, y]$ to denote the consecutive vertices on P from x to y in the direction specified by $\overrightarrow{P}$. The same vertices, in reverse order, are given by $\overleftarrow{P}[y, x]$. We use x^+ and x^- to denote respectively the successor and predecessor of a vertex x on P along the direction of P. We also use x^{++} and x^{--} to denote $(x^+)^+$ and $(x^-)^-$, respectively. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

Li in [4] presented the following sufficient condition for the Hamiltonicity of graphs.

Theorem 1.1. *Let G be a k-connected ($k \geq 2$) graph of order n. If $\gamma(G^c) \geq n - k$, then G is Hamiltonian or $K_k \vee K^c_{k+1}$.*

In this note, we present the following sufficient condition for the traceability of graphs.

Theorem 1.2. *Let G be a k-connected ($k \geq 1$) graph of order n. If $\gamma(G^c) \geq n - k - 1$, then G is traceable or $K_k \vee K^c_{k+2}$.*

2. The lemmas

The following two lemmas were used in the proofs of Theorem 1.1. The first one is from Theorem 1 of [3] and the second one is the main result of [1].

Lemma 2.1. *Let G be a graph of order n. Then, $\gamma(G) + \chi(G) \leq n + 1$.*

Lemma 2.2. *Let G be a k-connected ($k \geq 2$) graph with independent number $\alpha = k + 1$. Let C be the longest cycle in G. Then, $G[V(G) - V(C)]$ is complete.*

In order to prove Theorem 1.2, we need to prove the following lemma (that is, Lemma 2.3). Lemma 2.1 and Lemma 2.3 will be used in the proof of Theorem 1.2. Notice that Lemma 2.3 is motivated by Lemma 2.2 and some ideas used in the proof of Lemma 2.2 will be used in the proof of Lemma 2.3.

Lemma 2.3. *Let G be a k-connected ($k \geq 1$) graph with independent number $\alpha = k + 2$. Let P be the longest path cycle in G. Then $G[V(G) - V(P)]$ is complete.*

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Proof. Let G be a graph satisfying the conditions in Lemma 2.3. Let P be any longest path in G. Assume the two end-vertices of P are y and z. We assign an orientation from y to z for $P[y,z]$. Suppose $G[V(G)-V(P)]$ is not complete and let H_1, H_2, ..., H_l be the components of $G[V(G)-V(P)]$.

Claim 1. $|N_P(V(H_i))| = k$ for each i with $1 \leq i \leq l$.

Proof of Claim 1. Assume that $N_P(V(H_1)) := \{a_1, a_2, ..., a_r\}$ and $b_i a_i \in E$ where $b_i \in V(H_1)$ for each i with $1 \leq i \leq r$. Notice that b_1, b_2, ..., b_r are not necessarily distinct. We further assume that the appearance of a_1, a_2, ..., a_r agrees with the orientation of P. Since G is k-connected, we have $r \geq k$. Since P is a longest path in G, we have $y \neq a_1$,$z \neq a_r$, and $\{x, y, a_1^+, a_2^+, ..., a_r^+\}$ is independent in G, where x is any vertex in H_1. Otherwise we can easily find paths in G which are longer than P. Thus $r+2 \leq \alpha = k+2$. Namely, $r \leq k$. Therefore $r = k$ and $|N_P(V(H_1))| = k$. Similarly, we can prove that $|N_P(V(H_i))| = k$ for each i with $2 \leq i \leq l$.

Claim 2. H_i is complete for each i with $1 \leq i \leq l$.

Proof of Claim 2. Suppose H_1 is not complete. From Claim 1, we have that $N_P(V(H_1)) := \{a_1, a_2, ..., a_r\} = \{a_1, a_2, ..., a_k\}$. Then we can find two vertices u and v in H_1 such that $uv \notin E$. Notice that $uy \notin E$, $ua_i^+ \notin E$ for each i with $1 \leq i \leq k$, $vy \notin E$, and $va_i^+ \notin E$ for each i with $1 \leq i \leq k$. Otherwise we can easily find paths in G which are longer than P. Thus $\{u, v, y, a_1^+, a_2^+, ..., a_k^+\}$ is an independent set with cardinality $(k+3)$, a contradiction. Therefore H_1 is complete. Similarly, we can prove that H_i is complete for each i with $2 \leq i \leq l$.

Claim 3. $l = 1$.

Proof of Claim 3. Suppose $l \geq 2$. From Claim 1, we have that $N_P(V(H_1)) := \{a_1, a_2, ..., a_r\} = \{a_1, a_2, ..., a_k\}$. Choose a vertex u in H_1 and a vertex v in H_2. Since P is a longest path in G, we have $uy \notin E$, $vy \notin E$, and $ua_i^+ \notin E$ for each i with $1 \leq i \leq k$. Notice that the set of

$$\{u, v, y, a_1^+, a_2^+, ..., a_k^+\}$$

has $(k+3)$ elements, it is not independent. Then, we have the following possible cases for the remaining proof of Claim 3.

Case 1. $va_k^+ \in E$. Namely, v is adjacent to the successor of the last element in $\{a_1, a_2, ..., a_k\}$, according to the orientation of P.

Since P is a longest path in G, $z \neq a_k^+$ and $ya_k^{++} \notin E$ otherwise we can easily find paths in G which are longer than P. If $k = 1$, then $\{u, v, y, a_1^{++}\}$ is an independent set of size 4, contradicting to the fact that $\alpha = k+2$. Now we have that $k \geq 2$. Notice that the set of

$$\{u, v, y, a_1^+, a_2^+, ..., a_{k-1}^+, a_k^{++}\}$$

has $(k+3)$ elements, it is not independent. Clearly, $va_k^{++} \notin E$. Then there exists an index j such that $a_j^+ a_k^{++} \in E$ and $1 \leq j \leq (k-1)$. Let b_j and b_k be two vertices in H_1 such that $b_j a_j \in E$ and $b_k a_k \in E$. Notice that it may happen that $b_j = b_k$. We use $P_{H_1}[b_j, b_k]$ to denote a path between b_j and b_k in H_1. Since the path

$$P_1 := \overrightarrow{P}[y, a_j] P_{H_1}[b_j, b_k] \overleftarrow{P}[a_k, a_j^+] \overrightarrow{P}[a_k^{++}, z]$$

is not shorter than P and P is a longest path in G, P_1 is also a longest path in G. Now $G[V(H_2) \cup \{a_k^+\}]$ is a component of $G[V(G)-V(P_1)]$ and $|N_{P_1}(V(H_2) \cup \{a_k^+\})| = |(N_{P_1}(V(H_2) - \{a_k^+\}) \cup \{a_k, a_k^{++}\}| = k-1+2 = (k+1)$. Since P is any longest path in G, we reach a contradiction to Claim 1.

Case 2. $va_j^+ \in E$ for some j with $1 \leq j \leq (k-1)$.

For this case, we have the following two subcases.

Case 2.1. $a_j^{++} = a_{j+1}$.

Let b_j and b_{j+1} be two vertices in H_1 such that $b_j a_j \in E$ and $b_{j+1} a_{j+1} \in E$. Notice that it may happen that $b_j = b_{j+1}$. We use $P_{H_1}[b_j, b_{j+1}]$ to denote a path between b_j and b_{j+1} in H_1. Since the path

$$P_2 := \overrightarrow{P}[y, a_j] P_{H_1}[b_j, b_{j+1}] \overrightarrow{P}[a_{j+1}, z]$$

is not shorter than P and P is a longest path in G, P_2 is also a longest path in G. Now $G[V(H_2) \cup \{a_j^+\}]$ is a component of $G[V(G)-V(P_2)]$ and $|N_{P_2}(V(H_2) \cup \{a_j^+\})| = |(N_{P_2}(V(H_2) - \{a_j^+\}) \cup \{a_j, a_{j+1}\}| = k-1+2 = (k+1)$, contradicting to Claim 1.

Case 2.2. $a_j^{++} \neq a_{j+1}$.

Since P is a longest path, $ya_j^{++} \notin E$ otherwise we can easily find a path in G which is longer than P. If $j = 1$, then $\{u, v, y, a_1^{++}\}$ is an independent set of size 4, contradicting to the fact that $\alpha = k + 2$. Now we have that $j \geq 2$. Notice that the set of

$$\{u, v, y, a_1^+, a_2^+, ..., a_{j-1}^+, a_j^{++}, a_{j+1}^+, ..., a_k^+\}$$

has $(k + 3)$ elements, it is not independent. Clearly, $va_j^{++} \notin E$. Then there exists an index s such that $a_s^+ a_j^{++} \in E$ where $1 \leq s \leq k$ and $s \neq j$. Let b_j and b_s be two vertices in H_1 such that $b_j a_j \in E$ and $b_s a_s \in E$. We use $P_{H_1}[b_s, b_j]$ (resp., $P_{H_1}[b_j, b_s]$) to denote a path between b_s and b_j (resp., b_j and b_s) in H_1.

When $s < j$, since the path

$$P_3 := \overrightarrow{P}[y, a_s]P_{H_1}[b_s, b_j]\overleftarrow{P}[a_j, a_s^+]\overrightarrow{P}[a_j^{++}, z]$$

is not shorter than P and P is a longest path in G, P_3 is also a longest path in G. Now $G[V(H_2) \cup \{a_j^+\}]$ is a component of $G[V(G) - V(P_3)]$ and $|N_{P_3}(V(H_2) \cup \{a_j^+\})| = |(N_{P_3}(V(H_2) - \{a_j^+\}) \cup \{a_j, a_j^{++}\}| = k - 1 + 2 = (k + 1)$, contradicting to Claim 1.

When $s > j$, since the path

$$P_4 := \overrightarrow{P}[y, a_j]P_{H_1}[b_j, b_s]\overleftarrow{P}[a_s, a_j^{++}]\overrightarrow{P}[a_s^+, z]$$

is not shorter than P and P is a longest path in G, P_4 is also a longest path in G. Now $G[V(H_2) \cup \{a_j^+\}]$ is a component of $G[V(G) - V(P_4)]$ and $|N_{P_4}(V(H_2) \cup \{a_j^+\})| = |(N_{P_4}(V(H_2) - \{a_j^+\}) \cup \{a_j, a_j^{++}\}| = k - 1 + 2 = (k + 1)$, contradicting to Claim 1.

The combination of Claim 2 and Claim 3 completes the proof of the lemma.

□

3. Proof of Theorem 1.2

Proof. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose that G is not traceable. Choose a longest path P in G and give an orientation on P. Let y and z be the two end vertices of P. Since G is not traceable, there exists a vertex $x_0 \in V(G) - V(P)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths Q_1, Q_2, ..., Q_s between x_0 and $V(P)$. Let u_i be the end vertex of Q_i on P, where $1 \leq i \leq s$. Since P is a longest path in G, $y \neq u_i$ and $z \neq u_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use u_i^+ to denote the successor of u_i along the orientation of P, where $1 \leq i \leq s$. Then $\{x_0, y, u_1^+, u_2^+, ..., u_s^+\}$ and $\{x_0, z, u_1^-, u_2^-, ..., u_s^-\}$ are independent otherwise G would have paths which are longer than P. Since $s \geq k$, we have an independent set $S := \{x_0, y, u_1^+, u_2^+, ..., u_k^+\}$ of size $k + 2$ in G and a clique S of size $k + 2$ in G^c. we also have an independent set $T := \{x_0, z, u_1^-, u_2^-, ..., u_k^-\}$ of size $k + 2$ in G and a clique T of size $k + 2$ in G^c. From Lemma 2.1, we have that

$$n + 1 = n - k - 1 + k + 2 \leq \gamma(G^c) + \alpha(G)$$

$$= \gamma(G^c) + \omega(G^c) \leq \gamma(G^c) + \chi(G^c) \leq n + 1.$$

Then $\gamma(G^c) = n - k - 1$ and $\alpha(G) = \omega(G^c) = \chi(G^c) = k + 2$. Next we will present three claims and their proofs.

Claim 1. $G^c[V(G) - S]$ (resp., $G^c[V(G) - T]$) is an empty graph. Namely, $G[V(G) - S]$ (resp., $G[V(G) - T]$) is a complete graph.

Proof of Claim 1. Suppose that $G^c[V(G) - S]$ is not an empty graph. Then there exist vertices u, $v \in V(G) - S$ such that $uv \in E(G^c)$. Notice that $G^c[S]$ is complete. Then $(V(G) - S - \{u\}) \cup \{w\}$ is a domination set in G^c, where w is a vertex in S. Thus $n - k - 1 = \gamma(G^c) \leq |V(G) - S| - 1 + 1 = n - k - 2$, a contradiction. Similarly, we can prove that $G^c[V(G) - T]$ is an empty graph.

Claim 2. There are no edges between S (resp., T) and $V(G) - S$ (resp. $V(G) - T$) in G^c. Namely, for any vertex $u \in S$ (resp., T) and any vertex $w \in V(G) - S$ (resp., $V(G) - T$), $uw \in E(G)$.

Proof of Claim 2. Suppose that there exist vertices $u \in S$ and $w \in V(G) - S$ such that $uw \in E(G^c)$. Notice that $G^c[S]$ is complete. Then $(V(G) - S - \{w\}) \cup \{u\}$ is a domination set in G^c. Thus $n - k - 1 = \gamma(G^c) \leq |V(G) - S| - 1 + 1 = n - k - 2$, a contradiction. Similarly, we can prove that there are no edges between T and $V(G) - T$ in G^c.

Claim 3. $y = u_1^-$ and $z = u_k^+$.

Proof of Claim 3. Suppose $y \neq u_1^-$. Then $u_1^- \notin S$. If $z \notin S$, then Claim 1 implies that $zu_1^- \in E$, contradicting to the fact T is independent. If $z \in S$, then Claim 2 implies that $zu_1^- \in E$. Then we can easily find a path in G which is longer than P, a contradiction. Thus $y = u_1^-$.

Suppose $z \neq u_k^+$. Then $u_k^+ \notin T$. If $y \notin T$, then Claim 1 implies that $yu_k^+ \in E$, contradicting to the fact S is independent. If $y \in T$, then Claim 2 implies that $yu_k^+ \in E$. Then we can easily find a path in G which is longer than P, a contradiction. Thus $z = u_k^+$.

Set $T_i := \overrightarrow{P}[u_i^{++}, u_{i+1}]$, where $1 \leq i \leq k-1$. Obviously, $|T_i| \geq 1$ for each i with $1 \leq i \leq k-1$. Set $T := \{\, i : |T_i| \geq 2 \,\}$. Next we, according to the different sizes of $|T|$, divide the remainder of the proofs into two cases.

Case 1. $|T| = 0$.

Since $|T| = 0$, we have $P = yu_1u_1^+u_2u_2^+ ... u_ku_k^+z$. Next we will prove that $V(G) - V(P) = \{\, x_0 \,\}$. Suppose that $V(G) - V(P) \neq \{\, x_0 \,\}$. Then there exists a vertex, say w, in $V(G) - V(P) - \{\, x_0 \,\}$. Since $\alpha(G) = k + 2$, we have, by Lemma 2.3, that $G[V(G) - V(P)]$ is complete. Thus $x_0w \in E(G)$. If $k = 1$, then the path yu_1x_0w is longer than P, a contradiction. Now we have that $k \geq 2$. Since $w \in V(G) - S$ and $u_2 \in V(G) - S$, we, by Claim 1, have that $wu_2 \in E$. Therefore G has a path $yQ_1[u_1, x_0]wu_2u_2^+ ... u_ku_k^+z$ which is longer than P, a contradiction. Now we, by Claim 1 and Claim 2, have that G is $K_k \vee K_{k+2}^c$.

Case 2. $|T| \geq 1$.

Since $|T| \geq 1$, there exists an index t such that $|T_t| = |\overrightarrow{P}[u_t^{++}, u_{t+1}]| \geq 2$. Thus $u_t^+ \neq u_{t+1}^-$ and therefore $u_{t+1}^- \notin S$. Notice that $z \in S$. Then Claim 2 implies that $zu_{t+1}^- \in E$, contradicting to the fact that T is independent.

So, the proof of Theorem 1.2 is completed.

□

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