Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **4** (2020) 23–26

On a conjecture involving the second largest signless Laplacian eigenvalue and the index of graphs

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(Received: 14 April 2020. Received in revised form: 1 June 2020. Accepted: 15 June 2020. Published online: 18 June 2020.)

Abstract

Let G be a simple connected graph of order $n \geq 9$. Let q_2 be the second largest signless Laplacian eigenvalue of G and λ_1 be the index of G. Cvetković *et al.* [*Publ. Inst. Math. (Beograd)* **81(95)** (2007) 11–27] conjectured that $1-\sqrt{n-1} \leq q_2-\lambda_1 \leq n-2-\sqrt{2n-4}$, where the left equality holds if and only if G is the star $K_{1,n-1}$, and the right equality holds if and only if G is the complete bipartite graph $K_{2,\,n-2}$. Das [*Linear Algebra Appl.* **435** (2011) 2420–2424] proved that $1-\sqrt{n-1} \leq q_2-\lambda_1$ and characterized the graphs attaining the equality. In this note, we prove that the inequality $q_2-\lambda_1 \leq n-2-\sqrt{2n-4}$ holds for a certain class of graphs.

Keywords: graph; signless Laplacian matrix; second largest signless Laplacian eigenvalue; index of graph.

2020 Mathematics Subject Classification: 05C35, 05C50.

1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set $E=E(G)$. Also let $d(v_i)$ be the degree of vertex v_i for $i=1,2\ldots,n$. If vertices v_i and v_j are adjacent, we denote that by $v_iv_j \in E(G)$. The adjacency matrix $A(G)$ of G is defined by its entries $a_{ij}=1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ denote the eigenvalues of $A(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees. Then the signless Laplacian matrix of G is $Q(G)=D(G)+A(G)$. Let $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ denote the eigenvalues of $Q(G)$.

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be given graphs and $V_1 \cap V_2 \neq \emptyset$. The union $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The join $G_1 \bigtriangledown G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. A bipartite graph G is a graph whose vertex set can be partitioned into two independent sets, and if both of these independent sets have the same cardinality then G is called a balanced bipartite graph. As usual, K_n and $K_{t,n-t}$ denote respectively the complete graph and complete bipartite graph with n vertices. In 2007, Cvetković et al. [2] gave the following conjecture involving the second largest signless Laplacian eigenvalue q_2 and the index λ_1 of graph G (see also [1]).

Conjecture 1.1. *Let G be a connected graph of order $n \geq 9$. Then*

$$1-\sqrt{n-1} \leq q_2-\lambda_1 \leq n-2-\sqrt{2n-4} \tag{1}$$

with equality if and only if G is the star $K_{1,n-1}$ for the lower bound, and if and only if G is the complete bipartite graph $K_{n-2,2}$ for upper bound.

In [4], Das proved that the lower bound of Conjecture 1.1 is true and characterized the corresponding extremal graphs. In this note, we prove that the upper bound holds for a certain class of graphs.

2. Main results

First we shall list some previously known results that will be needed in the main result.

Lemma 2.1. [5] *Let G be a graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$. If G has $k-1$ duplicate pairs (v_i,v_{i+1}), $i=1,2,\ldots,k-1$, then G has at least $k-1$ signless Laplacian eigenvalues equal to the cardinality of the neighbor set. Also the corresponding $k-1$ eigenvectors are*

$$(\underbrace{1,-1}_{2},0,\ldots,0)^T,(\underbrace{1,0,-1}_{3},0,\ldots,0)^T,\ldots,(\underbrace{1,0,\ldots,-1}_{k},0,\ldots,0)^T.$$

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Lemma 2.2. [3] *If G' is a graph obtained from a graph G by deleting any edge then $\lambda_1(G') \leq \lambda_1(G)$. Moreover, this inequality is strict when G is connected.*

Lemma 2.3. [7] *Let G be a graph of order $n \geq 2$. Then $q_2(G) \leq n-2$.*

Lemma 2.4. [6] *Let G be a graph of order n and let H be a subgraph of G obtained by deleting an edge in G. Then*

$$q_1(G) \geq q_1(H) \geq q_2(G) \geq q_2(H) \geq q_3(G) \geq \cdots \geq q_{n-1}(H) \geq q_n(G) \geq q_n(H)$$

where $q_i(G)$ is the i-th largest signless Laplacian eigenvalue of G and $q_i(H)$ is the i-th largest signless Laplacian eigenvalue of H.

Let m and p be integers. For simplicity of notation, we write pK_m instead of $\underbrace{K_m \cup \cdots \cup K_m}_{p}$.

Lemma 2.5. *Let m and n be two integers and H be a graph nK_1 or K_n. Then*

$$q_2(H \bigtriangledown 2K_m) = n + 2m - 2.$$

Proof. First, let $H = nK_1$. As the consequence of Lemma 2.1 we conclude that the signless Laplacian eigenvalues of $nK_1 \bigtriangledown 2K_m$ are

$$\{\underbrace{n+m-2, n+m-2, \ldots, n+m-2}_{2m-2}, \underbrace{2m, 2m, \ldots, 2m}_{n-1}\} \tag{2}$$

and the remaining three eigenvalues satisfy the following system:

$$\begin{cases} qx_1 = (n+2m-2)x_1 + nx_3 \\ qx_2 = (n+2m-2)x_2 + nx_3 \\ qx_3 = mx_1 + mx_2 + 2mx_3 \end{cases}$$

where $(\underbrace{x_1, \ldots, x_1}_{m}, \underbrace{x_2, \ldots, x_2}_{m}, \underbrace{x_3, \ldots, x_3}_{n})^T$ is the eigenvector corresponding to eigenvalue q.

Thus, the remaining three eigenvalues are

$$\frac{n + 4m - 2 \pm \sqrt{(n-2)^2 + 8mn}}{2}, \qquad n + 2m - 2. \tag{3}$$

From (2) and (3), we get

$$q_2(nK_1 \bigtriangledown 2K_m) = n + 2m - 2.$$

Now, let $H = K_n$. As the consequence of Lemma 2.1 we conclude that the signless Laplacian eigenvalues of $K_n \bigtriangledown 2K_m$ are

$$\{\underbrace{n+m-2, n+m-2, \ldots, n+m-2}_{2m-2}, \underbrace{n+2m-2, n+2m-2, \ldots, n+2m-2}_{n-1}\} \tag{4}$$

and the remaining three eigenvalues satisfy the following system:

$$\begin{cases} qx_1 = (n+2m-2)x_1 + nx_3 \\ qx_2 = (n+2m-2)x_2 + nx_3 \\ qx_3 = mx_1 + mx_2 + (2n+2m-2)x_3 \end{cases}$$

where $(\underbrace{x_1, \ldots, x_1}_{m}, \underbrace{x_2, \ldots, x_2}_{m}, \underbrace{x_3, \ldots, x_3}_{n})^T$ is the eigenvector corresponding to eigenvalue q.

Thus, the remaining three eigenvalues are

$$\frac{3n + 4m - 2 \pm \sqrt{n^2 + 8mn}}{2}, \qquad n + 2m - 2. \tag{5}$$

From (4) and (5), we get

$$q_2(K_n \bigtriangledown 2K_m) = n + 2m - 2.$$

This completes the proof. □

Lemma 2.6. *Let G be a graph of order $n > 2$. If the complement of G has a balanced bipartite component then $q_2(G) = n-2$.*

Proof. Let (X, Y) with $|X| = |Y| = m$ be the balanced bipartite component in the complement of G. If there is an edge xy in G such that $x \in X$ and $y \in Y$, then we delete this edge and the obtained graph is denoted by G_1. Then by Lemma 2.4, we have

$$q_2(G) \geq q_2(G_1). \tag{6}$$

Repeating this procedure sufficient number of times, we arrive at a graph $H \bigtriangledown 2K_m$ where H is a graph of order $n-2m$. Therefore, we have

$$q_2(G) \geq q_2(G_1) \geq \cdots \geq q_2(H \bigtriangledown 2K_m) \tag{7}$$

by using Lemma 2.4.
If $E(H) \neq \emptyset$, we obtain a graph H_1 from H by deleting an edge. Then by Lemma 2.4

$$q_2(H \bigtriangledown 2K_m) \geq q_2(H_1 \bigtriangledown 2K_m).$$

Repeating this procedure sufficient number of times, we arrive at a graph $(n-2m)K_1 \bigtriangledown 2K_m$. Then we have

$$q_2(H \bigtriangledown 2K_m) \geq q_2(H_1 \bigtriangledown 2K_m) \geq \cdots \geq q_2((n-2m)K_1 \bigtriangledown 2K_m) = n-2 \tag{8}$$

by Lemma 2.4 and Lemma 2.5. From (7) and (8), we get

$$q_2(G) \geq n-2.$$

On the other hand, we have $q_2(G) \leq n-2$, by Lemma 2.3. Hence we get the required result. $\square$

Lemma 2.7. *Consider the inequality*

$$\sqrt{(x-1)^2 + 8x(n-2x)} \geq 2\sqrt{2n-4} + 1 - x \tag{9}$$

where n is an integer and x is a real number.
(i) Let $n \geq 10$. If $x \in \left[1, \frac{n}{2}\right)$, then (9) holds with the equality if and only if $x = 1$.
(ii) Let $2 < n \leq 9$. If $x \in \left[1, \frac{n-1}{2}\right]$, then (9) holds with the equality if and only if $x = 1$ or $n = 4, x = 3/2$.

Proof. Denote $f(n, x) = 4x^2 - (2n + \sqrt{2n-4})x + ((2n-4) + \sqrt{2n-4})$.
(i) Let $10 \leq n \leq 34$. Then it is clear that $2\sqrt{2n-4} + 1 \geq \frac{n}{2}$. Hence $2\sqrt{2n-4} + 1 - x \geq 0$ for $x \in \left[1, \frac{n}{2}\right)$. If we square the both sides of (9) then we get

$$f(n, x) \leq 0 \tag{10}$$

for $x \in \left[1, \frac{n}{2}\right)$. Now, it is sufficient to prove that the inequality (10) holds. Then, we have $f(n, 1) = 0$ and $f\left(n, \frac{n}{2}\right) = \dfrac{(2n-4)(4-\sqrt{2n-4})}{4} \leq 0$ for $n \geq 10$. Since $f(n, x)$ is the quadratic function for x, we get the required inequality.

Now, let $n > 34$. Then, $2\sqrt{2n-4}+1 < \frac{n}{2}$. If $2\sqrt{2n-4}+1 < x < \frac{n}{2}$ then clearly (9) holds. Otherwise $1 \leq x \leq 2\sqrt{2n-4}+1$. Then we get the inequality (10) by squaring the both sides of (9). Hence, we get

$$f(n, x) \leq 0, \quad x \in \left[1, 2\sqrt{2n-4} + 1\right]$$

because $f(n, 1) = 0$ and $f\left(n, 2\sqrt{2n-4} + 1\right) = 2\sqrt{2n-4} \cdot (55 - (2\sqrt{2n-4} - 7)^2) < 0$ for $n > 34$.

From the above, one can easily seen that the equality in (9) holds if and only if $x = 1$.

(ii) Since $2 < n \leq 9$, we have $2\sqrt{2n-4} + 1 - x \geq 0$ for $x \in \left[1, \frac{n-1}{2}\right]$. Then we get the inequality (10) by squaring the both sides of (9). Similarly as the above argument, we also get

$$f(n, x) \leq 0, \quad x \in \left[1, \frac{n-1}{2}\right]$$

because $f(n, 1) = 0$ and $f\left(n, \frac{n-1}{2}\right) = \dfrac{(n-3)(2-\sqrt{2n-4})}{2} \leq 0$ for $2 < n \leq 9$.

From the above we obtain the equality in (9) if and only if $x = 1$ or $n = 4, x = 1.5$. This completes the proof. $\square$

Theorem 2.1. *Let G be a graph of order $n > 2$. If the complement of G has a balanced bipartite component then the upper bound in (1) holds for the graph G.*

Proof. Let (X,Y) with $|X| = |Y| = m$ be the balanced bipartite component in the complement of G. If there is an edge xy in G such that $x \in X$ and $y \in Y$, then we delete this edge and the obtained graph is denoted by G'. Then, clearly the complement of G' also contains a balanced bipartite component. Hence $q_2(G) = q_2(G') = n-2$ by Lemma 2.6 and

$$q_2(G) - \lambda_1(G) \le q_2(G') - \lambda_1(G')$$

by Lemma 2.2. Using similar argument as in the proof of Lemma 2.6, we get

$$q_2(G) - \lambda_1(G) \le \cdots \le q_2((n-2m)K_1 \triangledown 2K_m) - \lambda_1((n-2m)K_1 \triangledown 2K_m). \tag{11}$$

On the other hand, it is easy to see that the eigenvalues of adjacency matrix of $((n-2m)K_1 \triangledown 2K_m$ are

$$\left\{ \frac{m-1 \pm \sqrt{(m-1)^2 + 8m(n-2m)}}{2}, m-1, \underbrace{0, 0, \ldots, 0}_{n-2m-1}, \underbrace{-1, -1, \ldots, -1}_{2m-2} \right\}.$$

Therefore, we have

$$q_2((n-2m)K_1 \triangledown 2K_m) - \lambda_1((n-2m)K_1 \triangledown 2K_m) = n-2 - \frac{m-1 + \sqrt{(m-1)^2 + 8m(n-2m)}}{2},$$

by using Lemma 2.5. Now, we prove that

$$n-2 - \frac{m-1 + \sqrt{(m-1)^2 + 8m(n-2m)}}{2} \le n-2-\sqrt{2n-4}. \tag{12}$$

If $n \ge 10$, then by Lemma 2.7 (i), the inequality (12) holds with equality if and only if $m = 1$. If $2 < n \le 9$, then by Lemma 2.7 (ii) the inequality (12) also holds with equality if and only if $m = 1$. From (11) and (12), we obtain

$$q_2(G) - \lambda_1(G) \le n-2-\sqrt{2n-4}$$

with equality if and only if G is isomorphic to $K_{n-2,2}$. This completes the proof. □

Acknowledgment

The authors are much grateful to two anonymous referees for their careful reading and helpful comments on our paper.

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