Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **4** (2020) 19–22

Seymour's second neighborhood conjecture for some oriented graphs with no sink

Moussa Daamouch*

Lebanese University, Faculty of Sciences I, Beirut, Lebanon

(Received: 24 April 2020. Received in revised form: 2 June 2020. Accepted: 15 June 2020. Published online: 18 June 2020.)

Abstract

Seymour's Second Neighborhood Conjecture (SNC) asserts that every oriented graph has a vertex whose second out-neighborhood is at least as large as its first out-neighborhood. Such a vertex is called a *Seymour vertex*. In this paper, we prove that every k-transitive oriented graph with minimum out-degree $\delta \geq k-2 \geq 1$ has at least two Seymour vertices, and we deduce that the SNC holds for every k-transitive oriented graph with $k \leq 9$. We also prove that every 3-quasi-transitive oriented graph with no sink has at least two Seymour vertices, where *sink* is a vertex having out-degree zero.

Keywords: Seymour's second neighborhood conjecture; 3-quasi-transitive digraph; k-transitive digraph; sink.

2020 Mathematics Subject Classification: 05C20.

1. Introduction

All the digraphs in this paper are finite and without loops. A digraph $D = (V, E)$ is a pair of two disjoint sets V and E, where V is non-empty and $E \subset V \times V$. The set V is called the vertex set of D and it is denoted by $V(D)$, while E is called the edge set (arc set) of D and it is denoted by $E(D)$. An oriented graph D is an asymmetric digraph (with no symmetric arcs), i.e., if $(x, y) \in E(D)$ then $(y, x) \notin E(D)$. The *out-neighborhood* of a vertex v is $N_D^+(v) = \{u \in V(D) : (v, u) \in E(D)\}$ and the *second out-neighborhood* of v is $N_D^{++}(v) = \{w \in V(D) \setminus N_D^+(v) : \exists\, x \in N_D^+(v), (x, w) \in E(D)\}$. The *out-degree* of v is $d_D^+(v) = |N_D^+(v)|$ and its *second out-degree* is $d_D^{++}(v) = |N_D^{++}(v)|$. A *sink* is a vertex having out-degree zero. For $A \subseteq V(D)$, $D[A]$ denotes the subdigraph of D induced by A, $N_A^+(v) = N_D^+(v) \cap A$ and $d_A^+(v) = |N_A^+(v)|$. We omit the subscript when it is clear from the context. We denote by $(x_0, x_1, ..., x_k)$ a directed x_0x_k-path of length k. Similarly, we denote by $(x_1, x_2, ..., x_k, x_1)$ a directed k-cycle of length k. We say that v is a *Seymour vertex* if $d^{++}(v) \geq d^+(v)$. In 1990, Paul Seymour proposed the following conjecture:

Conjecture 1.1 (Second Neighborhood Conjecture (SNC))**.** *In every finite oriented graph, there exists a Seymour vertex.*

It soon became an important topic of interest in graph theory. Extensive research was conducted in this field but the SNC still remains open. It was proven only for some very specific classes of digraphs. In 1995, Dean and Latka [3] conjectured a similar statement for tournaments. This problem, known as Dean's conjecture, has been solved in 1996 by Fisher [5]. In 2000, Havet and Thomassé [14] gave a short proof of Dean's conjecture using median orders. Their proof also yields the following stronger result.

Theorem 1.1. [14] *Let T be a tournament with no sink. Then T has at least two Seymour vertices.*

In 2007, Fidler and Yuster [4] used median orders and another tool called dependency digraph to prove that the SNC holds for tournaments missing a matching. In 2012, 2013 and 2015, Ghazal [10–12] also used median orders, dependency digraph and good digraphs to show that the conjecture holds for some new classes of digraphs (tournaments missing n-generalized stars and other classes of oriented graphs). In 2001, Kaneko and Locke [16] proved the following theorem.

Theorem 1.2. [16] *Let D be an oriented graph, and let δ be its minimum out-degree. If $\delta \leq 6$, then D has a Seymour vertex.*

A digraph D is called *transitive* if for any directed path (x_0, x_1, x_2) of length two in D we have $(x_0, x_2) \in E(D)$. A digraph D is called *quasi-transitive* if for any directed path (x_0, x_1, x_2) of length two in D we have $(x_0, x_2) \in E(D)$ or $(x_2, x_0) \in E(D)$. Clearly, the class of quasi-transitive digraphs is a generalization of tournaments and transitive digraphs. In 2012, Galena-Sánchez and Hernández-Cruz [8] introduced the class of k-transitive digraphs as a generalization of transitive digraphs, and the class of k-quasi-transitive digraphs as an extension of quasi-transitive digraphs. D is a *k-transitive digraph* if for

*Email address: moussadaamouch1923@gmail.com

any $u, v \in V(D)$, the existence of a directed uv-path of length k implies $(u, v) \in E(D)$. D is a *k-quasi-transitive digraph* if for any $u, v \in V(D)$, the existence of a directed uv-path of length k implies $(u, v) \in E(D)$ or $(v, u) \in E(D)$. Since their introduction, k-transitive digraphs and k-quasi-transitive digraphs have received a fair amount of attention (see [7]). In 2017, García-Vásquez and Hernández-Cruz [9] proved the SNC for 4-transitive oriented graphs using a characterization of strong 4-transitive digraphs. Also in 2017, Gutin and Li [13] proved the SNC for quasi-transitive (or 2-quasi-transitive) oriented graphs using a characterization of quasi-transitive digraphs. Recently, in [1], the SNC has been proved, by combinatorial methods, for some cases of k-transitive oriented graphs (for $k \leq 6$ and in case of forbidden directed cycles of length at most $\frac{k}{2} - 1$) and other classes of oriented graphs. It is seen that the difficulty of the SNC for k-transitive digraphs and k-quasi-transitive digraphs is increasing with respect to k, but the existence of characterizations, descriptions or properties for such class of digraphs, may reduce the difficulty of the problem (see [9, 13]). Unfortunately, for $k > 3$, there are no known structural characterizations for k-quasi-transitive digraphs even in case of strong digraphs. In addition, as mentioned by Galena-Sánchez and Hernández-Cruz [7], it is hard to obtain a complete description of k-quasi-transitive digraphs for $k > 3$. Also, k-transitive digraphs deal with similar difficulty for $k > 4$. In fact, 4 is the largest value of k such that strong k-transitive digraphs are characterized. However, there is some information about strong k-transitive digraphs e.g., Hernández-Cruz and Montellano-Ballesteros [15] characterized strong k-transitive digraphs having a cycle of length at least k. Their characterization will be used in this paper and, fortunately, it renders easy proofs of the SNC for some cases of k-transitive digraphs.

A related problem to the SNC is to find more than one Seymour vertex. Obviously, this is not always possible, e.g., a transitive tournament has exactly one Seymour vertex. This is why we consider digraphs with no sink when finding at least two Seymour vertices. Note that this problem was proved for some classes of oriented graphs, e.g., for tournaments, for tournaments missing a matching, for bipartite tournaments, for quasi-transitive oriented graphs and for k-transitive oriented graphs with $k \leq 4$ (see [1, 2, 13, 14, 17]). In 2019, Dara, Francis, Jacub and Narayanan [2] proposed the following conjectures.

Conjecture 1.2. [2] *In every oriented graph with no sink, there are at least two Seymour vertices.*

Conjecture 1.3. [2] *If an oriented graph contains exactly one Seymour vertex, then that vertex is a sink.*

Moreover, they proved that Conjecture 1.2 and Conjecture 1.3 are equivalent and both imply the SNC. This result further adds importance and interest to the problem of finding at least two Seymour vertices in digraphs with no sink.

In this paper, we verify Conjecture 1.2 for 3-quasi-transitive oriented graphs with no sink and for k-transitive oriented graphs with minimum out-degree $\delta \geq k - 2 \geq 1$. As a consequence, the SNC holds for every k-transitive oriented graph with $k \leq 9$.

2. Main results

A digraph is *strongly connected* (or *strong*) if for every pair of vertices u and v, there exists a uv-directed path. A *strong component* of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph D^* with $V(D^*)$ equal to the set of all strong components of D, and $(S, T) \in E(D^*)$ if and only if there is $(s, t) \in E(D)$ such that $s \in S$ and $t \in T$. Clearly, D^* is an acyclic digraph (a digraph without directed cycles), and thus, it has a vertex of out-degree zero. A *terminal component* of D is a strong component S of D such that $d^+_{D^*}(S) = 0$.

Lemma 2.1. *Let D be a simple digraph with no loops or symmetric arcs. Let S be a terminal strong component of D. If v is a Seymour vertex in the subdigraph $D[S]$ induced by S, then v is a Seymour vertex in D.*

Proof. For all $x \in S$, we have $N^+_S(x) = N^+_D(x)$ since S is a terminal strong component of D, so $d^+_S(v) = d^+_D(v)$ and $d^{++}_S(v) = d^{++}_D(v)$. □

2.1 The SNC for k-transitive digraphs

We will use a characterization given in [15] by Hernández-Cruz and Montellano-Ballesteros for strong k-transitive digraphs having a cycle of length at least k to prove that every k-transitive oriented graph with minimum out-degree $\delta \geq k - 2 \geq 1$ has at least two Seymour vertices.

Definition 2.1. *Let D be a general digraph. We say that D is a complete (respectively semicomplete) digraph if for any $x, y \in V(D)$ we have $(x, y) \in E(D)$ and $(y, x) \in E(D)$ (respectively $(x, y) \in E(D)$ or $(y, x) \in E(D)$).*

Definition 2.2. *Let X and Y be two disjoint sets. We say that $D(X \cup Y, E)$ is a complete bipartite (respectively semicomplete bipartite) digraph if for all $(x, y) \in X \times Y$ we have $(x, y) \in E(D)$ and $(y, x) \in E(D)$ (respectively $(x, y) \in E(D)$ or $(y, x) \in E(D)$).*

Definition 2.3. *Let $V_0, V_1, ..., V_{n-1}$ be pairwise disjoint vertex sets. The asymmetric digraph with vertex set $V_0 \cup V_1 \cup ... \cup V_{n-1}$ and arc set $\bigcup_{i=0}^{n-1} \{(v_i, v_{i+1}) : v_i \in V_i, v_{i+1} \in V_{i+1}\}$, where the subscripts are taken modulo n, is called an extended n-cycle and denoted by $C[V_0, V_1, ..., V_{n-1}]$.*

In [15] Hernández-Cruz and Montellano-Ballesteros characterized strong k-transitive digraphs (general digraphs) having a cycle of length at least k in the following two theorems.

Theorem 2.1. [15] *Let k be an integer, $k \geq 2$. Let D be a strong k-transitive digraph. Suppose that D contains a directed cycle of length n such that the greatest common divisor of n and $k-1$ is equal to d and $n \geq k+1$. Then the following hold:*

1. *If $d = 1$, then D is a complete digraph.*
2. *If $d \geq 2$, then D is either a complete digraph, a complete bipartite digraph, or an extended d-cycle.*

Theorem 2.2. [15] *Let k be an integer, $k \geq 2$. Let D be a strong k-transitive digraph of order at least $k+1$. If D contains a directed cycle of length k, then D is a complete digraph.*

Lemma 2.2. *Let n be an integer, $n \geq 3$. Then every extended n-cycle $C[V_0, V_1, ..., V_{n-1}]$ has at least two Seymour vertices.*

Proof. Let V_i be a smallest set of the partition $\{V_0, V_1, ..., V_{n-1}\}$, i.e., $|V_i| \leq |V_j|$ for all $0 \leq j \leq n-1$. Note that for all $0 \leq j \leq n-1$ we have $|V_j| \geq 1$. Let $x \in V_{i-1}$, where the subscripts are taken modulo n, we have $d^+(x) = |V_i| \leq |V_{i+1}| = d^{++}(x)$, so x is a Seymour vertex. Hence, if $|V_{i-1}| \geq 2$, then there are at least two Seymour vertices. If $|V_{i-1}| = 1$, then $|V_i| = 1$. Let $y \in V_{i-2}$, so $d^+(y) = |V_{i-1}| = 1 = |V_i| = d^{++}(y)$, therefore x and y are two Seymour vertices in $C[V_0, V_1, ..., V_{n-1}]$. □

It is well known that every 2-transitive digraph has a sink, which is a trivial Seymour vertex. This is why we consider k-transitive digraphs with $k \geq 3$.

Theorem 2.3. *Let k be an integer, $k \geq 3$. Let D be a k-transitive oriented graph (asymmetric digraph) with minimum out-degree $\delta \geq k-2$. Then D has at least two Seymour vertices.*

Proof. Let S be a terminal strong component of D. For all $x \in S$, we have $d^+(x) \geq \delta \geq k-2$. It is easy to prove that the digraph $D[S]$ induced by S contains a directed cycle C of length $l(C) \geq k$. In fact, let $(x_1, ..., x_t)$ be a longest directed path in $D[S]$, we have $N^+(x_t) \subseteq \{x_1, ..., x_{t-2}\}$. Let i be minimal such that $x_i \in N^+(x_t) \cap \{x_1, ..., x_{t-2}\}$, then $N^+(x_t) \subseteq \{x_i, ..., x_{t-2}\}$, so $\delta \leq d^+(x_t) \leq |\{x_i, ..., x_{t-2}\}| = |\{x_i, ..., x_t\}| - 2 = l(C) - 2$, thus $l(C) \geq \delta + 2 \geq k$. Suppose that $l(C) = k$. We have $|S|\delta \leq \sum_{x \in S} d^+(x) \leq \frac{|S|(|S|-1)}{2}$, then we get $|S| \geq 2\delta + 1 \geq 2k-1$, so $|S| \geq k+1$ for all $k \geq 2$. Hence, we can apply Theorem 2.2, so $D[S]$ is a complete digraph, but D is asymmetrical, then $D[S] = \{v\}$ with $d^+(v) = 0$ which is a contradiction to $\delta \geq 1$. Thus, $l(C) \geq k+1$, and we can apply Theorem 2.1. Since D is an asymmetric digraph and $\delta \geq 1$, then there is only one possibility for $D[S]$, which is an extended d-cycle for some $d \geq 3$, so by Lemma 2.2 $D[S]$ has at least two Seymour vertices. Therefore, by Lemma 2.1 D has two Seymour vertices. □

Corollary 2.1. *Let k be an integer, $3 \leq k \leq 9$. Let D be a k-transitive oriented graph. Then D has a Seymour vertex.*

Proof. Let δ be the minimum out-degree of D. If $\delta \geq k-2$, then by Theorem 2.3 D has a Seymour vertex. If $\delta \leq k-3$, then $\delta \leq 6$, so by Theorem 1.2 D has a Seymour vertex. □

2.2 The SNC for 3-quasi-transitive digraphs

We will use a characterization given in [6] by Ghaleana-Sánchez, Goldfeder and Urritia for strong 3-quasi-transitive digraphs to prove that every 3-quasi-transitive oriented graph with no sink has at least two Seymour vertices.

Consider the digraph F_n with vertex set $V(F_n) = \{x_i : i = 1, 2, ..., n\}$ and arc set $E(F_n) = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\} \bigcup \{(x_1, x_{i+3}), (x_{i+3}, x_2) : i = 1, 2, ..., n-3\}$.

Theorem 2.4. [6] *Let D be a strong 3-quasi-transitive digraph of order n. Then D is either a semicomplete digraph, a semicomplete bipartite digraph, or isomorphic to F_n.*

Theorem 2.5. [17] *A bipartite tournament with no sink has at least two Seymour vertices.*

Lemma 2.3. *Let D be a digraph isomorphic to F_n. Then D has at least two Seymour vertices.*

Proof. It is easy to see that x_2 and x_3 are two Seymour vertices. In fact, we have $N^+(x_2) = \{x_3\}$ and $N^{++}(x_2) = \{x_1\}$, so $d^+(x_2) = d^{++}(x_2) = 1$. $N^+(x_3) = \{x_1\}$ and $N^{++}(x_3) = \{x_2, x_4, ..., x_n\}$, so $d^+(x_3) = 1 \leq d^{++}(x_3) = n - 2$. □

Theorem 2.6. *Let D be a 3-quasi-transitive oriented graph with no sink. Then D has at least two Seymour vertices.*

Proof. Let S be a terminal strong component of D. The subdigraph $D[S]$ induced by S is also a 3-quasi-transitive digraph, so by Theorem 2.4 $D[S]$ is either a semicomplete digraph, a semicomplete bipartite digraph, or isomorphic to F_n, but D is asymmetrical, then $D[S]$ is either a tournament, a bipartite tournament, or isomorphic to F_n. Note that $D[S]$ has no sink. If $D[S]$ is a tournament then by Theorem 1.1 it has two Seymour vertices. If $D[S]$ is a bipartite tournament then by Theorem 2.5 it has two Seymour vertices. If $D[S]$ isomorphic to F_n then by Lemma 2.3 it has two Seymour vertices. Therefore, by Lemma 2.1 D has at least two Seymour vertices. □

Acknowledgment

I would like to thank the anonymous referees for several helpful comments.

References

[1] M. Daamouch, Seymour's second neighborhood conjecture for m-free, k-transitive, k-anti-transitive digraphs and some approaches, Submitted.
[2] S. Dara, M. C. Francis, D. Jacob, N. Narayanan, Extending some results on the second neighborhood conjecture, *arXiv*: 1808.02247v2, 2019.
[3] N. Dean, B. J. Latka, Squaring the tournament-an open problem, *Congr. Numer.* **109** (1995) 73–80.
[4] D. Fidler, R. Yuster, Remarks on the second neighborhood problem, *J. Graph Theory* **55** (2007) 208–220.
[5] D. Fisher, Squaring a tournament: a proof of Dean's conjecture, *J. Graph Theory* **23** (1996) 43–48.
[6] H. Galeana-Sánchez, I. A. Goldfeder, I. Urritia, On the structure of strong 3-quasi-transitive digraphs, *Discrete Math.* **310** (2010) 2495–2498.
[7] H. Galeana-Sánchez, C. Hernández-Cruz, Quasi-transitive Digraphs and Their Extensions, In: J. Bang-Jensen, G. Gutin (Eds.), *Classes of Directed Graphs*, Springer, Cham, 2018, pp. 341–404.
[8] H. Galeana-Sánchez, C. Hernández-Cruz, k-kernels in k-transitive and k-quasi-transitive digraphs, *Discrete Math.* **312** (2012) 2522–2530.
[9] P. R. García-Vásquez, C. Hernández-Cruz, Some results on 4-transitive digraphs, *Discuss. Math. Graph Theory* **37** (2017) 117–129.
[10] S. Ghazal, Seymour's second neighborhood conjecture for tournaments missing a generalized star, *J. Graph Theory* **71** (2012) 89–94.
[11] S. Ghazal, A contribution to the second neighborhood problem, *Graphs Combin.* **29** (2013) 1365–1375.
[12] S. Ghazal, A remark on the second neighborhood conjecture, *Electron. J. Graph Theory Appl.* **3** (2015) 182–190.
[13] G. Gutin, R. Li, Seymour's second neighbourhood conjecture for quasi-transitive oriented graphs, *arXiv*: 1704.01389v2, 2017.
[14] F. Havet, S. Thomassé, Median orders of tournaments: a tool for the second neighborhood problem and Sumner's conjecture, *J. Graph Theory* **35** (2000) 244–256.
[15] C. Hernández-Cruz, J. J. Montellano-Ballesteros, Some remarks on the structure of strong k-transitive digraphs, *Discuss. Math. Graph Theory* **34** (2014) 651–672.
[16] Y. Kaneko, S. C. Locke, The minimum degree approach for Paul Seymour's distance 2 conjecture, *Congr. Numer.* **148** (2001) 201–206.
[17] R. Li, B. Sheng, The second neighborhood for bipartite tournaments, *Discuss. Math. Graph Theory* **39** (2019) 555–565.