Maximal hyper-Zagreb index of trees, unicyclic and bicyclic graphs with a given order and matching number

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(Received: 8 February 2020. Received in revised form: 28 April 2020. Accepted: 12 June 2020. Published online: 16 June 2020.)

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Abstract

Let $G$ be a simple connected graph. The hyper-Zagreb index is defined as $HM(G) = \sum_{uv \in E(G)}(d_G(u) + d_G(v))^2$. In this paper, the sharp upper bounds of the hyper-Zagreb index for trees, unicyclic and bicyclic graphs with a given order $n$ and matching number $\alpha$ are determined, and the graphs attaining these bounds are characterized.

Keywords: hyper-Zagreb index; tree; unicyclic graph; bicyclic graph; matching number.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C92.

1. Introduction

In this paper, all graphs we considered are finite, undirected, and simple. Let $G$ be an $n$-vertex graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V|$ and $|E|$ be the number of vertices and edges of $G$, respectively. For a vertex $u \in V(G)$, the degree of $u$, denoted by $d_u(G)$ (or shortly by $d_u$), is the number of vertices which are adjacent to $u$. Let $N_G(u)$ (or shortly $N(u)$) be the set of all neighbours of $u$ in $G$. Call a vertex $u$ a pendant vertex of $G$ if $d_u = 1$ and denote by $PV$ the set of pendant vertices of $G$, and call an edge $uv$ a pendant edge of $G$, if $d_u = 1$ or $d_v = 1$. Denote by $C_n$ and $S_n$ the cycle and star on $n$ vertices, respectively. Let $d_{G}(u,v)$ be the distance between vertices $u$ and $v$ in $G$. For $v \in V(G)$, let $G - v$ be a subgraph of $G$ obtained from $G$ by deleting a vertex $v$ and its incident edges.

A connected graph $G$ is called a unicyclic graph if it has a unique cycle. Bicyclic graphs are connected graphs with $n$ vertices and $n + 1$ edges. For a unicyclic or bicyclic graph $G$, the forest obtained from $G$ by deleting the edges of cycle(s) consists of several vertex-disjoint trees, each containing a vertex of the cycle(s), which is called the root of this tree in $G$.

A subset $M \subseteq E$ is called a matching in $G$ if no two elements of $M$ are adjacent. A matching $M$ of $G$ is said to be maximum, if for any other matching $M'$ of $G$, $|M'| \leq |M|$. The matching number of $G$ is the number of edges of a maximum matching in $G$. If $M$ is a matching of $G$ and vertex $v \in V(G)$ is incident with an edge of $M$, then $v$ is said to be $M$-saturated, and if every vertex of $G$ is $M$-saturated, then $M$ is a perfect matching.

For a molecular graph $G$, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are defined [8, 9] as

$$M_1(G) = \sum_{uv \in V(G)} (d(u) + d(v)),$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

The first and second Zagreb index were first suggested by Gutman et al., which absorbed attention of many scientists in different fields. See for instance [4, 5, 10, 17] and the references therein.

In 2013, Shirdel et al. [20] introduced a new degree-based topological index named hyper-Zagreb index as

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$ The hyper-Zagreb index is an important tool as it integrates the first and the second Zagreb indices. Gao et al. [6] found sharp bounds of the hyper-Zagreb index for acyclic, unicyclic, and bicyclic graphs. Liu et al. [15] obtained the maximum hyper-Zagreb index among cacti with perfect matchings. For more detail about this index, see [1, 7, 13, 18].

Recently, the bounds of various indices for cacti, bicyclic graphs and other graphs with perfect matchings or with a given matching number have been studied. The lower bounds on augmented Zagreb index of trees and unicyclic graphs

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with perfect matchings are presented by Sun et al. [19]. Liu et al. [14, 16] obtained the minimum value of Szeged index and revised edge Szeged index among trees and unicyclic graphs with perfect matchings. Zhong et al. [23] determined minimum general sum-connectivity index of trees with given matching number. For other related results, see [3, 12].

In this paper, we determine the sharp upper bounds of the hyper-Zagreb index for trees, unicyclic and bicyclic graphs with a given order $n$ and matching number $\alpha'$, and characterize the graphs attaining these bounds.

## 2. Main results

For the integers $n$ and $\alpha'$ satisfying $n \geq 2\alpha'$, $\alpha' \geq 2$, let $T(n, \alpha'), U(n, \alpha')$ and $B(n, \alpha')$, respectively, be the set of trees, unicyclic graphs and bicyclic graphs with $n$ vertices and matching number $\alpha'$. Firstly, we introduce some useful lemmas which will be used frequently.

**Lemma 2.1.** [11] Let $G \in T(2\alpha', \alpha')$, where $\alpha' \geq 2$, then $G$ has at least two pendent vertices such that they are adjacent to vertices of degree 2, respectively.

**Lemma 2.2.** [11] Let $G \in T(n, \alpha')$, where $n > 2\alpha'$, then there is an $\alpha'$-matching $M$ and a pendent vertex $u$ such that $u$ is not $M$-saturated.

**Lemma 2.3.** [2] Let $G \in U(2\alpha', \alpha')$, where $\alpha' \geq 3$, and let $T$ be a branch of $G$ with root $r$. If $u \in V(T)$ is a pendent vertex which is furthest from the root $r$ with $d(u, r) \geq 2$, then $u$ is a adjacent to a vertex of degree two.

**Lemma 2.4.** [21] Let $G \in U(n, \alpha')$, where $n > 2\alpha'$, and $G \neq C_n$, then there exists a maximum matching $M$ and a pendant vertex $u$ in $G$ such that $u$ is not $M$-saturated.

**Lemma 2.5.** [22] Let $G \in B(2\alpha', \alpha')$, and $\alpha' \geq 3$, and $T$ be a tree in $G$ attached to a root $r$. If $v \in V(T)$ is a vertex furthest from the root $r$ with $d_G(v, r) \geq 2$, then $v$ is a pendent vertex and adjacent to a vertex $u$ of degree two.

**Lemma 2.6.** [22] Let $G \in B(n, \alpha')$, and $n > 2\alpha' \geq 6$, and $G$ contains at least one pendent vertex, then there exist an $\alpha'$-matching $M$ and a pendant vertex $u$ in $G$ such that $u$ is not $M$-saturated.

![Figure 1: The graphs $T_{n,\alpha'}, U_{n,\alpha'}$, and $B_{n,\alpha'}$.](image)

For $n \geq 2\alpha'$, $\alpha' \geq 2$, let $T_{n,\alpha'}$ (shown in Figure 1) be the tree obtained by attaching a pendent vertex to $\alpha' - 1$ noncentral vertices of the star $S_{n-\alpha'+1}$, and let $U_{n,\alpha'}$ (shown in Figure 1) be the unicyclic graph obtained by attaching $n-2\alpha'+1$ pendent vertices and $\alpha'-2$ paths on two vertices to one vertex of a triangle, and let $B_{n,\alpha'}$ (shown in Figure 1) be the bicyclic graph of order $n$ obtained by attaching $n-2\alpha'+1$ pendent vertices and $\alpha'-3$ paths on two vertices to the common vertex of the two triangles. Obviously, $T_{n,\alpha'} \in T(n, \alpha')$, $U_{n,\alpha'} \in U(n, \alpha')$, and $B_{n,\alpha'} \in B(n, \alpha')$. By the definition of the hyper-Zagreb index, we have that

$$HM(T_{n,\alpha'}) = (n-2\alpha'+1)(n-\alpha'+1)^2 + (\alpha'-1)(n-\alpha'+2)^2 + 9\alpha' - 9.$$  

$$HM(U_{n,\alpha'}) = \alpha'(n-\alpha'+3)^2 + (n-2\alpha'+1)(n-\alpha'+2)^2 + 9\alpha' - 2.$$  

$$HM(B_{n,\alpha'}) = (n-2\alpha'+1)(n-\alpha'+3)^2 + (\alpha'+1)(n-\alpha'+4)^2 + 9\alpha' + 5.$$
Lemma 2.8. \[15\] If \( G \in \mathcal{T}(2\alpha', \alpha') \), where \( \alpha' \geq 2 \), then
\[
HM(G) \leq (\alpha')^3 + 4(\alpha')^2 + 11\alpha' - 12,
\]
with equality if and only if \( G \cong T_{2\alpha', \alpha'} \).

In the following, we give sharp upper bound for the hyper-Zagreb index of tree with a given matching number.

Theorem 2.1. If \( G \in \mathcal{T}(n, \alpha') \), where \( n \geq 2\alpha', \alpha' \geq 2 \), then
\[
HM(G) \leq (n - 2\alpha' + 1)(n - \alpha' + 1)^2 + (\alpha' - 1)(n - \alpha' + 2)^2 + 9\alpha' - 9,
\]
with equality if and only if \( G \cong T_{n, \alpha'} \) (shown in Figure 1).

Proof. We prove this result by using induction on \( n \). When \( n = 2\alpha' \), by Lemma 2.7, the theorem holds. Next, we consider \( n > 2\alpha' \), and assume that the result holds for the graphs in \( T_{n-1, \alpha'} \). By Lemma 2.2, there exist a maximum matching \( M \) and a pendent vertex \( u \) such that \( u \) is not \( M \)-saturated. Let \( w \) be the unique neighbor of \( u \) and \( N(w) \cap PV = \{u, u_1, u_2, \ldots, u_{t-1}\} \), where \( PV \) is the set of all pendent vertices in \( G \). Let \( d_w = d \) and \( N(w) \setminus PV = \{x_1, x_2, \ldots, x_{d-1}\} \).

As \( M \) contains one edge incident with \( w \) and there are \( n - \alpha' - 1 \) edges outside \( M \), so we have that \( d - 1 \leq n - \alpha' - 1 \), i.e., \( d \leq n - \alpha' \). We known that \( \sum_{v \in V(G)} d_v = 2|E| \), so we have
\[
\sum_{i=1}^{d-t} d_{x_i} + d + t + (n - d - 1) \leq 2|E| = 2n - 2,
\]
so
\[
\sum_{i=1}^{d-t} d_{x_i} \leq n - t - 1.
\]
Let \( G^* = G - u \), as \( u \) is not \( M \)-saturated, then \( G^* \in T_{n-1, \alpha'} \). By the inductive assumption, \( HM(G^*) \leq HM(T_{n-1, \alpha'}) \).

\[
HM(G) = HM(G^*) + (t - 1)[(d + 1)^2 - d^2] + (d + 1)^2 + \sum_{i=1}^{d-t} [(d + d_{x_i})^2 - (d + d_{x_i} - 1)^2]
\]
\[
\leq HM(T_{n-1, \alpha'}) + 3d^2 - d + 2n - 2
\]
\[
= HM(T_{n, \alpha'}) + 3(d^2 - (n - \alpha')^2) + [(n - \alpha') - d]
\]
\[
\leq HM(T_{n, \alpha'}).\]

The equality \( HM(G) = HM(T_{n, \alpha'}) \) holds if and only if equalities hold throughout the above inequalities, i.e., \( HM(G^*) = HM(T_{n-1, \alpha'}) \), \( d = n - \alpha' \), and \( V(G) \setminus \{N(w) \cup \{w\}\} \) are pendent vertices. So we have that \( HM(G) \leq HM(T_{n, \alpha'}) \) with equality if and only if \( G \cong T_{n, \alpha'} \). The proof is completed.

Lemma 2.8. \([6]\) If \( G \) is a unicyclic graph with \( n \) vertices, then
\[
HM(C_n) \leq HM(G),
\]
with equality if and only if \( G \cong C_n \).

Lemma 2.9. \([15]\) If \( G \in U(2\alpha', \alpha') \), where \( \alpha' \geq 2 \), then
\[
HM(G) \leq (\alpha')^3 + 7(\alpha')^2 + 22\alpha' + 2,
\]
with equality if and only if \( G \cong U_{2\alpha', \alpha'} \).

Next we give sharp upper bound for the hyper-Zagreb index of unicyclic graph with a given matching number.

Theorem 2.2. If \( G \in U(n, \alpha') \), where \( n \geq 2\alpha', \alpha' \geq 2 \), then
\[
HM(G) \leq \alpha'(n - \alpha' + 3)^2 + (n - 2\alpha' + 1)(n - \alpha' + 2)^2 + 9\alpha' - 2,
\]
with equality if and only if \( G \cong U_{n, \alpha'} \) (shown in Figure 1).
Proof. Again we use induction on \( n \). When \( n = 2\alpha' \), by Lemma 2.9, the theorem holds. Next, we consider \( n > 2\alpha' \), and assume that the result holds for the graphs in \( \mathcal{U}(n-1,\alpha') \).

Case 1 \( G = C_n \).

By Lemma 2.8, one knows that \( C_n \) is the graph with minimum hyper-Zagreb index.

Case 2 \( G \neq C_n \).

By Lemma 2.4, there exist a maximum matching \( M \) and a pendent vertex \( u \) such that \( u \) is not \( M \)-saturated. Let \( w \) be the unique neighbor of \( u \) and \( N(w) \cap PV = \{ u, u_1, u_2, \ldots, u_{t-1} \} \), where \( PV \) is the set of all pendent vertices in \( G \). Let \( d_w = d \) and \( N(w) \cap PV = \{ x_1, x_2, \ldots, x_{d-1} \} \).

As \( M \) contains one edge incident with \( w \) and there are \( n - \alpha' \) edges outside \( M \), so we have that \( d - 1 \leq n - \alpha' \), i.e., \( d \leq n - \alpha' + 1 \).

We know that \( \sum_{v \in V(G)} d_v = 2|E| \), so we have

\[
\sum_{i=1}^{d-t} dx_i + d + t + (n - d - 1) \leq 2|E| = 2n,
\]

so

\[
\sum_{i=1}^{d-t} dx_i \leq n - t + 1.
\]

Let \( G^* = G - u \), as \( u \) is not \( M \)-saturated, then \( G^* \in \mathcal{U}(n-1,\alpha') \). By the inductive assumption, \( HM(G^*) \leq HM(U_{n-1,\alpha'}) \).

\[
HM(G) = HM(G^*) + (t-1)(d+1)^2 - d^2 + (d+1)^2 + \sum_{i=1}^{d-t} [(d+dx_i)^2 - (d+dx_i - 1)^2]
\]

\[
\leq HM(U_{n-1,\alpha'}) + 3d^2 - d + 2n + 2
\]

\[
= HM(U_{n,\alpha'}) + 3(d^2 - (n - \alpha' + 1)^2) + [(n - \alpha' + 1) - d]
\]

\[
\leq HM(U_{n,\alpha'}).
\]

The equality \( HM(G) = HM(U_{n,\alpha'}) \) holds if and only if equalities hold throughout the above inequalities, i.e., \( HM(G^*) = HM(U_{n-1,\alpha'}) \), \( d = n - \alpha' + 1 \), and \( V(G) \setminus \{ N(w) \cup \{ w \} \} \) are pendent vertices. So we have that \( HM(G) \leq HM(U_{n,\alpha'}) \) with equality if and only if \( G \cong U_{n,\alpha'} \). The proof is completed.

Theorem 2.3. If \( G \in B(2\alpha', \alpha') \), \( \alpha' \geq 3 \), then

\[
HM(G) \leq (\alpha' + 3)^2 + (\alpha' + 1)(\alpha' + 4)^2 + 9\alpha' + 5,
\]

with equality if and only if \( G \cong B_{2\alpha', \alpha'} \).
Proof. Let \( f(2\alpha', \alpha') = (\alpha' + 3)^2 + (\alpha' + 1)(\alpha' + 4)^2 + 9\alpha' + 5 \). If \( PV(G) = \emptyset \), then \( G \in \{ F_i : 1 \leq i \leq 5 \} \), (see Figure 2), and \( n = 2\alpha' \). By calculating directly, we have that

\[
\text{HM}(F_1) = 32\alpha' + 80, \quad \text{HM}(F_2) = 32\alpha' + 56, \quad \text{HM}(F_3) = 32\alpha' + 54, \quad \text{HM}(F_4) = 32\alpha' + 56, \quad \text{HM}(F_5) = 32\alpha' + 54.
\]

Obviously, we have that \( \text{HM}(F_i) < \text{HM}(B_{2\alpha', \alpha'}) (\alpha' \geq 3) \), for \( 1 \leq i \leq 5 \).

Next, we assume that \( PV(G) \neq \emptyset \).

We prove the result by induction on \( \alpha' \). When \( \alpha' = 3 \), all graphs of the class \( B(6, 3) = \{ H_i : 1 \leq i \leq 11 \} \) are shown in Figure 3. Though calculating directly, we have that

\[
\text{HM}(F_1) = 212, \quad \text{HM}(F_2) = 236, \quad \text{HM}(F_3) = 192, \quad \text{HM}(F_4) = 190, \quad \text{HM}(F_5) = 212, \\
\text{HM}(F_6) = 188, \quad \text{HM}(F_7) = 264, \quad \text{HM}(F_8) = 224, \quad \text{HM}(F_9) = 226, \quad \text{HM}(F_{10}) = \text{HM}(F_{11}) = 188.
\]

We known that \( H(G) \leq f(6, 3) \) with equality if and only if \( G \cong B_{6, 3} \).

Next, we assume that \( \alpha' \geq 4 \), and the conclusion is true for \( B(2k, k) (k < \alpha') \). Let \( T_i \) be a tree in \( G \) which attached at the root \( r_i \) \((i = 1, 2, \ldots)\). Let \( v_i \in PV(T_i) \) be farthest from the root \( r_i \). We consider the following two cases to prove our results.

Case 1 \( d_{T_i}(r_i, v_i) = 1 \) for all \( T_i \in G \).

Subcase 1.1 \( d_v \neq 2 \) for all vertex \( v \in V(G) \).

As \( d_v \neq 2 \) for all vertex \( v \in V(G) \), one has \( G \in \{ B_i : 1 \leq i \leq 7 \} \), bicyclic graphs \( B_i (1 \leq i \leq 7) \) are shown in Figure 4. By calculating directly, we have that \( \text{HM}(B_1) = 52\alpha' + 168, \quad \text{HM}(B_2) = 52\alpha' + 56, \quad \text{HM}(B_3) = 52\alpha' + 134, \quad \text{HM}(B_4) = 52\alpha' + 132, \quad \text{HM}(B_5) = 52\alpha' + 56, \quad \text{HM}(B_6) = 52\alpha' + 134, \quad \text{HM}(B_7) = 52\alpha' + 132 \).

We have that \( \text{HM}(B_i) < \text{HM}(B_{2\alpha', \alpha'}) (\alpha' \geq 4) \), for \( 1 \leq i \leq 7 \).

Subcase 1.2 \( d_v = 2 \) for several \( v \in V(G) \).

Subcase 1.2.1 There is no vertex of degree two which lie in any cycle of \( G \).

As \( d_{T_i}(r_i, v_i) = 1 \) for all \( T_i \in G \), and there is no vertex of degree two which lie in any cycle of \( G \). Note that there exist \( u_2u_3 \in E(G) \) which belongs to one of the cycles in \( G \) such that \( d_{u_2} = d_{u_3} = 3 \). Let \( N(u_2) = \{ u_1, u_3, v_2 \} \), \( N(u_3) = \{ u_2, u_4, v_3 \} \).

Without loss of generality, we suppose that \( d_{u_2} = d_{v_3} = 1, \quad 3 \leq d_{u_1} \leq 4, \quad 3 \leq d_{u_4} \leq 4 \). Let \( G^* = G - u_2u_3, \) then \( G^* \in U(2\alpha', \alpha')(\alpha' \geq 6) \). By the inductive assumption, we have that

\[
\text{HM}(G) = \text{HM}(G^*) + ((d_{u_1} + 3)^2 - (d_{u_1} + 2)^2) + ((d_{u_4} + 3)^2 - (d_{u_4} + 2)^2) \\
\leq \text{HM}(U_{2\alpha', \alpha'}) + 2d_{u_1} + 2d_{u_4} + 60 \\
= \text{HM}(B_{2\alpha', \alpha'}) - [3(\alpha')^2 + 17\alpha' + 19] + [2d_{u_1} + 2d_{u_4} + 60] \\
< \text{HM}(B_{2\alpha', \alpha'}).\]

Subcase 1.2.2 There exists a vertex of degree two which lies on one of the cycles of \( G \).

Suppose that the vertex \( u_2 \) with degree 2 lie in one of cycles of \( G \), and \( N(u_2) = \{ u_1, u_3 \} \). As \( G \in B(2\alpha', \alpha') \), there exists an edge between \( u_1u_2 \) and \( u_2u_3 \) that are not belong to an \( \alpha' \)-matching. Without loss of generality, we suppose that edge \( u_2u_3 \) is not belong to the \( \alpha' \)-matching. Let \( d_{u_3} = d \), and \( N(u_3) \setminus \{ u_2 \} = \{ x_1, x_2, \ldots, x_{d-1} \} \). Obviously, one has \( 2 \leq d \leq 5, \quad 2 \leq d_{u_1} \leq 5 \).
Let $G^* = G - u_2u_3$, then $G^* \in U_{2\alpha', \alpha'}$. We know that $\sum_{v \in V(G)} d_v = 2|E|$, so we have

$$\sum_{i=1}^{d-1} d_{x_i} + d + d_{u_4} + 2 + (2\alpha' - d - 2) \leq 2|E| = 2(2\alpha' + 1),$$

so $\sum_{i=1}^{d-1} d_{x_i} \leq 2\alpha' - d_{u_4} + 2$. By the inductive assumption, $HM(G^*) \leq HM(U_{2\alpha', \alpha'})$, and hence we have that

$$HM(G) = HM(G^*) + \sum_{i=1}^{d-1} [(d + d_{x_i})^2 - (d + d_{x_i} - 1)^2] + (d_{u_4} + 2)^2 - (d_{u_4} + 1)^2 + (d + 2)^2$$

$$\leq HM(U_{2\alpha', \alpha'}) + 3d^2 + d + 12$$

$$= HM(B_{2\alpha', \alpha'}) - (3(\alpha')^2 + 17\alpha' + 19) + 3d^2 + d + 12$$

$$< HM(B_{2\alpha', \alpha'}).$$

where the last inequality holds for $\alpha' \geq 4$ and $2 \leq d \leq 5$.

**Case 2** $d_{T_i}(r_i, v_i) \geq 2$ for several $T_i \in G$.

Since $v_i \in PV(G)$, let $N(v_i) = u$. By Lemma 2.5, one has $d_u = 2$. Let $N(u) = \{v_i, w\}$, $N(w) \setminus PV = \{x_1, x_2, \ldots, x_t\}$, $N(w) \setminus PV = \{x_{t+1}, x_{t+2}, \ldots, x_{d-1}, x_d = u\}$. As $M$ contains exactly one edge incident with $w$ and there $\alpha'$ edges of $G$ outside $M$, we have that $d-1 \leq \alpha' - 1$, i.e., $d \leq \alpha' + 2$. $d_{x_i} \geq 2, i = t + 1, \ldots, d - 1$.

Let $G^* = G - v_i - u$, then $G^* \in B(2(\alpha' - 1), \alpha' - 1)$. By the inductive assumption, $HM(G^*) \leq f(2(\alpha' - 1), \alpha' - 1)$. We know that $\sum_{v \in V(G)} d_v = 2|E|$, so we have $\sum_{i=t+1}^{d-1} d_{x_i} + d + t + 3 + (2\alpha' - d - 2) \leq 2|E| = 2(2\alpha' + 1)$, so $\sum_{i=t+1}^{d-1} d_{x_i} \leq 2\alpha' - t + 1$.

$$HM(G) = HM(G^*) + \sum_{i=t+1}^{d-1} [(d + d_{x_i})^2 - (d + d_{x_i} - 1)^2] + (d_{u_4} + 2)^2 - (d_{u_4} + 1)^2 + (d + 2)^2$$

$$\leq f(2\alpha', \alpha') + [f(2(\alpha' - 1), \alpha' - 1) - f(2\alpha', \alpha')] + 3d^2 + d + 4\alpha' + 16$$

$$= f(2\alpha', \alpha') + 3(d^2 - (\alpha' + 2)^2) + (d - (\alpha' + 2))$$

$$\leq f(2\alpha', \alpha').$$

The equality $HM(G) = f(2\alpha', \alpha')$ holds if and only if equalities hold throughout the above inequalities, i.e., $HM(G^*) = f(2(\alpha' - 1), \alpha' - 1)$, $d = \alpha' + 2$, and $V(G) \setminus \{N(w) \setminus \{w \cup \{v_i\}\}\}$ are pendent vertices. So we have that $HM(G) \leq f(2\alpha', \alpha')$ with equality if and only if $G \cong B_{2\alpha', \alpha'}$ (shown in Figure 1). The proof is completed. \hfill \Box

![Figure 4: The graphs of the class $\{B_i : 1 \leq i \leq 7\}$](image)

Next we give sharp upper bound for the hyper-Zagreb index of bicyclic graph with a given matching number.

**Theorem 2.4.** If $G \in B(n, \alpha')$, where $n \geq 2\alpha', \alpha' \geq 3$, then

$$HM(G) \leq (n - 2\alpha' + 1)(n - \alpha' + 3)^2 + (\alpha' + 1)(n - \alpha' + 4)^2 + 9\alpha' + 5,$$

with equality if and only if $G \cong B_{n, \alpha'}$. 


**Proof.** Let \( g(n, \alpha') = (n - 2\alpha' + 1)(n - \alpha' + 3)^2 + (\alpha' + 1)(n - \alpha' + 4)^2 + 9\alpha' + 5 \). We prove the result by induction on \( n \). If \( n = 2\alpha' \), the result follows from Theorem 2.3. Next, we assume that \( n > 2\alpha' \) and the result holds for all bicyclic graphs on fewer than \( n \) vertices. Let \( G \in B(n, \alpha') \).

**Case 1** \( PV(G) = \emptyset \).

As there is no pendant vertex in \( G \) which has an \( \alpha' \)-matching, then \( G \in \{ F_i : 1 \leq i \leq 5 \} \) (see Figure 3), and \( n = 2\alpha' + 1 \). By direct calculations, one has \( HM(F_1) = 32\alpha' + 112, HM(F_2) = 32\alpha' + 88, HM(F_3) = 32\alpha' + 86, HM(F_4) = 32\alpha' + 88, HM(F_5) = 32\alpha' + 86 \). By Lemma 2.6, \( G \) has \( \alpha' \)-matching, \( g(2\alpha' + 1, \alpha') \), for \( 1 \leq i \leq 5 \).

**Case 2** \( PV(G) \neq \emptyset \).

By Lemma 2.6, \( G \) has an \( \alpha' \)-matching \( M \) and there exist \( v \in PV(G) \) such that \( v \) is \( M \)-unsaturated. Let \( N(v) = \{ v \} \), and \( d_u = d \). Let \( N(v) \cap PV = \{ v_1, v_2, \ldots, v_{d-1}, v \} \), and \( N(v) \setminus PV = \{ x_t, x_{t+1}, \ldots, x_{d-1} \} \). As \( M \) contains exactly one edge incident with \( v \) and there \( n + 1 - \alpha' \) edges of \( G \) outside \( M \), we have that \( d - 1 \leq n + 1 - \alpha' \), i.e., \( d \leq n - \alpha' + 2 \).

Let \( G^* = G - v \), then \( G^* \in B(n - 1, \alpha') \). By the inductive assumption, \( HM(G^*) \leq g(n - 1, \alpha') \). We know that \( \sum_{v \in V(G)} d_v = 2|E| \), so we have \( \sum_{i=1}^{d-1} d_{x_i} + d + t + (n - d - 1) \leq 2|E| = 2(n + 1) \), so \( \sum_{i=1}^{d-1} d_{x_i} \leq n - t + 3 \).

\[
HM(G) = HM(G^*) + \sum_{i=1}^{d-1} [(d + d_{x_i})^2 - (d + d_{x_i} - 1)^2] + (d + 1)^2 + (t - 1)(d + 1)^2 - d^2
\leq g(n, \alpha') + [g(n - 1, \alpha') - g(n, \alpha' - 1)] + 3d^2 - d - 2n + 6
= g(n, \alpha') + 3(d^2 - (n - \alpha' + 2)^2) + ((n - \alpha' + 2) - d)
\leq g(n, \alpha').
\]

The equality \( HM(G) = g(n, \alpha') \) holds if and only if equalities hold throughout the above inequalities, i.e., \( HM(G^*) = g(n - 1, \alpha') \), \( d = n - \alpha' + 2 \), and \( V(G) \setminus \{ N(u) \cup \{ v \} \} \) are pendant vertices. So we have that \( HM(G) \leq g(n, \alpha') \) with equality if and only if \( G \cong B_{n,\alpha'} \). The proof is completed.

\[\square\]

### 3. Conclusion

In this paper, we determine the sharp upper bounds of the hyper-Zagreb index for trees, unicyclic and bicyclic graphs with a given order \( n \) and matching number \( \alpha' \), and characterize the graphs attaining these bounds. Motivated by [23], it is also interesting to obtain the bounds of general sum-connectivity index for trees, unicyclic and bicyclic graphs with a given order \( n \) and matching number \( \alpha' \). We intend to consider this problem in the near future.

### Acknowledgments

This work is supported by the Hunan Provincial Natural Science Foundation of China through grant number 2020JJ4423. The authors are much grateful to the anonymous referees for their helpful comments and suggestions on our paper, which have considerably improved the presentation of this paper.

### References


