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*Research Article*

# The $r$-Hued Coloring of Complements of Cycles

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**Abstract**

Given a graph $G$, an $r$-hued coloring of $G$ is a proper vertex coloring such that for every vertex $v$, the number of colors appearing in its neighborhood is at least $\min\{d_G(v), r\}$, where $d_G(v)$ denotes the degree of $v$ in $G$. The $r$-hued chromatic number $\chi_r(G)$ is the smallest number of colors needed for an $r$-hued coloring of $G$. In this paper, we study the $r$-hued coloring of complements of cycles. For any positive integer $r$, we completely determine the $r$-hued chromatic number of $\overline{C_n}$ by constructing explicit coloring schemes and analyzing the sizes of independent sets of color classes.

**Keywords:** $r$-hued coloring; $r$-hued chromatic number; complements of cycles.

**2020 Mathematics Subject Classification:** 05C15, 05C76.

## 1. Introduction

All graphs considered in this paper are simple and finite. Undefined notation and terminology can be found in [1]. Let $\mathbb{N}$ be the set of natural numbers and $2^{\mathbb{N}}$ its power set. For $k \in \mathbb{N}$, let $[k] = \{1, 2, \ldots, k\}$. For a graph $G$, let $E(G)$, $V(G)$ and $\Delta(G)$ denote its edge set, vertex set and maximum degree, respectively. The order $n = |V(G)|$ of $G$ is the number of its vertices, while the size $m = |E(G)|$ of $G$ is the number of its edges. For any $v \in V(G)$, the neighborhood of $v$ in $G$ is $N_G(v)$, where $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$, and the degree of $v$ is $d_G(v)$, where $d_G(v) = |N_G(v)|$. For a mapping $c : V(G) \to [k]$, let $c(N_G(v))$ be the set of colors of the neighbors of $v$ in $G$. The diameter of $G$ is the maximum distance between any two vertices in $G$, denoted by $\mathrm{diam}(G)$.

Graph coloring is one of the most classic and highly valuable core branches in graph theory, and it is also an important area of research in discrete mathematics, operations research, and theoretical computer science. Its origin as a research problem can be traced back to the Four Color Conjecture in the 19th century [10]. In recent decades, with the extensive application of graph theory in computer networks, frequency assignment and scheduling problems, many variants of classical coloring have been proposed and attracted much attention. The $r$-hued coloring is one such coloring model with important theoretical value and practical background. The concept of $r$-hued coloring was first introduced in [11]. The special case $r = 2$ was studied in [6] and is also known as dynamic coloring. The first paper on $r$-hued coloring for general $r$ is [5].

**Definition 1.1.** *Let $k, r$ be positive integers. A $(k, r)$-coloring of a graph $G$ (also known as an $r$-hued coloring of a graph $G$) is a mapping $c : V(G) \to [k]$ satisfying:*

*(C1) For any adjacent vertices $u, v \in V(G)$, $c(u) \neq c(v)$.*

*(C2) For every $v \in V(G)$, $|c(N_G(v))| \geq \min\{d_G(v), r\}$.*

If a mapping $c : V(G) \to [k]$ satisfies only (C1), it is called a proper $k$-coloring of $G$.

The chromatic number $\chi(G)$ of $G$ is the smallest $k$ for which $G$ admits a proper $k$-coloring. The $r$-hued chromatic number $\chi_r(G)$ is the smallest $k$ for which $G$ admits a $(k, r)$-coloring.

By Definition 1.1, $\chi(G) = \chi_1(G)$, so $\chi_r(G)$ can be regarded as a generalization of classical graph coloring. The following series of inequalities on the $r$-hued chromatic number were first established in [5]:

Let $r \geq 1$ be an integer. Then

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_{r-1}(G) \leq \chi_r(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi_{\Delta(G)+1}(G) = \cdots.$$

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Li et al. [7] proved that, given a graph $G$ and positive integers $k$ and $r$, deciding whether $G$ admits a $(k, r)$-coloring is NP-complete. Since every finite graph with $n$ vertices always admits an $(n, r)$-coloring, the upper bound $k$ on the number of colors is an essential part of the input. In view of this computational complexity, research has focused on estimating upper and lower bounds of the chromatic number, or on determining exact values for graph classes with specific structural properties. Moreover, concerning $r$-hued coloring, the following lower bound and upper bound for every graph $G$ were given in [5].

**Theorem 1.1.** *Let $r \geq 1$ be an integer. Then* $\min\{r, \Delta(G)\} + 1 \leq \chi_r(G) \leq |V(G)|$.

In [5], the following two theorems were given for $\chi_r(G)$ achieving the upper and lower bounds, respectively.

**Theorem 1.2** (see [5]). *Let $G$ be a connected graph and $|V(G)| = n$. For any $r \geq 2$, $\chi_r(G) = |V(G)|$ if and only if any two nonadjacent vertices of $G$ are adjacent to a vertex of degree at most $r$.*

**Theorem 1.3** (see [5]). *If $G$ is a tree, then $\chi_r(G) = \min\{r, \Delta(G)\} + 1$.*

Brooks' Theorem [2] is a fundamental result in the classical vertex coloring theory of graphs. For any connected graph $G$, its chromatic number $\chi(G)$ does not exceed the maximum degree $\Delta(G)$ plus one, and equality holds if and only if $G$ is a complete graph or an odd cycle. This result has been extended to the $r$-hued chromatic number and related parameters. In [6], Lai, Montgomery and Poon proved that if $\Delta(G) \leq 3$ and $G \neq C_5$, then $\chi_2(G) \leq 4$. Otherwise, when $\Delta(G) \geq 4$, $\chi_2(G) \leq \Delta(G) + 1$. Furthermore, Lin [8] and Wang [9] established that $\chi_r(G) \leq r\Delta(G) + 1$, with equality if and only if $r = \Delta(G)$ and $G$ is a Moore graph. In [4], Karpov proved that if $\Delta(G) \geq 8$, then $\chi_2(G) \leq \Delta(G)$ for connected graphs without vertices of degree 2.

For the cycle graph $C_n$ of order $n$, its chromatic number is given by the following theorem.

**Theorem 1.4.** *Let $n \geq 3$ be an integer. Then*

$$\chi(C_n) = \begin{cases} 2, & n \equiv 0 \pmod 2; \\ 3, & n \equiv 1 \pmod 2. \end{cases}$$

The $r$-hued chromatic number of cycles is known as follows.

**Theorem 1.5** (see [5]). *Let $n \geq 3$ be an integer and $C_n$ the cycle of order $n$. For $r \geq 2$,*

$$\chi_r(C_n) = \begin{cases} 5, & n = 5; \\ 3, & n \equiv 0 \pmod 3; \\ 4, & \textit{otherwise}. \end{cases}$$

A comprehensive overview of existing results on $r$-hued colorings can be found in the recent survey presented by Chen, Fan, Lai and Xu [3].

**Definition 1.2.** *The complement of a graph $G$, denoted by $\overline{G}$, is the simple graph with vertex set $V(G)$, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.*

In 1956, Nordhaus and Gaddum [12] studied the chromatic number of a graph $G$ and its complement $\overline{G}$ together. Motivated by the $r$-hued coloring of cycles and properties of complements, we obtain the following main results.

**Theorem 1.6.** *Let $n \geq 3$ be an integer and $\overline{C_n}$ the complement of the cycle $C_n$. Then*

$$\chi(\overline{C_n}) = \begin{cases} 1, & n = 3; \\ \left\lceil \frac{n}{2} \right\rceil, & n \geq 4. \end{cases}$$

**Theorem 1.7.** *Let $n \geq 3$ be an integer and $\overline{C_n}$ the complement of the cycle $C_n$. For $r \geq 2$,*

$$\chi_r(\overline{C_n}) = \begin{cases} 1, & n = 3; \\ 2, & n = 4; \\ n, & 5 \leq n \leq r+3; \\ r+3, & r+4 \leq n < \left\lceil \frac{4r+8}{3} \right\rceil; \\ r+2, & \left\lceil \frac{4r+8}{3} \right\rceil \leq n \leq 2r+1; \\ \left\lceil \frac{n}{2} \right\rceil, & n \geq 2r+2. \end{cases}$$

## 2. The $r$-Hued Chromatic Number of the Complement of Cycles

According to the definition of complement and the structure of $\overline{C_n}$, we have the following simple observations.

**Lemma 2.1.** *For positive integers $n$ and $r$, $\overline{C_n}$ satisfies the following properties.*

*(i) If $n \geq 3$, then $\overline{C_n}$ is $(n-3)$-regular.*

*(ii) If $n \geq 4$, then in any proper coloring of $\overline{C_n}$, at most two vertices can share the same color. In other words, each color appears at most twice.*

*(iii) If $n \geq 4$, then $\chi_r(\overline{C_n}) \geq \left\lceil \frac{n}{2} \right\rceil$.*

*(iv) If $n \geq 5$, then $\operatorname{diam}(\overline{C_n}) \leq 2$.*

**Proof.** We prove each part separately.

(i) The cycle $C_n$ is 2-regular, so $d_{C_n}(v) = 2$ for any $v \in V(C_n)$. By Definition 1.2, for any $v \in V(C_n)$,

$$d_{\overline{C_n}}(v) = n - 1 - d_{C_n}(v) = n - 3.$$

(ii) Suppose, to the contrary, that three distinct vertices $x, y, z$ share the same color. Since the coloring is proper, $x, y, z$ are pairwise nonadjacent in $\overline{C_n}$. Nonadjacency in $\overline{C_n}$ is equivalent to adjacency in $C_n$. Thus $x, y, z$ form a triangle in $C_n$, which is impossible because $C_n$ contains no triangles for $n \geq 4$. Hence each color can appear at most twice.

(iii) From part (ii), each color class contains at most two vertices. With $n$ vertices in total, any proper coloring requires at least $\left\lceil \frac{n}{2} \right\rceil$ colors, then $\chi_r(\overline{C_n}) \geq \chi(\overline{C_n}) \geq \left\lceil \frac{n}{2} \right\rceil$. Hence, for any integer $r \geq 1$, $\chi_r(\overline{C_n}) \geq \left\lceil \frac{n}{2} \right\rceil$.

(iv) Take any two distinct vertices $u, v \in V(\overline{C_n})$. If they are adjacent in $\overline{C_n}$, their distance is 1. If they are nonadjacent in $\overline{C_n}$, then they are adjacent in $C_n$ by Definition 1.2. In $C_n$, let $u'$ be the neighbor of $u$ different from $v$, and $v'$ the neighbor of $v$ different from $u$. Since $n \geq 5$, the set $V(C_n) \setminus \{u, v, u', v'\}$ is nonempty. Choose any $w$ in this set, then $w$ is adjacent to neither $u$ nor $v$ in $C_n$, so in $\overline{C_n}$ it is adjacent to both. Thus $uwv$ is a path of length 2, showing that the distance between $u$ and $v$ is 2. Thus, $\operatorname{diam}(\overline{C_n}) \leq 2$.

□

**Proof of Theorem 1.6.** When $n = 3$, $\overline{C_3}$ consists of three isolated vertices, so $\chi(\overline{C_3}) = 1$. Now we discuss the case $n \geq 4$. By Lemma 2.1(iii), for $r \geq 1$, $\chi_r(\overline{C_n}) \geq \lceil \frac{n}{2} \rceil$. Also, $\chi(\overline{C_n}) = \chi_1(\overline{C_n})$. Hence $\chi(\overline{C_n}) \geq \lceil \frac{n}{2} \rceil$.

Let $V(\overline{C_n}) = \{v_1, v_2, \ldots, v_n\}$ be arranged in clockwise order. If $n$ is even, define

$$c(v_1) = c(v_2) = 1,\ c(v_3) = c(v_4) = 2,\ \ldots,\ c(v_{n-1}) = c(v_n) = \frac{n}{2} = \left\lceil \frac{n}{2} \right\rceil.$$

By the definition of $c$, (C1) holds. If $n$ is odd, define

$$c(v_1) = c(v_2) = 1,\ c(v_3) = c(v_4) = 2,\ \ldots,\ c(v_{n-2}) = c(v_{n-1}) = \frac{n-1}{2},\ c(v_n) = \frac{n-1}{2} + 1 = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil.$$

By the definition of $c$, (C1) holds. Thus, $\chi(\overline{C_n}) \leq \left\lceil \frac{n}{2} \right\rceil$. In conclusion, $\chi(\overline{C_n}) = \left\lceil \frac{n}{2} \right\rceil$. □

**Proof of Theorem 1.7.** Let $V(\overline{C_n}) = \{v_1, v_2, \ldots, v_n\}$ be arranged in clockwise order.

**Case 1:** $n = 3$.
The graph $\overline{C_3}$ consists of three isolated vertices, so $\chi_r(\overline{C_3}) = 1$.

**Case 2:** $n = 4$.
For $\overline{C_4}$, we denote $V(\overline{C_4}) = \{v_1, v_2, v_3, v_4\}$ and $E(\overline{C_4}) = \{v_1v_3, v_2v_4\}$. Define a coloring $c$ of $\overline{C_4}$ as $c(v_1) = c(v_2) = 1$, $c(v_3) = c(v_4) = 2$. Conditions (C1) and (C2) hold under coloring $c$, so $\chi_r(\overline{C_4}) \leq 2$. By Theorem 1.1, $\chi_r(\overline{C_4}) \geq \min\{r, 1\} + 1 = 2$. So $\chi_r(\overline{C_4}) = 2$.

**Case 3:** $5 \leq n \leq r + 3$.
By Lemma 2.1 (iv), $\operatorname{diam}(\overline{C_n}) \leq 2$, and so any two nonadjacent vertices have a common neighbor. By Lemma 2.1 (i), for any vertex v of $\overline{C_n}$, $d_{\overline{C_n}}(v) = n - 3 \leq r$. It follows that any two nonadjacent vertices have a common neighbor of degree at most $r$. By Theorem 1.2, $\chi_r(\overline{C_n}) = |V(\overline{C_n})| = n$.

**Case 4:** $r+4 \le n \le 2r+1$.

Let $c$ be a coloring of $\overline{C_n}$. By Lemma 2.1 (ii), at most two vertices can share the same color. A color class of size two consists of two consecutive vertices in $C_n$; we call such a pair a *vertex block*. A color class of size one is called a *singleton color class*. Before proving Case 4, we first prove the following claim.

**Claim.** If $r+4 \le n \le 2r+1$, then $\chi_r(\overline{C_n}) \ge r+2$.

When $n \ge r+4$, by Theorem 1.1 and Lemma 2.1 (i), we have

$$\chi_r(\overline{C_n}) \ge \min\{r, \Delta(\overline{C_n})\} + 1 = \min\{r, n-3\} + 1 = r+1.$$

Suppose that an $(r+1,r)$-coloring exists. Since every color class has size at most two and $n \le 2r+1 < 2(r+1)$, there must be a singleton color class; let $v_i$ be its vertex. Consider $v_{i-1}$. The color of $v_i$ is absent from $N_{\overline{C_n}}(v_{i-1})$. The color of $v_{i-1}$ is also absent from this neighborhood: if that color is repeated, its second occurrence can only be at $v_{i-2}$, because $v_i$ is a singleton. Hence at least two of the $r+1$ colors are absent from $N_{\overline{C_n}}(v_{i-1})$, so $|c(N_{\overline{C_n}}(v_{i-1}))| \le r-1$, contradicting (C2). Thus no $(r+1,r)$-coloring exists.

Therefore, the claim is proved.

**Case 4.1:** $r+4 \le n < \lceil \frac{4r+8}{3} \rceil$.

Let $a = n-r-3 \ge 1$. Since $n < \lceil \frac{4r+8}{3} \rceil$, we have

$$n - 4a = 4r - 3n + 12 > 4r - 3 \cdot \left\lceil \frac{4r+8}{3} \right\rceil + 12.$$

Let $k$ be a positive integer. If $r = 3k$, then

$$\left\lceil \frac{4r+8}{3} \right\rceil = \left\lceil \frac{12k+8}{3} \right\rceil = \left\lceil \frac{12k+8+1}{3} \right\rceil = \frac{4r+8}{3} + \frac{1}{3}.$$

If $r = 3k+1$, then

$$\left\lceil \frac{4r+8}{3} \right\rceil = \left\lceil \frac{12k+12}{3} \right\rceil = \frac{12k+12}{3} = \frac{4r+8}{3}.$$

If $r = 3k+2$, then

$$\left\lceil \frac{4r+8}{3} \right\rceil = \left\lceil \frac{12k+16}{3} \right\rceil = \left\lceil \frac{12k+16+2}{3} \right\rceil = \frac{4r+8}{3} + \frac{2}{3}.$$

Hence, $\left\lceil \frac{4r+8}{3} \right\rceil \le \frac{4r+8}{3} + \frac{2}{3}$. Thus,

$$n - 4a > 4r - 3 \cdot \left\lceil \frac{4r+8}{3} \right\rceil + 12 \ge 4r - 3 \cdot \left( \frac{4r+8}{3} + \frac{2}{3} \right) + 12 = 2.$$

We provide explicit colorings for each parity of $n$.

**Case 4.1.1:** $n$ is even.

Here, we denote $S_1 = \{v_1, v_2\}$, $T_1 = \{v_3, v_4\}$, $\ldots$, $S_a = \{v_{4a-3}, v_{4a-2}\}$, $T_a = \{v_{4a-1}, v_{4a}\}$, $T_{a+1} = \{v_{4a+1}, v_{4a+2}\}$, $\ldots$, $T_b = \{v_{n-1}, v_n\}$. Since $2a + 2b = n$, $a = n-r-3$, we have $b = \frac{2r+6-n}{2}$. Define

$$\begin{cases} c(S_i) = \{i\}, & 1 \le i \le a; \\ c(T_j) = \{a+2j-1,\ a+2j\}, & 1 \le j \le b. \end{cases}$$

Here $S_i$ is a vertex block, $T_j$ is a set of singleton color classes. And $|c(V(\overline{C_n}))| = a + 2b = n - r - 3 + 2 \cdot \frac{2r+6-n}{2} = r+3$. By the definition of $c$, two adjacent vertices in $\overline{C_n}$ have different colors, so (C1) holds. For any vertex $v \in V(\overline{C_n})$, if $v \in S_i$ $(1 \le i \le a)$, then among the two vertices not adjacent to $v$ in $\overline{C_n}$, one lies in the same vertex block as $v$ and the other is a singleton color class, so $|c(N_{\overline{C_n}}(v))| = r+3-2 = r+1 > r$, and (C2) holds. If $v \in T_j$ $(1 \le j \le b)$, then the two vertices not adjacent to $v$ in $\overline{C_n}$ are either both singleton color classes or one from a vertex block and one singleton color class. Hence $|c(N_{\overline{C_n}}(v))| \ge r+3-3 = r$, and (C2) holds. Thus, $c$ is an $(r+3,r)$-coloring and $\chi_r(\overline{C_n}) \le r+3$.

**Case 4.1.2:** $n$ is odd.

Here, we denote $S_1 = \{v_1, v_2\}$, $T_1 = \{v_3, v_4\}$, $\ldots$, $S_a = \{v_{4a-3}, v_{4a-2}\}$, $T_a = \{v_{4a-1}, v_{4a}\}$, $T_{a+1} = \{v_{4a+1}, v_{4a+2}\}$, $\ldots$, $T_{b-1} = \{v_{n-2}, v_{n-1}\}$, $T_b = \{v_n\}$. Since $2a + 2b - 1 = n$, $a = n-r-3$, we have $b = \frac{2r+7-n}{2}$. Define

$$\begin{cases} c(S_i) = \{i\}, & 1 \le i \le a; \\ c(T_j) = \{a+2j-1,\ a+2j\}, & 1 \le j \le b-1; \\ c(T_j) = \{a+2b-1\}, & j = b. \end{cases}$$

Here $S_i$ is a vertex block, $T_j$ is a set of singleton color classes. Also, $|c(V(\overline{C_n}))| = a+2b-1 = n-r-3+2\cdot\frac{2r+7-n}{2}-1 = r+3$. By the definition of $c$, two adjacent vertices in $\overline{C_n}$ have different colors, so (C1) holds. For any vertex $v \in V(\overline{C_n})$, if $v \in S_i$ $(1 \le i \le a)$, then among the two vertices not adjacent to $v$ in $\overline{C_n}$, one lies in the same vertex block as $v$ and the other is a singleton color class, so $|c(N_{\overline{C_n}}(v))| = r+3-2 = r+1 > r$, and (C2) holds. If $v \in T_j$ $(1 \le j \le b)$, then the two vertices not adjacent to $v$ in $\overline{C_n}$ are either both singleton color classes or one from a vertex block and one singleton color class. Hence $|c(N_{\overline{C_n}}(v))| \ge r+3-3 = r$, and (C2) holds. Thus, $c$ is an $(r+3,r)$-coloring and $\chi_r(\overline{C_n}) \le r+3$.

Consequently, we have $\chi_r(\overline{C_n}) \le r+3$ when $r+4 \le n < \lceil\frac{4r+8}{3}\rceil$. By Claim, $\chi_r(\overline{C_n}) \ge r+2$. If $\chi_r(\overline{C_n}) = r+2$, then there is an $(r+2,r)$-coloring $c_0$ of $\overline{C_n}$. By Lemma 2.1(ii), each color appears at most twice. Let $x$ be the number of vertex blocks and let $y$ be the number of vertices in singleton color classes. Then $n = 2x+y$ and $x+y = r+2$, which yield $x = n-r-2$ and $y = 2r+4-n$. If no three consecutive vertices are all in singleton color classes, then the $y$ vertices in singleton color classes must be separated by the $x$ vertex blocks. Since each vertex block consists of two consecutive vertices, it can separate at most two singleton vertices, one on each side. Hence $y \le 2x$. Substituting the expressions for $x$ and $y$ gives $2r+4-n \le 2(n-r-2)$, so $n \ge \frac{4r+8}{3}$ (which implies $n \ge \lceil\frac{4r+8}{3}\rceil$), contradicting the hypothesis $n < \lceil\frac{4r+8}{3}\rceil$. Therefore, there exist three consecutive vertices, say $v_{i-1}, v_i, v_{i+1}$, each being in a singleton color class. Hence $|c_0(N_{\overline{C_n}}(v_i))| = (r+2)-3 = r-1 < r$, contradicting (C2). Thus, $\chi_r(\overline{C_n}) \ge r+3$. Consequently, when $r+4 \le n < \lceil\frac{4r+8}{3}\rceil$, we obtain $\chi_r(\overline{C_n}) = r+3$.

To facilitate the understanding of the proof process, we give Figure 2.1, where numbers denote the color assigned to each vertex.

**Case 4.2:** $\lceil\frac{4r+8}{3}\rceil \le n \le 2r+1$.

**Case 4.2.1:** $n$ is even.

Here $\lceil\frac{4r+8}{3}\rceil \le n \le 2r$. Let $b = \frac{2r-n+4}{2}$. Then $b \ge \frac{2r-2r+4}{2} = 2$ and

$$n-4b = 3n-4r-8 \ge 3\cdot\left\lceil\frac{4r+8}{3}\right\rceil - 4r-8 \ge 3\cdot\frac{4r+8}{3} - 4r-8 = 0.$$

As $n$ is even. Denote

$$S_1 = \{v_1, v_2\},\ T_1 = \{v_3, v_4\},\ S_2 = \{v_5, v_6\},\ T_2 = \{v_7, v_8\},\ \ldots,$$
$$S_b = \{v_{4b-3}, v_{4b-2}\},\ T_b = \{v_{4b-1}, v_{4b}\},\ S_{b+1} = \{v_{4b+1}, v_{4b+2}\},$$
$$S_{b+2} = \{v_{4b+3}, v_{4b+4}\},\ \ldots,\ S_a = \{v_{n-1}, v_n\}.$$

If $n-4b = 0$, then $T_b = \{v_{n-1}, v_n\}$. Since $2a+2b = n$ and $b = \frac{2r-n+4}{2}$, $a = n-r-2$. Define

$$\begin{cases} c(S_i) = \{i\}, & 1 \le i \le a; \\ c(T_j) = \{a+2j-1,\ a+2j\}, & 1 \le j \le b. \end{cases}$$

Here $S_i$ is a vertex block, $T_j$ is a set of singleton color classes. Also, $|c(V(\overline{C_n}))| = a+2b = n-r-2+2\cdot\frac{2r-n+4}{2} = r+2$. By the definition of $c$, two adjacent vertices in $\overline{C_n}$ have different colors, so (C1) holds. For any vertex $v \in V(\overline{C_n})$, if $v \in S_i$ $(1 \le i \le a)$, the two vertices not adjacent to $v$ in $\overline{C_n}$ are either both from vertex blocks or one from a block and one singleton color class. Thus, $|c(N_{\overline{C_n}}(v))| \ge r+2-2 = r$, and (C2) holds. If $v \in T_j$ $(1 \le j \le b)$, then the two vertices not adjacent to $v$ in $\overline{C_n}$ are one from a vertex block and one singleton color class, so $|c(N_{\overline{C_n}}(v))| = r+2-2 = r$, and (C2) holds. Hence $c$ is an $(r+2,\ r)$-coloring, and $\chi_r(\overline{C_n}) \le r+2$.

**Case 4.2.2:** $n$ is odd.

Let $b = \frac{2r-n+5}{2}$. Then $b \ge \frac{2r-(2r+1)+5}{2} = 2$ and $n-4b+1 = 3n-4r-9$. Let $k$ be any positive integer.

If $r = 3k$, then

$$n \ge \left\lceil\frac{4r+8}{3}\right\rceil = \left\lceil\frac{12k+8}{3}\right\rceil = 4k + \left\lceil\frac{8}{3}\right\rceil = 4k+3.$$

Thus, $n-4b+1 = 3n-4r-9 \ge 3(4k+3)-12k-9 = 0$.

If $r = 3k+1$, then

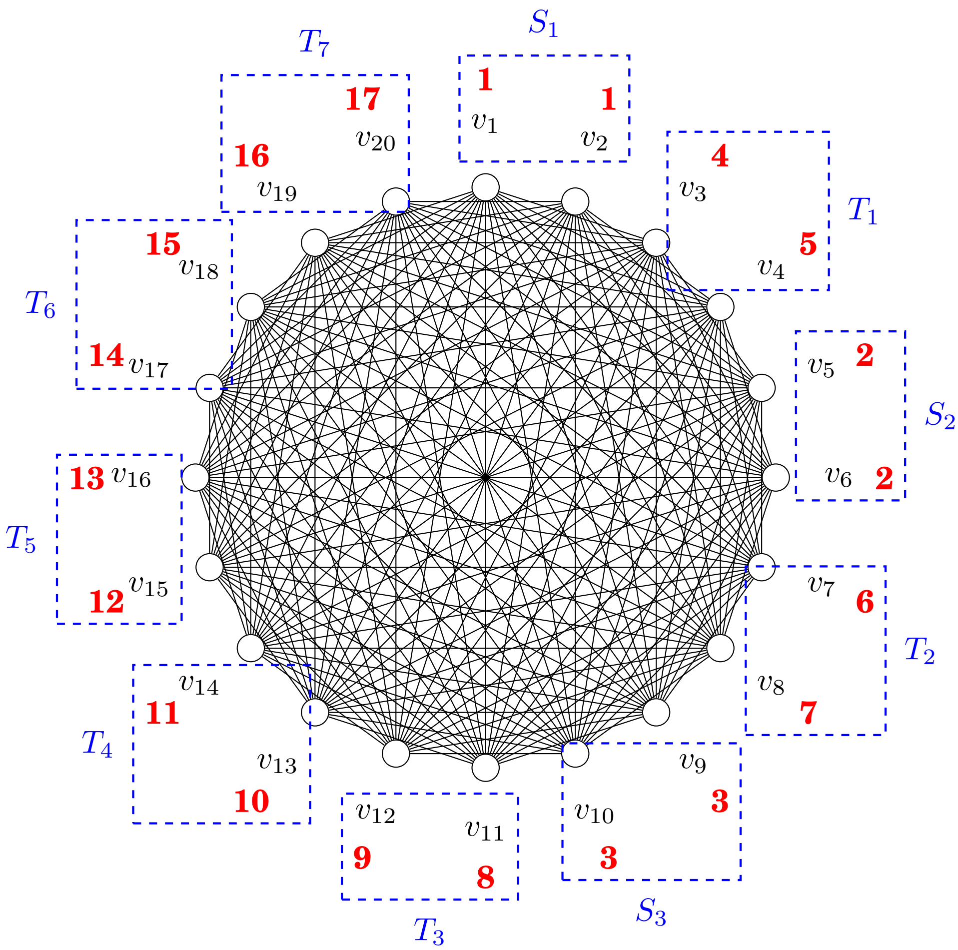
$$n \ge \left\lceil\frac{4r+8}{3}\right\rceil = \left\lceil\frac{12k+12}{3}\right\rceil = 4k+4.$$

Since $n$ is an odd integer, we have $n \ge 4k+5$. Then $n-4b+1 = 3n-4r-9 \ge 3(4k+5)-12k-4-9 = 2 > 0$.

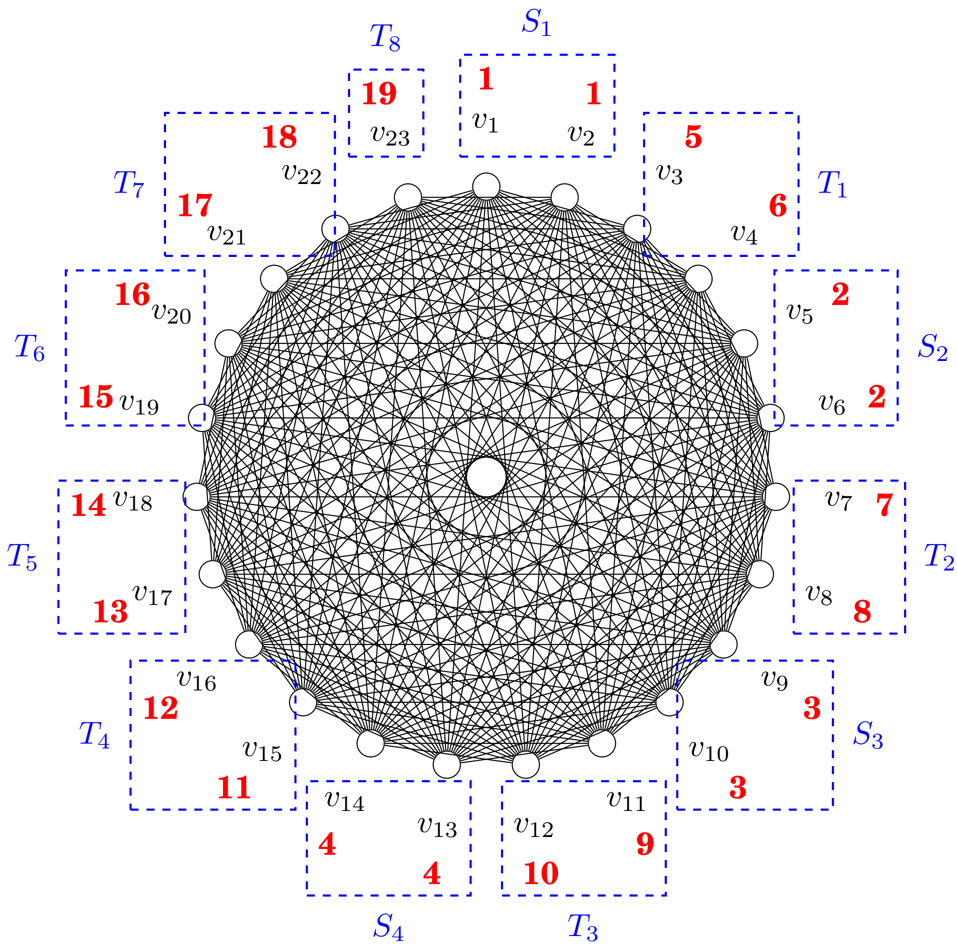
If $r = 3k+2$, then

$$n \ge \left\lceil\frac{4r+8}{3}\right\rceil = \left\lceil\frac{12k+16}{3}\right\rceil = 4k + \left\lceil\frac{16}{3}\right\rceil = 4k+6.$$

Since $n$ is an odd integer, we have $n \ge 4k+7$. Then $n-4b+1 = 3n-4r-9 \ge 3(4k+7)-12k-8-9 = 4 > 0$.

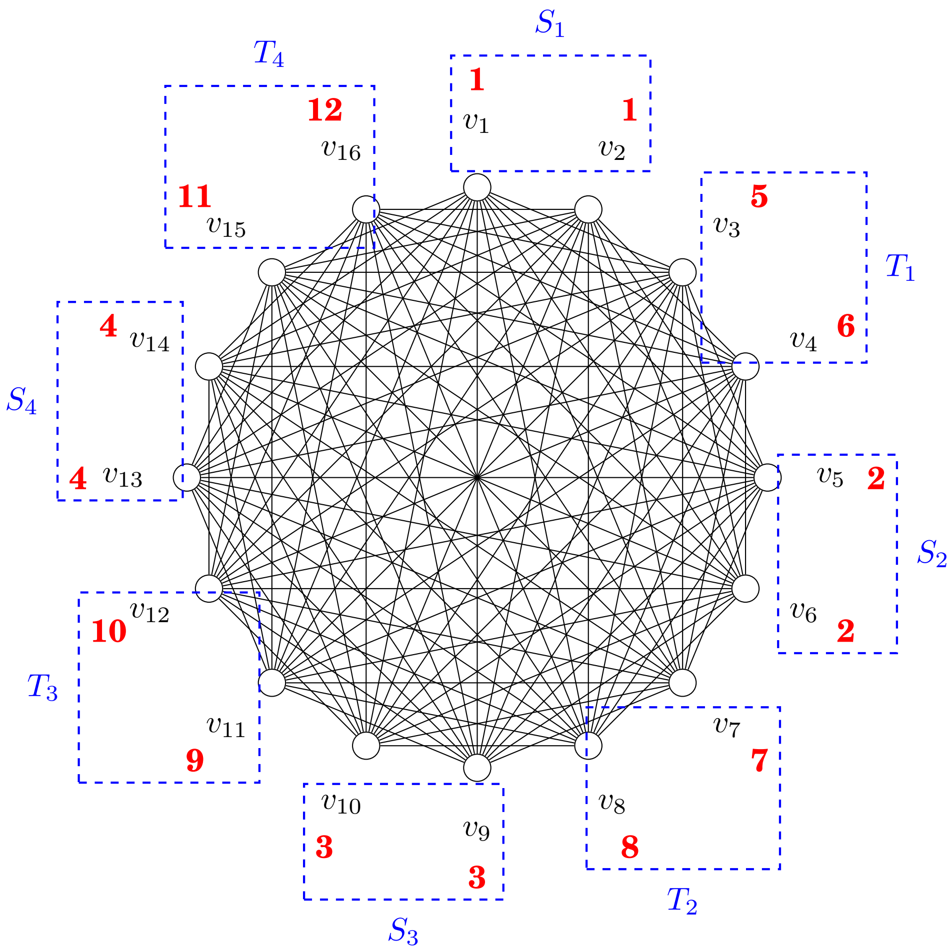



**(a)** $\overline{C}_{20}$, $r = 14$

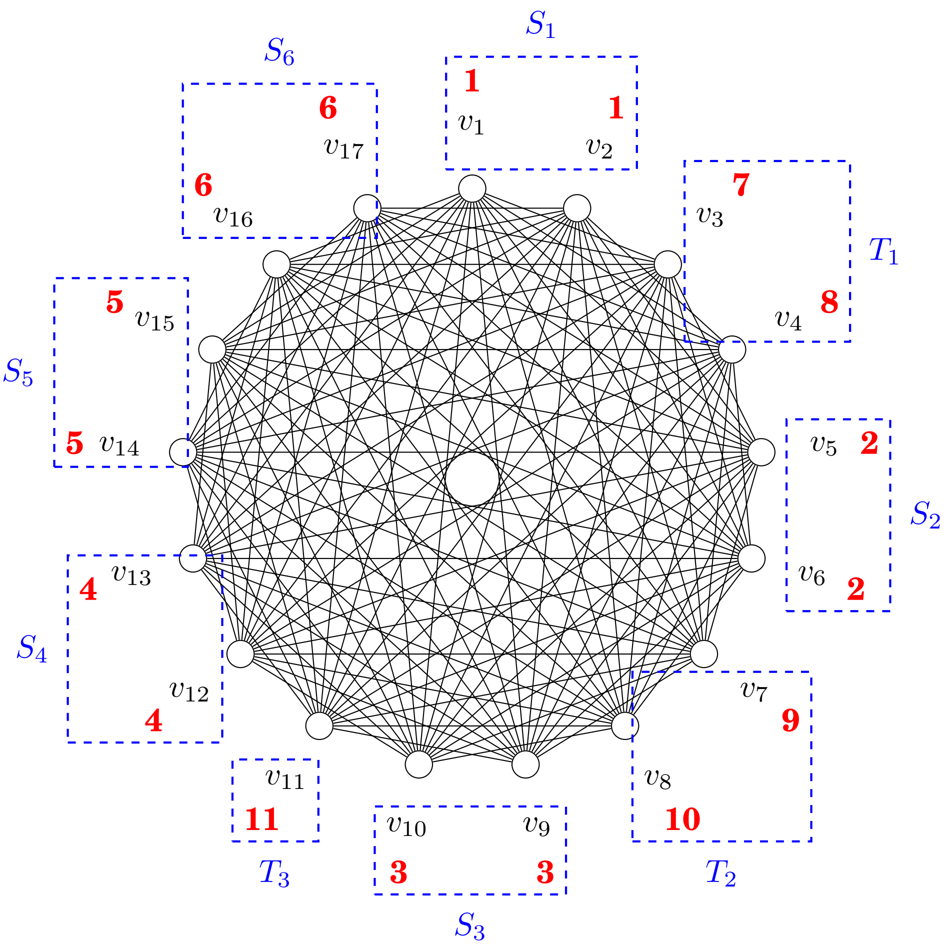



**(b)** $\overline{C}_{23}$, $r = 16$

**Figure 2.1:** The $r$-hued coloring for the case $r + 4 \leq n < \lceil \frac{4r+8}{3} \rceil$.




**(a)** $\overline{C}_{16}$, $r=10$




**(b)** $\overline{C}_{17}$, $r=9$

**Figure 2.2:** The $r$-hued coloring for the case $\lceil \frac{4r+8}{3} \rceil \le n \le 2r+1$.

Thus, when $\lceil\frac{4r+8}{3}\rceil \le n \le 2r+1$, we have $n-4b+1\ge 0$. Denote

$$S_1=\{v_1,v_2\},\ T_1=\{v_3,v_4\},\ S_2=\{v_5,v_6\},\ T_2=\{v_7,v_8\},\ \ldots,$$
$$S_b=\{v_{4b-3},v_{4b-2}\},\ T_b=\{v_{4b-1}\},\ S_{b+1}=\{v_{4b},v_{4b+1}\},$$
$$S_{b+2}=\{v_{4b+2},v_{4b+3}\},\ \ldots,\ S_a=\{v_{n-1},v_n\}.$$

If $n-4b+1=0$, then $T_b=\{v_n\}$. Since $2a+2b-1=n$ and $b=\frac{2r-n+5}{2}$, $a=n-r-2$. Recall that $n$ is odd. Define

$$\begin{cases} c(S_i)=\{i\}, & 1\le i\le a;\\ c(T_j)=\{a+2j-1,\ a+2j\}, & 1\le j\le b-1;\\ c(T_j)=\{a+2j-1\}, & j=b. \end{cases}$$

Here $S_i$ is a vertex block and $T_j$ is a set of singleton color classes. Also, $|c(V(\overline{C_n}))|=a+2b-1=n-r-2+2\cdot\frac{2r-n+5}{2}-1=r+2$. By the definition of $c$, two adjacent vertices in $\overline{C_n}$ have different colors, so (C1) holds. For any vertex $v\in V(\overline{C_n})$, if $v\in S_i$ $(1\le i\le a)$, the two vertices not adjacent to $v$ in $\overline{C_n}$ are either both from vertex blocks or one from a vertex block and one singleton color class. Therefore $|c(N_{\overline{C_n}}(v))|\ge r+2-2=r$, and (C2) holds. If $v\in T_j$ $(1\le j\le b-1)$, then the two vertices not adjacent to $v$ in $\overline{C_n}$ are one from a vertex block and one singleton color class, so $|c(N_{\overline{C_n}}(v))|=r+2-2=r$, and (C2) holds. If $v\in T_b$, then $|c(N_{\overline{C_n}}(v))|=r+2-1=r+1>r$, and (C2) holds. Thus, $c$ is an $(r+2,r)$-coloring, and $\chi_r(\overline{C_n})\le r+2$.

Consequently, we have $\chi_r(\overline{C_n})\le r+2$ for $\lceil\frac{4r+8}{3}\rceil\le n\le 2r+1$. By Claim, $\chi_r(\overline{C_n})\ge r+2$. So we obtain $\chi_r(\overline{C_n})=r+2$.

To facilitate the understanding of the proof process, we give Figure 2.2, where numbers denote the color assigned to each vertex.

**Case 5:** $n\ge 2r+2$.

**Case 5.1:** $n$ is even.

Let $a=\frac{n}{2}$. Then $a\ge\frac{2r+2}{2}=r+1$. Denote $S_1=\{v_1,v_2\},\ S_2=\{v_3,v_4\},\ \ldots,\ S_a=\{v_{n-1},v_n\}$. Define $c(S_i)=\{i\}$ for $1\le i\le a$, so $|c(V(\overline{C_n}))|=a=\frac{n}{2}=\lceil\frac{n}{2}\rceil$. By the definition of $c$, two adjacent vertices in $\overline{C_n}$ have different colors, so (C1) holds. For any vertex $v\in V(\overline{C_n})$, $v\in S_i(1\le i\le a)$ and $|c(N_{\overline{C_n}}(v))|=a-1\ge r+1-1=r$. So (C2) holds. Hence $c$ is a $(\lceil\frac{n}{2}\rceil,r)$-coloring, and $\chi_r(\overline{C_n})\le\lceil\frac{n}{2}\rceil$.

**Case 5.2:** $n$ is odd.

Now $n\ge 2r+3$. Let $a=\frac{n-1}{2}$. Then $a\ge\frac{2r+3-1}{2}=r+1$. Denote $S_1=\{v_1,v_2\},\ T_1=\{v_3\},\ S_2=\{v_4,v_5\},\ \ldots,\ S_a=\{v_{n-1},v_n\}$. Define

$$\begin{cases} c(S_i)=\{i\}, & 1\le i\le a;\\ c(T_1)=c(v_3)=\{a+1\}. \end{cases}$$

Then $|c(V(\overline{C_n}))|=a+1=\frac{n-1}{2}+1=\frac{n+1}{2}=\lceil\frac{n}{2}\rceil$. By the definition of $c$, two adjacent vertices in $\overline{C_n}$ have different colors, so (C1) holds. For any vertex $v\in V(\overline{C_n})$, if $v\in S_i$ $(1\le i\le a)$, the two vertices not adjacent to $v$ are either both from vertex blocks or one singleton color class and one from a vertex block, therefore $|c(N_{\overline{C_n}}(v))|\ge\frac{n+1}{2}-2=\frac{n-3}{2}\ge\frac{2r+3-3}{2}=r$, so (C2) holds. If $v=v_3\in T_1$, then the two vertices not adjacent to $v$ are both from vertex blocks, and we obtain $|c(N_{\overline{C_n}}(v))|=\frac{n+1}{2}-1\ge r+1>r$. Thus (C2) holds. Consequently, $c$ is a $(\lceil\frac{n}{2}\rceil,r)$-coloring. So $\chi_r(\overline{C_n})\le\lceil\frac{n}{2}\rceil$.

For $n\ge 2r+2$, we obtain $\chi_r(\overline{C_n})\le\lceil\frac{n}{2}\rceil$, and by Lemma 2.1(iii), $\chi_r(\overline{C_n})\ge\lceil\frac{n}{2}\rceil$. Thus, $\chi_r(\overline{C_n})=\lceil\frac{n}{2}\rceil$. □

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