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Research Article

Hitting-Time Index of Trees: Bounds, Relationships, and Exact Formulas

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Abstract

The hitting-time index is a recently introduced topological molecular descriptor based on graph distance. Its lower and upper bounds for trees, in terms of other graph invariants, are derived. In particular, the hitting-time index is related to the eccentric connectivity index, total eccentricity, eccentric distance sum, and degree distance index. Additionally, a closed formula for the hitting-time index of double brooms is obtained.

Keywords: hitting-time index; distance; tree; double broom.

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1. Introduction

The hitting-time index is a recently introduced topological descriptor related to the well-known Kirchhoff index and related indices. Distinct features of this index have initiated a more detailed investigation. One of the common questions that appear in this kind of investigation is whether there exists a relation between the hitting-time index and the other already introduced topological invariants. Some relations have already been established in [6, 11]. Here, the hitting index is compared with several well-known distance-based topological invariants, and achievable bounds are derived.

The results in this article are valid for a simple connected graph G with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex v in a graph G, denoted by $deg(v)$, is the number of neighbors of v in G. The *minimum* and *maximum degree* of the vertices of G are denoted by δ and Δ, respectively. For any two vertices $u, v \in V(G)$, the *distance* $d(u, v)$ is defined as the length of a shortest path between u and v in G. For integers s, t, we define the set $[s, t] = \{x \in \mathbb{Z} \mid s \leqslant x \leqslant t\}$.

The simple random walk in G is defined as the Markov chain $\{X_n \mid n \geqslant 0\}$ whose state space is V and whose transition probabilities are defined as uniform, from a vertex v to any of its neighboring vertices. The *hitting time* T_u of the vertex u is defined as the smallest number of jumps needed by the random walk to reach the vertex u: $T_u = \inf\{n \geqslant 0 : X_n = u\}$. Its expected value when the process starts in state v is denoted by $E_v T_u$. Remark that $E_u T_u = 0$, and this should not be confused with the mean return time to vertex u, $E_u T_u^+ = \frac{2|E|}{deg(u)}$, which involves $T_u^+ = \inf\{n \geqslant 1 : X_n = u\}$. More information on hitting times of Markov chains can be found in [4]. A recent probabilistic/electrical index, *the random walk index*, was proposed in [2].

For a vertex pair v and u, a battery is placed between v and u so that $1\,\mathrm{A}$ of current enters v and exits u. This generates a voltage U_w^{vu} at each vertex $w \in V(G)$ and a potential difference across any edge $xy \in E(G)$ given by $U_x^{vu} - U_y^{vu} = \Delta U_{xy}^{vu}$. The voltage at u is $U_u^{vu} = 0\,\mathrm{V}$. Let R_{vu} be the effective resistance between the vertices v and u when the graph is considered as an electrical network, where all edges have unit resistance, that is, the resistance of any edge xy is $R_{xy} = 1\,\Omega$. For $x, y \in V(G)$, we have $\Delta U_{xy}^{vu} = I_{xy} R_{xy}$, where I_{xy} is the current from x to y.

The hitting time of random walks on simple graphs in [13] was expressed as $\sum_{w \in V(G)} deg(w) \cdot U_w^{vu} = E_v\, T_u$, and can be rewritten as $\sum_{xy \in E(G)} (U_x^{vu} + U_y^{vu}) = E_v\, T_u$. The *hitting-time index* $HT(G)$ of a graph G was defined as [11],

$$HT(G) = \sum_{\{v,u\} \subseteq V(G)} D(v, u)$$

where $D(u, v) = \max\{E_v\, T_u, E_u\, T_v\}$.

Suppose that xy is a cut edge of G and $G_x = (V_x, E_x)$ is the connected subgraph of $G - xy$ that contains x.

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Theorem 1.1. [13] *Let T be a tree. For any $v, w \in V(T)$, let P be the unique path between v and w, then*

$$E_vT_w = d(v,w) + 2 \sum_{x \neq w \in V(P)} |E_x|. \tag{1}$$

In the proof of Theorem 1.1, the following results were used.

Theorem 1.2. [13]

(i) If vw is a cut edge of G, then $E_vT_w = 2|E_v| + 1$;

(ii) If $P = u_1u_2 \ldots u_m$ is a path, then $E_{u_1}T_{u_m} = \sum_{i=1}^{m-1} E_{u_i}T_{u_{i+1}}$.

A *double broom* $D(d,a,b)$ (see [3]) is a tree with $a+b+d-1$ vertices, consisting of a path P_{d-1} with $d-1$ vertices and $a+b$ pendent vertices, where a and b pendent vertices are adjacent to the respective end vertices of the path P_{d-1}. The double broom graph $D(d,a,b)$ has diameter d.

Let $B(a,b)$ (briefly B) be a *bistar graph* (see [6]) with vertex set $\{v_i \mid 1 \leqslant i \leqslant a+b\}$ where v_a and v_{a+1} are the central vertices with $deg(v_a) = a$ and $deg(v_{a+1}) = b$. Obviously, double broom $D(3, a-1, b-1)$ is the bistar graph $B(a,b)$. Mengya He et al. obtained the following:

Theorem 1.3. [6] *Let $b \geqslant a \geqslant 1$ be two integers. Then $HT(B(a,b)) = a^3 + b^3 - 2a^2 - 4b^2 + 5ab^2 + 3a^2b - 3ab - 5a - b + 6$.*

A *broom graph* $B_{n,d}$ (see [12]) with n vertices is made up of a path P_d with d vertices and $(n-d)$ pendant vertices attached to one of the ends of the path P_d. Note that the double broom $D(d,1,n-d)$ is the broom graph $B_{n,d}$. Palacios established the following:

Theorem 1.4. [12] *For $n \geq 6$, we have $HT(B_{n,3}) = n^3 - 7n + 2$.*

Theorem 1.5. [12] *For $n \geq 8$, we have $HT(B_{n,4}) = n^3 + 4n^2 - 33n + 34$.*

Palacios [11] exhibited the relations between the hitting-time index and previous probabilistic/electric indices such as the RW index and the Kirchhoff index. Additionally, he provided methods for computing this index for certain graph families.

The *Wiener index* [15] is the sum of the distances over all unordered pairs of vertices in a graph. More formally, for any vertices u, v, the Wiener index was defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)$. Here, the sharp bounds for the hitting-time index in terms of the eccentric connectivity index, the total eccentricity, and the eccentric distance sum are presented.

The *eccentricity* $ec(u)$ of a vertex u is the maximum distance from u to any other vertex in graph G. The *radius* $\mathrm{rad}(G)$ and *diameter* $\mathrm{diam}(G)$ of G are the minimum and maximum eccentricity among all vertices in G, respectively; that is, $\mathrm{rad}(G) = \min_{u \in V(G)} ec(u)$, $\mathrm{diam}(G) = \max_{u \in V(G)} ec(u)$. A vertex c of G is called *central* if $ec(c) = \mathrm{rad}(G)$. The *center* C is the set of all central vertices in G, that is, $C = \{c \in V(G) \mid ec(c) = \mathrm{rad}(G)\}$. If $S \subseteq V$, then the distance $d(u,S)$ from a vertex u to S is the minimum distance from u to some vertex in S, that is, $d(u,S) = \min\{d(v,u) \mid v \in S\}$. The sum of all vertex eccentricities in a graph G is the *total eccentricity* [3]: $\xi(G) = \sum_{u \in V(G)} ec(u)$. The *eccentric connectivity index* (see [1,8,9]) of a connected graph G, denoted by $\xi^c(G)$, is defined as $\xi^c(G) = \sum_{u \in V(G)} ec(u) \cdot deg(u)$. The *eccentric distance sum* [7,10] is defined as $\xi^d(G) = \sum_{u \in V(G)} ec(u) \cdot D(u)$, where $D(u)$ is the sum of distances between u and all other vertices in G. The *eccentric distance sum* has been used as a predictor of biological and physical properties [5]. It can be written as

$$\xi^d(G) = \sum_{v \in V(G)} ec(v) \cdot D(v) = \sum_{\{u,v\} \subseteq V(G)} (ec(v) + ec(u)) \cdot d(u,v).$$

The center of a graph is a vertex or set of vertices with minimal eccentricity. The following known results will be essential in proving our results.

Theorem 1.6. [14] *Let T be a tree, and $C(T)$ be its center. For any $v \in V(T)$,*

$$\sum_{u \in V(T)} d(u, C(T)) \leqslant \sum_{u \in V(T)} d(u,v).$$

Theorem 1.7. [3] *Let T be a tree. The eccentricity of any vertex $u \in V(T)$ is $ec(u) = \mathrm{rad}(T) + d(u, C(T))$.*

Theorem 1.8. [6] *Let T be a tree with vertex set $\{u_i \mid 1 \leqslant i \leqslant n\}$. Then*

$$HT(T) = W(T) + 2 \sum_{i<j} \max \left\{ \sum_{u_x \neq u_j \in V(P)} |E_{u_x}|, \sum_{u_y \neq u_i \in V(P)} |E_{u_y}| \right\},$$

where P is the unique path between u_i and u_j.

It is evident that $|E_{u_y}| = n - |E_{u_{y-1}}| - 2$ for all $u_y \neq u_i \in V(P)$. Hence,

$$\sum_{u_y \neq u_i \in V(P)} |E_{u_y}| = \sum_{u_x \neq u_j \in V(P)} (n - |E_{u_x}| - 2), \tag{2}$$

Combining Theorem 1.8 and Equation (2), the following corollary is obtained.

Corollary 1.1. *Let T be a tree with n vertices. For any $u_i, u_j \in V(T)$, if P is the unique path between u_i and u_j, then*

$$HT(T) = W(T) + 2\sum_{i<j} \max\left\{ \sum_{u_x \neq u_j \in V(P)} |E_{u_x}|, \sum_{u_x \neq u_j \in V(P)} (n - |E_{u_x}| - 2) \right\}.$$

For convenience, we define

$$X = \sum_{i<j} \max\left\{ \sum_{u_x \neq u_j \in V(P)} |E_{u_x}|, \sum_{u_x \neq u_j \in V(P)} (n - |E_{u_x}| - 2) \right\}. \tag{3}$$

Then $HT(T) = W(T) + 2X$.

2. Hitting-Time Index vs. Eccentricity-Based Indices

The lower and upper bounds of the hitting index of trees in terms of the total eccentricity index are given in Theorem 2.1.

Theorem 2.1. *Let T be a tree with $n \geqslant 2$ vertices. Then*

$$\frac{n\,\xi(T) - n^2\,\mathrm{rad}(T)}{2} + 2X < HT(T) < \frac{n\,\xi(T)}{2} + 2X.$$

Proof. First, we show the lower bound. For any $v \in V(T)$, one gets

$$\xi(T) = \sum_{u\in V(T)} ec(u) = \sum_{u \in V(T)} [\mathrm{rad}(T) + d(u, C(T))] < n\,\mathrm{rad}(T) + \sum_{u\in V(T)} d(u,v).$$

Furthermore, we have $\sum_{v\in V(T)} \xi(T) = n\,\xi(T) < n^2\,\mathrm{rad}(T) + 2\,W(T)$. According to Theorem 1.8, we get

$$HT(T) > \frac{n\,\xi(T) - n^2\,\mathrm{rad}(T)}{2} + 2X.$$

To show the upper bound, we take any $w \in V(T)$ and $\xi(T) = \sum_{u\in V(T)} ec(u) > \sum_{u\in V(T)} d(u,w)$. Furthermore, we obtain that

$$\sum_{w\in V(T)} \xi(T) = n\,\xi(T) > \sum_{w\in V(T)} \sum_{u \in V(T)} d(u,w) = 2\,W(T) = 2\,(HT(T) - 2X).$$

Hence, $HT(T) < \frac{n\,\xi(T)}{2} + 2X$. □

Theorem 2.2 shows the lower and upper bounds for the hitting-time index in terms of the eccentric connectivity index.

Theorem 2.2. *Let T be a tree with $n \geqslant 2$ vertices, the maximum degree Δ and the minimum degree δ. Then*

$$\frac{n\,\xi^c(T) - 2n(n-1)\,\mathrm{rad}(T)}{2\,\Delta} + 2X < HT(T) < \frac{n\,\xi^c(T)}{2\,\delta} + 2X.$$

Proof. Using Theorems 1.6 and 1.7 we obtain the lower bound:

$$\begin{aligned}\xi^c(T) &= \sum_{u\in V(T)} ec(u)\,deg(u) = \sum_{u\in V(T)} [\mathrm{rad}(T) + d(u, C(T))]\,deg(u) < 2(n-1)\,\mathrm{rad}(T) + \Delta \sum_{u\in V(T)} d(u, C(T)) \\ &< 2(n-1)\,\mathrm{rad}(T) + \Delta \sum_{u\in V(T)} d(u,v).\end{aligned}$$

Then, $\sum_{v\in V(T)} \xi^c(T) = n\,\xi^c(T) < 2n(n-1)\,\mathrm{rad}(T) + \Delta \sum_{v\in V(T)} \sum_{u\in V(T)} d(u,v) = 2n(n-1)\,\mathrm{rad}(T) + 2\,\Delta\,W(T)$. By applying Corollary 1.1, we obtain $n\,\xi^c(T) < 2n(n-1)\,\mathrm{rad}(T) + 2\,\Delta\,(HT(T) - 2X)$. Finally, we get the lower bound

$$HT(T) > \frac{n\,\xi^c(T) - 2n(n-1)\,\mathrm{rad}(T)}{2\,\Delta} + 2X.$$

The upper bound is derived from an obvious inequality. For any $w \in V(T)$, the following is valid:

$$\xi^c(T) = \sum_{u\in V(T)} ec(u)\,deg(u) > \sum_{u\in V(T)} d(u,w)\,deg(u).$$

Then, $\sum_{w\in V(T)} \xi^c(T) = n\,\xi^c(T) > \sum_{w\in V(T)} \sum_{u\in V(T)} d(u,w)\,deg(u) \geqslant 2\,\delta\,W(T)$. Using Corollary 1.1, the upper bound is obtained $HT(T) < \frac{n\,\xi^c(T)}{2\,\delta} + 2\,X$. □

This section concludes by establishing a relationship between the hitting-time index and the eccentric distance sum. The lower and upper bounds are presented in Theorem 2.3.

Theorem 2.3. *Let T be a tree with $n \geqslant 2$ vertices. Then*

$$\frac{n\,\xi^d(T)}{2n\,\mathrm{rad}(T) + 2(n-1)\,\mathrm{diam}(T)} + 2\,X < HT(T) \leqslant \frac{\xi^d(T)}{2\,\mathrm{rad}(T)} + 2\,X.$$

Proof. From the definition of the eccentric distance sum, it follows

$$\begin{aligned}
\xi^d(T) &= \sum_{u\in V(T)} ec(u)\,D(u) = \sum_{\{u,v\}\subseteq V(T)} (ec(u)+ec(v))\,d(u,v) = \sum_{\{u,v\}\subseteq V(T)} (2\,\mathrm{rad}(T) + d(u,C(T)) + d(v,C(T)))\,d(u,v)\\
&< 2\,\mathrm{rad}(T) \sum_{\{u,v\}\subseteq V(T)} d(u,v) + \mathrm{diam}(T) \sum_{\{u,v\}\subseteq V(T)} [d(u,C(T)) + d(v,C(T))]\\
&= 2\,\mathrm{rad}(T)\,W(T) + (n-1)\,\mathrm{diam}(T) \sum_{u\in V(T)} d(u,C(T)).
\end{aligned}$$

By Theorem 1.6, for any $w \in V(T)$ the following is valid: $\xi^d(T) < 2\,\mathrm{rad}(T)\,W(T) + (n-1)\,\mathrm{diam}(T) \sum_{u\in V(T)} d(u,w)$. Then,

$$\sum_{w\in V(T)} \xi^d(T) = n\,\xi^d(T) < 2n\,\mathrm{rad}(T)\,W(T) + 2(n-1)\,\mathrm{diam}(T)\,W(T).$$

The lower bound is obtained using Corollary 1.1,

$$HT(T) > \frac{n\,\xi^d(T)}{2n\,\mathrm{rad}(T) + 2(n-1)\,\mathrm{diam}(T)} + 2\,X.$$

The proof for the upper bound starts from the inequality

$$\xi^d(T) \geqslant \mathrm{rad}(T) \sum_{u\in V(T)} D(u) = 2\,\mathrm{rad}(T)\,W(T) = 2\,\mathrm{rad}(T)(HT(T) - 2\,X)\,,$$

which can be rearranged to $HT(T) \leqslant \frac{\xi^d(T)}{2\,\mathrm{rad}(T)} + 2\,X$. This bound is attained for the P_2 graph. □

3. Hitting-Time Index of Double Brooms

A formula for the calculation of the hitting-time index of double broom $D(d,a,b)$ is presented here. Firstly, it may be assumed without loss of generality that $b \geqslant a$. Then, the hitting-time index of a double broom is given in Theorem 3.1.

Theorem 3.1. *Let $b \geqslant a \geqslant 1, d \geqslant 3$. If $(a+b+d)^2 - 4\,(ba+bd-2b+2) \leqslant 0$ (or $(a+b+d)^2 - 4\,(ba+bd-2b+2) > 0\,,\ i_2 \leqslant a+1$ or $i_1 \geqslant a+d-1$) then,*

$$\begin{aligned}
HT(D(d,a,b)) &= a^3 + 3a^2b + 3a^2d - 5a^2 + 2ab^2d - ab^2 + 2abd^2 - 3abd - 2ab + 4a + b^3 + 3b^2d^2 - 4b^2d - b^2 + \frac{2ad^3}{3} - \frac{3ad^2}{2}\\
&\quad - \frac{13ad}{6} + \frac{7bd^3}{3} - \frac{11bd^2}{2} + \frac{25bd}{6} + \sum_{i=a+1}^{a+d-2} \sum_{j=i+1}^{h} (i^2 - j^2 - Ci + Cj) + \sum_{i=a+1}^{a+d-2} \sum_{j=l}^{a+d-1} (j^2 - i^2 - 2j + 2i).
\end{aligned}$$

On the other hand, if $(a+b+d)^2 - 4\,(ba+bd-2b+2) > 0,\ i_2 > a+1$ and $i_1 < a+d-1$ then

$$\begin{aligned}
HT(D(d,a,b)) &= a^3 + 3a^2b + 3a^2d - 5a^2 + 2ab^2d - ab^2 + 2abd^2 - 3abd - 2ab + 4a + b^3 + b^2d - 3b^2 - bd + 2b + \frac{2ad^3}{3} - \frac{3ad^2}{2}\\
&\quad - \frac{13ad}{6} + b\sum_{i=l_1}^{l_2} [(a+d-i)(a+d+i-2) + 2b - 2] + b\sum_{i\in H} [(a+d-i)(a+2b+d-i-2) - 2b + 2]\\
&\quad + \sum_{i=a+1}^{a+d-2} \sum_{j=i+1}^{h} (i^2 - j^2 - Ci + Cj) + \sum_{i=a+1}^{a+d-2} \sum_{j=l}^{a+d-1} (j^2 - i^2 - 2j + 2i),
\end{aligned}$$

where $h=\min\{a+d-1,a+b+d-i\}$, $l=\max\{a+b+d-i+1,i+1\}$, $C=2a+2b+2d-2$ *and* $l_1=\max\{a+1,i_1\}$, $l_2=\min\{a+d-1,i_2\}$ *with* $H=[a+1,a+d-1]\setminus[l_1,l_2]$,

$$i_1=\left\lceil\left(a+b+d-\sqrt{(a+b+d)^2-4(ba+bd-2b+2)}\right)/2\right\rceil,\ \textit{and}\ i_2=\left\lfloor\left(a+b+d+\sqrt{(a+b+d)^2-4(ba+bd-2b+2)}\right)/2\right\rfloor.$$

Proof. Let $T=D(d,a,b)$ be a double broom with vertex set $\{u_i\mid 1\leqslant i\leqslant a+b+d-1\}$ where $deg(u_{a+1})=a+1$ and $deg(u_{a+d-1})=b+1$. Since $D(d,a,b)$ is a tree, using Theorem 1.1, the $E_{u_i}T_{u_j}$ can be computed for any vertex pair $u_i,u_j\in V(D(d,a,b))$.

Let $P_{i,j}=u_iu_{a+1}u_j$ be a unique path between two leaves u_i and u_j where $1\leqslant i<j\leqslant a$. Then,

$$E_{u_i}T_{u_j}=d(u_i,u_j)+2\sum_{u_x\neq u_j\in V(P_{i,j})}|E_{u_x}|=2+2(a+b+d-3)=2a+2b+2d-4=E_{u_j}T_{u_i},\ \text{due to symmetry.}$$

Then $D(u_i,u_j)=2a+2b+2d-4$. The same holds for the leaves u_i and u_j where $a+d\leqslant i<j\leqslant a+b+d-1$.

Let $P_{i,j}=u_iu_{a+1}\cdots u_{j-1}u_j$ be the unique path between leaf u_i and vertex u_j, where $1\leqslant i\leqslant a$ and $a+1\leqslant j\leqslant a+d-1$. Then, by Eq. (1), we have

$$E_{u_i}T_{u_j}=d(u_i,u_j)+2\sum_{u_x\neq u_j\in V(P_{i,j})}|E_{u_x}|=(j-a)+2\sum_{l=1}^{j-a-1}(a+l-1)=(j-a)(a+j-1)-a-j+2,\tag{4}$$

$$\begin{aligned}E_{u_j}T_{u_i}&=d(u_j,u_i)+2\sum_{u_x\neq u_i\in V(P_{i,j})}|E_{u_x}|=(j-a)+2\sum_{l=0}^{j-a-1}[(a+d+b-2)-(a+l)]+2(a+b+d-3)\\&=(j-a)(a+2b+2d-j-2)+2a-2.\end{aligned}\tag{5}$$

From Eqs. (4) and (5), it follows $E_{u_j}T_{u_i}-E_{u_i}T_{u_j}=(j-a)(2b+2d-2j)+4a-4$. Since $a+1\leqslant j\leqslant a+d-1$ and $b\geqslant a$, then $j\leqslant b+d-1$. Hence, $E_{u_j}T_{u_i}\geqslant E_{u_i}T_{u_j}$, and $D(u_i,u_j)=(j-a)(a+2b+2d-j-2)+2a-2$.

Let $P_{i,j}=u_iu_{i+1}\cdots u_{a+d-1}u_j$ be the unique path between vertex u_i and leaf u_j, where $a+1\leqslant i\leqslant a+d-1$ and $a+d\leqslant j\leqslant a+d+b-1$. Then, by Eq. (1),

$$\begin{aligned}E_{u_i}T_{u_j}&=d(u_i,u_j)+2\sum_{u_x\neq u_j\in V(P_{i,j})}|E_{u_x}|=(a+d-i)+2\sum_{l=1}^{a+d-i-1}(i+l-2)+2(a+b+d-3)\\&=(a+d-i)(a+d+i-2)+2b-2,\end{aligned}\tag{6}$$

$$E_{u_j}T_{u_i}=d(u_j,u_i)+2\sum_{u_x\neq u_i\in V(P_{i,j})}|E_{u_x}|=(a+d-i)+2\sum_{l=0}^{a+d-i-2}(b+l)=(a+d-i)(a+2b+d-i-2)-2b+2.\tag{7}$$

From Eqs. (6) and (7), it follows $E_{u_j}T_{u_i}-E_{u_i}T_{u_j}=2\left[i^2-(a+d+b)\,i+ba+bd-2b+2\right]$. If $(a+b+d)^2-4(ba+bd-2b+2)\leqslant 0$, then $E_{u_i}T_{u_j}\leqslant E_{u_j}T_{u_i}$ and hence $D(u_i,u_j)=(a+d-i)(a+2b+d-i-2)-2b+2$.
If $(a+b+d)^2-4(ba+bd-2b+2)>0$, we let $l_1=\max\{a+1,i_1\}$ and $l_2=\min\{a+d-1,i_2\}$, where

$$i_1=\left\lceil\left(a+b+d-\sqrt{(a+b+d)^2-4(ba+bd-2b+2)}\right)/2\right\rceil,\ \text{and}\ i_2=\left\lfloor\left(a+b+d+\sqrt{(a+b+d)^2-4(ba+bd-2b+2)}\right)/2\right\rfloor.$$

If $l_1\leqslant i\leqslant l_2$, then $E_{u_i}T_{u_j}\geqslant E_{u_j}T_{u_i}$, and so, $D(u_i,u_j)=(a+d-i)(a+d+i-2)+2b-2$. If $i\in H=[a+1,a+d-1]\setminus[l_1,l_2]$, then $E_{u_i}T_{u_j}\leqslant E_{u_j}T_{u_i}$, and hence, $D(u_i,u_j)=(a+d-i)(a+2b+d-i-2)-2b+2$.

Let $P_{i,j}=u_iu_{a+1}u_{a+2}\cdots u_{a+d-1}u_j$ be the unique path between a leaf u_i and leaf u_j, where $a+d\leqslant j\leqslant a+b+d-1$ and $1\leqslant i\leqslant a$. Then, by Eq. (1), we have

$$E_{u_i}T_{u_j}=d(u_i,u_j)+2\sum_{u_x\neq u_j\in V(P_{i,j})}|E_{u_x}|=d+2\sum_{l=0}^{d-3}(a+l)+2(a+b+d-3)=(d-3)(2a+d+1)+4a+2b+3.$$

It is easy to see that $E_{u_i}T_{u_j}-E_{u_j}T_{u_i}=(d-2)(2a-2b)$. Since $b\geqslant a$ and $d\geqslant 3$, we have $E_{u_j}T_{u_i}\geqslant E_{u_i}T_{u_j}$, and hence $D(u_i,u_j)=(d-3)(2b+d+1)+4b+2a+3$. Let $P_{i,j}=u_iu_{i+1}u_{i+2}\cdots u_{j-1}u_j$ be the unique path between two vertices u_i and u_j where $a+1\leqslant i<j\leqslant a+d-1$. Then, by Eq. (1), we have

$$E_{u_i}T_{u_j}=d(u_i,u_j)+2\sum_{u_x\neq u_j\in V(P_{i,j})}|E_{u_x}|=(j-i)+2\sum_{l=1}^{j-i}(i+l-2)=(j-i)(i+j-2)$$

and

$$E_{u_j}T_{u_i}=d(u_j,u_i)+2\sum_{u_x\neq u_i\in V(P_{i,j})}|E_{u_x}|=(j-i)+2\sum_{l=1}^{j-i}[(a+b+d-2)-(j-l)]=(j-i)(2a+2b+2d-i-j-2).$$

Therefore, we have $E_{u_j}T_{u_i} - E_{u_i}T_{u_j} = (j-i)(2a+2b+2d-2i-2j)$.

Recall that $j > i$. If $a+b+d \geqslant i+j$, then $E_{u_j}T_{u_i} \geqslant E_{u_i}T_{u_j}$. Hence, $D(u_i,u_j) = (j-i)(2a+2b+2d-i-j-2)$. If $a+b+d < i+j$, then $E_{u_j}T_{u_i} < E_{u_i}T_{u_j}$, and $D(u_i,u_j) = (j-i)(i+j-2)$. Since $|V(T)| = n = a+b+d-1$, we have

$$HT(T) = \sum_{i<j} D(u_i,u_j) = \sum_{1\leqslant i<j\leqslant a} D(u_i,u_j) + \sum_{a+d\leqslant i<j\leqslant n} D(u_i,u_j) + \sum_{\substack{1\leqslant i\leqslant a\\ a+1\leqslant j\leqslant a+d-1}} D(u_i,u_j)$$
$$+ \sum_{\substack{a+1\leqslant i\leqslant a+d-1\\ a+d\leqslant j\leqslant n}} D(u_i,u_j) + \sum_{\substack{1\leqslant i\leqslant a\\ a+d\leqslant j\leqslant n}} D(u_i,u_j) + \sum_{a+1\leqslant i<j\leqslant a+d-1} D(u_i,u_j).$$

If $(a+b+d)^2 - 4(ba+bd-2b+2) \leqslant 0$, then

$$HT(T) = \binom{a}{2}(2a+2b+2d-4) + \binom{b}{2}(2a+2b+2d-4) + \binom{a}{1}\sum_{j=a+1}^{a+d-1}[(j-a)(a+2b+2d-j-2)+2a-2]$$
$$+ \binom{b}{1}\sum_{i=a+1}^{a+d-1}[(a+d-i)(a+2b+d-i-2)-2b+2] + \binom{a}{1}\binom{b}{1}[(d-3)(2b+d+1)+4b+2a+3]$$
$$+ \sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j\leqslant a+b+d}} (j-i)(2a+2b+2d-i-j-2) + \sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j>a+b+d}} (j-i)(i+j-2).$$

On the other hand, if $(a+b+d)^2 - 4(ba+bd-2b+2) > 0$ and either $i_2 \leqslant a+1$ or $i_1 \geqslant a+d-1$, then we

$$HT(T) = \binom{a}{2}(2a+2b+2d-4) + \binom{b}{2}(2a+2b+2d-4) + \binom{a}{1}\sum_{j=a+1}^{a+d-1}[(j-a)(a+2b+2d-j-2)+2a-2]$$
$$+ \binom{b}{1}\sum_{i=a+1}^{a+d-1}[(a+d-i)(a+2b+d-i-2)-2b+2] + \binom{a}{1}\binom{b}{1}[(d-3)(2b+d+1)+4b+2a+3]$$
$$+ \sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j\leqslant a+b+d}} (j-i)(2a+2b+2d-i-j-2) + \sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j>a+b+d}} (j-i)(i+j-2).$$

In all other cases

$$HT(T) = \binom{a}{2}(2a+2b+2d-4) + \binom{b}{2}(2a+2b+2d-4) + \binom{a}{1}\sum_{j=a+1}^{a+d-1}[(j-a)(a+2b+2d-j-2)+2a-2]$$
$$+ \binom{b}{1}\sum_{i=l_1}^{l_2}[(a+d-i)(a+d+i-2)+2b-2] + \binom{b}{1}\sum_{i\in H}[(a+d-i)(a+2b+d-i-2)-2b+2]$$
$$+ \binom{a}{1}\binom{b}{1}[(d-3)(2b+d+1)+4b+2a+3] + \sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j\leqslant a+b+d}} (j-i)(2a+2b+2d-i-j-2)$$
$$+ \sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j>a+b+d}} (j-i)(i+j-2).$$

To simplify the above expressions, let $k = j-a$. Then,

$$\begin{aligned}\sum_{j=a+1}^{a+d-1}[(j-a)(a+2b+2d-j-2)+2a-2] &= \sum_{k=1}^{d-1}[-k^2+(2b+2d-2)\,k+2a-2]\\ &= -\frac{d\,(d-1)(2d-1)}{6} + d\,(d-1)(b+d-1) + (2a-2)(d-1).\end{aligned}$$

Then, let $k = i-a$,

$$\begin{aligned}\sum_{i=a+1}^{a+d-1}[(a+d-i)(a+2b+d-i-2)-2b+2] &= \sum_{k=1}^{d-1}[k^2-(2b+2d-2)\,k+d^2+2bd-2d-2b+2]\\ &= (d-1)\left[\frac{d\,(2d-1)}{6} - d\,(b+d-1) + d^2 + 2bd - 2d - 2b + 2\right].\end{aligned}$$

Let $h = \min\{a+d-1, a+b+d-i\}$ and $C = 2a+2b+2d-2$. Then,

$$\sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j\leqslant a+b+d}} (j-i)(2a+2b+2d-i-j-2) = \sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj).$$

Let $l=\max\{a+b+d-i+1,i+1\}$. Then

$$\sum_{\substack{a+1\leqslant i<j\leqslant a+d-1\\ i+j>a+b+d}}(j-i)(i+j-2)=\sum_{i=a+1}^{a+d-2}\sum_{j=l}^{a+d-1}(j^2-i^2-2j+2i).$$

Thence, if $(a+b+d)^2-4(ba+bd-2b+2)\leqslant 0$,

$$HT(T)=a^3+3a^2b+3a^2d-5a^2+2ab^2d-ab^2+2abd^2-3abd-2ab+4a+b^3+3b^2d^2-4b^2d-b^2+\frac{2ad^3}{3}-\frac{3ad^2}{2}-\frac{13ad}{6}+\frac{7bd^3}{3}-\frac{11bd^2}{2}+\frac{25bd}{6}+\sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj)+\sum_{i=a+1}^{a+d-2}\sum_{j=l}^{a+d-1}(j^2-i^2-2j+2i).$$

If $(a+b+d)^2-4(ba+bd-2b+2)>0$ and either $i_2\leqslant a+1$ or $i_1\geqslant a+d-1$, then

$$HT(T)=a^3+3a^2b+3a^2d-5a^2+2ab^2d-ab^2+2abd^2-3abd-2ab+4a+b^3+3b^2d^2-4b^2d-b^2+\frac{2ad^3}{3}-\frac{3ad^2}{2}-\frac{13ad}{6}+\frac{7bd^3}{3}-\frac{11bd^2}{2}+\frac{25bd}{6}+\sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj)+\sum_{i=a+1}^{a+d-2}\sum_{j=l}^{a+d-1}(j^2-i^2-2j+2i).$$

Otherwise, the hitting-time index is equal to

$$HT(T)=a^3+3a^2b+3a^2d-5a^2+2ab^2d-ab^2+2abd^2-3abd-2ab+4a+b^3+b^2d-3b^2-bd+2b+\frac{2ad^3}{3}-\frac{3ad^2}{2}-\frac{13ad}{6}+b\sum_{i=l_1}^{l_2}[(a+d-i)(a+d+i-2)+2b-2]+b\sum_{i\in H}[(a+d-i)(a+2b+d-i-2)-2b+2]+\sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj)+\sum_{i=a+1}^{a+d-2}\sum_{j=l}^{a+d-1}(j^2-i^2-2j+2i).\qquad\square$$

By setting $d=3$ in the formula given in Theorem 3.1, one obtains the result published in [6], which is also outlined in this article as Theorem 1.3.

Remark 3.1. *For $d=3$, we have $(a+b+d)^2-4(ba+bd-2b+2)=(a-b)^2+6a+2b+1=(b-a)^2+6a+2b+1>0$. Since $i_1\leqslant\lceil(a+b+3-(b-a))/2\rceil<a+2$ and $i_2\geqslant\lfloor(a+b+3)/2\rfloor>a+1$, we have $l_1=\max\{a+1,i_1\}=a+1$ and $l_2=\min\{a+2,i_2\}=a+2$. Hence, $H=\emptyset$ and $b\sum_{i=l_1}^{l_2}[(a+d-i)(a+d+i-2)+2b-2]=b\,(6a+4b+3)$.*

For $i=a+1$, we have $h=\min\{a+2,b+2\}=a+2$, $l=\max\{b+3,a+2\}=b+3$ and $C=2a+2b+2d-2$. Since $a+1\leqslant i<j\leqslant a+2$, the case when $i=a+2$ is impossible. Thus,

$$\sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj)+\sum_{i=a+1}^{a+d-2}\sum_{j=l}^{a+d-2}(j^2-i^2-2j+2i)=\sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj)=2b+1.$$

Therefore, $HT(D(3,a-1,b-1))=a^3+3a^2b-2a^2+5ab^2-3ab-5a+b^3-4b^2-b+6=HT(B(a,b))$.

On the other hand, by setting $d=3$, $a=1$ and $b=n-3$ in the formula from Theorem 3.1, one obtains the result published in [12], which is also outlined in this article as Theorem 1.4.

Remark 3.2. *For $d=3$, $a=1$, $b=n-3$ and $n\geq 6$, we have $(a+b+d)^2-4(ba+bd-2b+2)=n^2-6n+17>0$. Since $[a+1,a+d-1]=[2,3]$ and*

$$i_1=\left\lceil\frac{n+1-\sqrt{n^2-6n+17}}{2}\right\rceil\leqslant 2 \text{ and } i_2=\left\lfloor\frac{n+1+\sqrt{n^2-6n+17}}{2}\right\rfloor\geqslant 4,$$

we have $l_1=\max\{2,i_1\}=2$, $l_2=\min\{3,i_2\}=3$. Hence $H=\emptyset$ and

$$b\sum_{i=2}^{3}[(a+d-i)(a+d+i-2)+2b-2]=(n-3)\,(4n-3).$$

Also $h=\min\{3,n+1-i\}$, $l=\max\{n+2-i,i+1\}$ and $C=2a+2b+2d-2=2n$. Thus,

$$\sum_{i=a+1}^{a+d-2}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj)+\sum_{i=a+1}^{a+d-2}\sum_{j=l}^{a+d-2}(j^2-i^2-2j+2i)=\sum_{i=2}^{2}\sum_{j=i+1}^{h}(i^2-j^2-2ni+2nj)=2n-5.$$

Therefore, $HT(D(3,1,n-3))=n^3-7n+2=HT(B_{n,3})$.

Setting $d = 4$, $a = 1$ and $b = n - 4$ in the formula from Theorem 3.1, one obtains the result published in [12], which is also outlined in this article as Theorem 1.5.

Remark 3.3. *For $d = 4$, $a = 1$, $b = n - 4$ and $n \geq 8$, we have $(a+b+d)^2 - 4(ba+bd-2b+2) = n^2 - 10n + 41 > 0$. Note that $[a+1, a+d-1] = [2, 4]$.*

For $n > 8$, we have

$$2 < i_1 = \left\lceil \frac{n+1-\sqrt{n^2-10n+41}}{2} \right\rceil \leqslant 3 \text{ and } i_2 = \left\lfloor \frac{n+1+\sqrt{n^2-10n+41}}{2} \right\rfloor \geqslant 5\,,$$

we have $l_1 = \max\{2, i_1\} = 3$, $l_2 = \min\{4, i_2\} = 4$. Hence $H = \{2\}$ and

$$b\sum_{i=3}^{4}[(a+d-i)(a+d+i-2)+2b-2] + b\sum_{i\in H}[(a+d-i)(a+2b+d-i-2)-2b+2] = (n-4)\,(8n-12).$$

Also, $h = \min\{4\,, n+1-i\}$, $l = \max\{n+2-i, i+1\}$ and $C = 2a+2b+2d-2 = 2n$. Thus,

$$\sum_{i=2}^{3}\sum_{j=i+1}^{h}(i^2-j^2-Ci+Cj) + \sum_{i=2}^{3}\sum_{j=l}^{4}(j^2-i^2-2j+2i) = \sum_{i=2}^{3}\sum_{j=i+1}^{h}(i^2-j^2-2ni+2nj) = 8n-24.$$

Therefore, $HT(D(4,1,n-4)) = n^3+4n^2-33n+34 = HT(B_{n,4})$. For $n = 8$, we have

$$i_1 = \left\lceil \frac{n+1-\sqrt{n^2-10n+41}}{2} \right\rceil = 2 \text{ and } i_2 = \left\lfloor \frac{n+1+\sqrt{n^2-10n+41}}{2} \right\rfloor \geqslant 5\,,$$

and hence, $l_1 = \max\{2, i_1\} = 2$, $l_2 = \min\{4, i_2\} = 4$. Thus, $H = \emptyset$ and

$$b\sum_{i=2}^{4}[(a+d-i)(a+d+i-2)+2b-2] = (n-4)\,(6n+4) = 208.$$

Also, $h = \min\{4\,, 9-i\}$, $l = \max\{10-i, i+1\}$ and $C = 2a+2b+2d-2 = 2n = 16$. Thus,

$$\sum_{i=2}^{3}\sum_{j=i+1}^{h}(i^2-j^2-16i+16j) + \sum_{i=2}^{3}\sum_{j=l}^{4}(j^2-i^2-2j+2i) = \sum_{i=2}^{3}\sum_{j=i+1}^{h}(i^2-j^2-2ni+2nj) = 8n-24 = 40.$$

It is easy to compute that $HT(D(4,1,4)) = HT(B_{8,4}) = 538$. Note that $HT(D(4,1,4)) = 8^3 + 4\times 8^2 - 33\times 8 + 34 = 538$. Therefore, for $n \geqslant 8$, we have $HT(D(4,1,n-4)) = HT(B_{n,4})$.

For $d \geqslant 3$, $a \geqslant 2$ and $b \geqslant 2$, we give the following example.

Remark 3.4. *For $d = 4\,, a = 2\,, b = 2$, we computed $HT(D(4,2,2))$ by using the definition of the hitting-time index and formulas of Theorem 3.1 respectively, and the result is 354 in both cases.*

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References

[1] A. R. Ashrafi, M. Saheli, M. Ghorbani, The eccentric connectivity index of nanotubes and nanotori, *J. Comput. Appl. Math.* **235** (2011) 4561–4566.

[2] E. Camby, G. Caporossi, M. H. M. Paiva, M. E. V. Segatto, Expected distance based on random walks, *J. Math. Chem.* **56** (2018) 618–629.

[3] P. Dankelmann, W. Goddard, C. S. Swart, The average eccentricity of a graph and its subgraphs, *Util. Math.* **65** (2004) 41–51.

[4] P. G. Doyle, J. L. Snell, *Random Walks and Electrical Networks*, Math. Assoc. Am., Washington, 1984.

[5] S. Gupta, M. Singh, A. K. Madan, Eccentric distance sum: A novel graph invariant for predicting biological and physical properties, *J. Math. Anal. Appl.* **275** (2002) 386–401.

[6] M. He, Y. Mao, B. Furtula, X. Zhang, On the hitting time index of trees, *J. Math. Chem.* **64** (2026) #7.

[7] H. Hua, K. Xu, S. Wen, A short and unified proof of Yu et al's two results on the eccentric distance sum, *J. Math. Anal. Appl.* **382** (2011) 364–366.

[8] A. Ilić, Eccentric connectivity index, In: I. Gutman, B. Furtula (Eds.), *Novel Molecular Structure Descriptors – Theory and Applications II*, Univ. Kragujevac, Kragujevac, 2010, 139–168.

[9] A. Ilić, I. Gutman, Eccentric connectivity index of chemical trees, *MATCH Commun. Math. Comput. Chem.* **65** (2011) 731–744.

[10] A. Ilić, G. Yu, L. Feng, On the eccentric distance sum of graphs, *J. Math. Anal. Appl.* **381** (2011) 590–600.

[11] J. L. Palacios, A new probabilistic molecular index, *MATCH Commun. Math. Comput. Chem.* **93** (2025) 499–507.

[12] J. L. Palacios, On the hitting time index of broom graphs, *Mathematics* **14** (2026) #1508.

[13] J. L. Palacios, E. Gómez, M. Del Río, Hitting times of walks on graphs through voltages, *J. Prob.* **2014** (2014) #852481.

[14] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 2001.

[15] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.