Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **18** (2026) 69–76
DOI: 10.47443/dml.2026.140

Research Article

New Combinatorial Identities Associated with Sun's Identity

Necdet Batır*

Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, 50300, Turkey

(Received: 16 May 2026. Received in revised form: 22 August 2026. Accepted: 29 August 2026. Published online: 2 September 2026.)

Abstract

In this paper, a variety of new combinatorial identities are established, including an alternating version of a well-known combinatorial identity due to Sun [*Sci. China Math.* **54** (2011) 2509–2536]. The obtained results extend existing formulas and provide deeper insight into the interplay between binomial coefficients, harmonic numbers, and alternating sums. Among the numerous derived identities, the following relations are presented, for instance:

$$\sum_{k=1}^{n}\frac{(-1)^k}{k\binom{n+k}{k}}=\sum_{k=1}^{n}\frac{2^{n-k}}{k}+3\sum_{k=1}^{n}\frac{(-1)^k2^{n-k}}{k\binom{2k}{k}}\quad\text{and}\quad\sum_{k=1}^{n}\frac{2^{n+k}H_{k-1}}{k\binom{n+k}{k}}=\sum_{k=1}^{n}\frac{(H_n-H_{k-1})2^k}{k}-\sum_{k=1}^{n}\frac{2^{2k}}{k^2\binom{2k}{k}},$$

which reveal a refined relationship involving harmonic numbers and central binomial coefficients. These identities illustrate the richness of the combinatorial structures under consideration and demonstrate how alternating techniques can be effectively used to derive nontrivial and elegant closed-form expressions.

Keywords: binomial coefficients; combinatorial identities; harmonic numbers; harmonic sums; central binomial coefficients; finite sums.

2020 Mathematics Subject Classification: 05A10, 05A19, 11B65.

1. Introduction

For $k, n \in \mathbb{N}$ the binomial coefficients are defined by

$$\binom{n}{k}=\begin{cases}\frac{n!}{k!(n-k)!}, & n\geq k,\\ 0, & n<k.\end{cases}$$

The binomial coefficients are fundamental mathematical objects, and play a significant role in a wide range of areas, including combinatorial analysis, graph theory, statistics and probability, and number theory. Finite sums involving reciprocals of the binomial coefficients have attracted considerable attention due to their rich combinatorial structure and their connections with harmonic numbers, generating functions, and finite hypergeometric sums. They have been studied extensively in the mathematical literature. Numerous identities involving them can be found in the works of Batır [5, 9], Batır and Choi [6, 7], Batır and Chen [8], Mansour [12], Pla [13], and Rockett [14]. By employing a known identity together with mathematical induction, Sun [19] derived the following elegant formula:

$$\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}=3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}}-\sum_{k=1}^{n}\frac{1}{k^2}. \tag{1}$$

Here and throughout, let $\mathbb{N}$ denote the set of positive integers. In [17], Sofo and Batır presented an alternative representation of (1) involving the harmonic numbers as follows: For $n \in \mathbb{N}$

$$\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}=-H_n^2+\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\binom{n}{k}H_{n+k}.$$

The symbol H_n denotes the classical harmonic numbers, defined by

$$H_n=\sum_{k=1}^{n}\frac{1}{k},\quad (n\in\mathbb{N}),$$

*E-mail address: nbatir@hotmail.com

with the convention $H_0 := 0$. Batır and Choi [6] made a systematic study on identities of type (1) and presented several related results. As examples, they, together with many others, proved the following identities:

$$\sum_{k=1}^{n} \frac{H_{k-1}}{k\binom{n+k}{k}} = \frac{1}{n^2} - \frac{nH_n + 1}{n^2\binom{2n}{n}} \quad (n \in \mathbb{N}) \tag{2}$$

and

$$\sum_{k=1}^{n} \frac{H_k}{\binom{n+k}{k}} = \frac{n}{(n-1)^2}\left(1 - \frac{(n-1)H_{n+1} + 1}{\binom{2n}{n-1}}\right) \quad (n \geq 2). \tag{3}$$

In [7] the authors established a general formula which can be specialized to (1), (2) and (3), one of which is chosen to be recalled here as follows [7, Corollary 6]:

$$\sum_{k=1}^{n} \frac{1}{k^3\binom{n+k}{k}} = \sum_{k=1}^{n} \frac{H_k^{(2)}}{k} - 3\sum_{k=1}^{n} \frac{1}{k}\sum_{\upsilon=1}^{k-1} \frac{1}{\upsilon^3\binom{2\upsilon}{\upsilon}} - \sum_{k=1}^{n} \frac{1}{k^3\binom{2k}{k}}.$$

In this paper, we derive elegant closed-form expressions involving the central binomial coefficients for the alternating versions of the sums given in (1), (2) and (3), together with several related sums. In fact, we establish more general identities from which all of these formulas and their alternating analogues follow as special cases. More precisely, for $p = 0, 1, 2$ and $q = 0, 1$, we obtain explicit evaluations of the following sums in terms of the central binomial coefficients: For any $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^p\binom{n+k}{k}} \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^k H_k}{k^q\binom{n+k}{k}}.$$

Our proofs rely mainly on the results established in [6]. Throughout the paper, we adopt the convention that an empty sum is equal to zero. For further examples of interesting finite binomial sums, we refer the reader to [3, 6–8] and the references therein.

2. Preliminary Lemmas

For our purpose we recall some special numbers. The generalized harmonic number of order α is defined as

$$H_n^{(\alpha)} := \sum_{k=1}^{n} \frac{1}{k^\alpha} \quad (\alpha \in \mathbb{C},\ n \in \mathbb{N}).$$

By convention, $H_0^{(\alpha)} := 0$, and when $\alpha = 1$, $H_n^{(1)} = H_n$ denotes the n th harmonic number. The binomial coefficients satisfy the following simple relations:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{4}$$

and

$$\frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}. \tag{5}$$

The central binomial coefficients are defined by $\binom{2n}{n}$ for $n \in \mathbb{N}$. The following lemmas will be needed in the proofs of our main results. The first two lemmas are recalled from [6]; see Theorems 1 and 2 therein.

Lemma 2.1. *Let $m, n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then*

$$\sum_{k=1}^{m} \frac{(-1)^k t^{n+k}}{k\binom{n+k}{k}} = (1+t)^n\left\{H_m - H_n + \sum_{k=1}^{n} \frac{1}{k\left(1+\frac{1}{t}\right)^k}\right\} - \frac{(-1)^n}{\binom{m+n}{n}} \sum_{k=0, k\neq n}^{m+n} (-1)^k \binom{m+n}{k} \frac{(1+t)^k}{n-k}. \tag{6}$$

Lemma 2.2. *Let $m, n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then*

$$\begin{aligned}\sum_{k=1}^{m} \frac{(-1)^k H_{k-1} t^{n+k}}{k\binom{n+k}{k}} =& \frac{1}{2}\left\{H_m^2 - H_n^2 - H_m^{(2)} - H_n^{(2)}\right\}(1+t)^n + \frac{(-1)^{n+1}}{\binom{m+n}{n}} \sum_{k=0, k\neq n}^{m+n} \frac{(-1)^k}{(n-k)^2}\binom{m+n}{n}(1+t)^k \\ &+ (1+t)^n \sum_{k=1}^{n} \frac{H_n - H_{k-1}}{k\left(1+\frac{1}{t}\right)^k} + \frac{(-1)^{n+1}H_m}{\binom{m+n}{n}} \sum_{k=0, k\neq n}^{m+n} \frac{(-1)^k}{n-k}\binom{m+n}{n}(1+t)^k.\end{aligned} \tag{7}$$

Lemma 2.3. *Let n be a non-negative integer and x any nonzero complex number. Then*

$$\sum_{k=1}^{n}\frac{1}{k^2}\binom{2n}{n+k}\left(x^k+x^{-k}\right)=\binom{2n}{n}\left[\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}}\frac{(1+x)^{2k}}{x^k}-H_n^{(2)}\right]. \tag{8}$$

Proof. We define

$$f_n(x):=\sum_{k=1}^{n}\frac{n!^2\left(x^k+x^{-k}\right)}{k^2(n+k)!(n-k)!}=\frac{1}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{1}{k^2}\binom{2n}{n-k}\left(x^k+x^{-k}\right).$$

We have

$$\begin{aligned}
f_n(x)-f_{n-1}(x)&=\sum_{k=1}^{n}\frac{n!^2\left(x^k+x^{-k}\right)}{k^2(n+k)!(n-k)!}-\sum_{k=1}^{n-1}\frac{(n-1)!^2\left(x^k+x^{-k}\right)}{k^2(n-1+k)!(n-1-k)!}\\
&=\frac{n!^2\left(x^n+x^{-n}\right)}{n^2(2n)!}+\sum_{k=1}^{n-1}\frac{(n-1)!^2\left(x^k+x^{-k}\right)}{k^2(n-1+k)!(n-1-k)!}\left(\frac{n^2}{(n-k)(n+k)}-1\right)\\
&=\sum_{k=1}^{n}\frac{(n-1)!^2\left(x^k+x^{-k}\right)}{(n+k)!(n-k)!}=\frac{(n-1)!^2}{2(2n)!}\sum_{k=-n}^{n}\binom{2n}{n+k}\left(x^k+x^{-k}\right)-\frac{1}{n^2}\\
&=\frac{(n-1)!^2}{2(2n)!}\sum_{k=0}^{2n}\binom{2n}{k}\left(x^{k-n}+x^{n-k}\right)-\frac{1}{n^2}\\
&=\frac{1}{n^2\binom{2n}{n}}\frac{(1+x)^{2n}}{x^n}-\frac{1}{n^2}.
\end{aligned} \tag{9}$$

Replacing n in (9) by k and summing both sides of the resultant identity from $k=1$ to $k=n$, we complete the proof. □

Differentiating both sides of (8) with respect to x, and simplifying, we get the following lemma:

Lemma 2.4. *Let $n\in\mathbb{N}$ and $x\in\mathbb{C}$ (with $x\neq 0,-1$). Then*

$$\sum_{k=1}^{n}\frac{1}{k}\binom{2n}{n+k}\left(x^k-x^{-k}\right)=\frac{x-1}{x+1}\binom{2n}{n}\sum_{k=1}^{n}\frac{1}{k\binom{2k}{k}}\frac{(1+x)^{2k}}{x^k}. \tag{10}$$

Remark 2.1. *Letting $x=-1$ and $x=1$ in (8), we get, respectively*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{2n}{n+k}=\binom{2n}{n}\frac{H_n^{(2)}}{2} \tag{11}$$

and

$$\sum_{k=1}^{n}\frac{1}{k^2}\binom{2n}{n+k}=\frac{1}{2}\binom{2n}{n}\left[\sum_{k=1}^{n}\frac{2^{2k}}{k^2\binom{2k}{k}}-H_n^{(2)}\right].$$

Equation (11) has been proved in a very recent paper [11, p. 9]. Thus, (8) gives a generalization of (11). Differentiating both sides of (10) with respect to x and then evaluating the resulting expression at $x=1$ and $x=-1$, we obtain the following two formulas: For $n\in\mathbb{N}$

$$\sum_{k=1}^{n}\binom{2n}{n+k}=\frac{1}{4}\binom{2n}{n}\sum_{k=1}^{n}\frac{2^{2k}}{k\binom{2k}{k}} \tag{12}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}=\frac{1}{2}\binom{2n}{n}.$$

By [18, Theorem 4.1] we have

$$\sum_{k=1}^{n}\frac{2^{2k}}{k\binom{2k}{k}}=2^{2n+1}\binom{2n}{n}^{-1}-2. \tag{13}$$

Using this formula, we get

$$\sum_{k=0}^{n}\binom{2n}{n+k}=\frac{1}{2}\left(2^{2n}+\binom{2n}{n}\right).$$

We emphasize that (10) is first extended continuously (or analytically) to $x = -1$, and only then is the resulting extension differentiated and evaluated at $x = -1$. Equivalently, the desired value may be obtained by taking the corresponding limit as $x \to -1$. Since

$$\sum_{k=1}^{n}\binom{2n}{n+k} = \sum_{k=n+1}^{2n}\binom{2n}{k} = \sum_{k=0}^{2n}\binom{2n}{k} - \sum_{k=0}^{n}\binom{2n}{k} = 2^{2n} - \sum_{k=0}^{n}\binom{2n}{k},$$

we get

$$\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n} - \frac{1}{4}\binom{2n}{n}\sum_{k=1}^{n}\frac{2^{2k}}{k\binom{2k}{k}}.$$

Applying (13), it follows that

$$\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}.$$

This formula can be regarded as a complement of the following summation formula [5, Theorem 1]:

$$\sum_{k=0}^{n}\frac{1}{\binom{2n}{k}} = \frac{2n+1}{2^{2n+2}}\sum_{k=1}^{2n+1}\frac{2^k}{k} + \frac{1}{2\binom{2n}{n}}.$$

Meanwhile, we also find it worthwhile to present the following counterpart of (11), which was proved in [11, p. 9]:

$$\sum_{k=1}^{n}\frac{1}{k}\binom{2n}{n+k} = \binom{2n}{n}\frac{H_n}{2}. \tag{14}$$

The following identity, an alternative form of (14), is an immediate consequence of Theorem 2.2 in [10] upon setting $\alpha = n$.

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{2n}{n+k} = \binom{2n}{n}\left(H_{2n} - H_n\right).$$

3. Main Results

We now proceed to present the main results of this work, which consist of several new and nontrivial identities involving harmonic numbers and central binomial coefficients. These results not only extend existing formulas in the literature but also reveal deeper structural relationships between these two important mathematical objects. Throughout this section, formulas involving t, $1/t$, or $1/(1+t)$ are understood for values of t for which the corresponding expressions are defined. In the cases where a removable singularity occurs, the corresponding value is understood by continuity.

Theorem 3.1. *Let $m, n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then,*

$$\begin{aligned}\sum_{k=1}^{m}\frac{(-1)^k t^{n+k}}{k\binom{n+k}{k}} &= (1+t)^n\Bigg\{H_m - H_n + \sum_{k=1}^{n}\frac{1}{k\left(1+\frac{1}{t}\right)^k} \\ &-\frac{1}{\binom{m+n}{n}}\left[\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{m+n}{m+k}(1+t)^{-k} - \sum_{k=1}^{m}\frac{(-1)^k}{k}\binom{n+m}{n+k}(1+t)^k\right]\Bigg\}.\end{aligned} \tag{15}$$

Proof. We use Lemma 2.1. Splitting the second sum on the right hand side of (6) into two parts, we arrive at

$$\sum_{k=0,k\neq n}^{m+n}(-1)^k\binom{m+n}{k}\frac{(1+t)^k}{n-k} = \sum_{k=0}^{n-1}(-1)^k\binom{m+n}{k}\frac{(1+t)^k}{n-k} + \sum_{k=n+1}^{m+n}(-1)^k\binom{m+n}{k}\frac{(1+t)^k}{n-k}. \tag{16}$$

By setting $n - k = k'$ in the first sum and $k - n = k'$ in the second sum on the right-hand side of (16), and then deleting the prime, we obtain

$$\sum_{k=0,k\neq n}^{m+n}(-1)^k\binom{m+n}{k}\frac{(1+t)^k}{n-k} = (-1)^n(1+t)^n\left[\sum_{k=1}^{n}(-1)^k\binom{m+n}{m+k}\frac{(1+t)^{-k}}{k} - \sum_{k=1}^{m}(-1)^k\binom{n+m}{n+k}\frac{(1+t)^k}{k}\right]. \tag{17}$$

Substituting this expression into the formula given in Lemma 2.1, we deduce the result. □

Theorem 3.2. *Let $n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then we have*

$$\sum_{k=1}^{n}\frac{(-1)^k t^{n+k}}{k\binom{n+k}{k}} = (1+t)^n\left\{\sum_{k=1}^{n}\frac{1}{k}\left(\frac{t}{t+1}\right)^k + \frac{t+2}{t}\sum_{k=1}^{n}\frac{(-1)^k}{k\binom{2k}{k}}\left(\frac{t^2}{t+1}\right)^k\right\}. \tag{18}$$

Proof. Setting $m = n$ in (15), we get

$$\sum_{k=1}^{n} \frac{(-1)^k t^{n+k}}{k\binom{n+k}{k}} = (1+t)^n \left\{ \sum_{k=1}^{n} \frac{1}{k} \left(\frac{t}{t+1}\right)^k + \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{2n}{n+k} \left\{(1+t)^k - (1+t)^{-k}\right\} \right\}. \tag{19}$$

Applying Lemma 2.4 with $x = -t-1$ in (19) the desired result follows. □

Corollary 3.1. *Putting $t = -2$ in (18), we get*

$$\sum_{k=1}^{n} \frac{2^{n+k}}{k\binom{n+k}{k}} = \sum_{k=1}^{n} \frac{2^k}{k}. \tag{20}$$

Remark 3.1. *Identity (20) agrees with Theorem 2.3 in [5].*

Corollary 3.2. *Putting $t = 1$ in (18), we get*

$$\sum_{k=1}^{n} \frac{(-1)^k}{k\binom{n+k}{k}} = 2^n \left(\sum_{k=1}^{n} \frac{1}{k2^k} + 3 \sum_{k=1}^{n} \frac{(-1)^k}{k2^k \binom{2k}{k}} \right).$$

The following corollary can be obtained by taking the limit as $t \to -1$ in (18), after simplification:

Corollary 3.3. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{k=1}^{n} \frac{1}{k\binom{n+k}{k}} = \frac{1}{n} - \frac{1}{n\binom{2n}{n}}. \tag{21}$$

Remark 3.2. *Identity (21) is not new and can be found in [17] and [6].*

Upon dividing both sides of (18) by t^n, differentiating the resulting expression with respect to t, and subsequently multiplying by t, we obtain the following theorem:

Theorem 3.3. *Let $n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then,*

$$\begin{aligned} \sum_{k=1}^{n} \frac{(-1)^k t^k}{\binom{n+k}{k}} = \left(1+\frac{1}{t}\right)^n \Bigg\{ &-\frac{n}{1+t} \sum_{k=1}^{n} \frac{1}{k} \left(\frac{t}{t+1}\right)^k - \left(\frac{n(t+2)}{t(t+1)} + \frac{2}{t}\right) \sum_{k=1}^{n} \frac{(-1)^k}{k\binom{2k}{k}} \left(\frac{t^2}{t+1}\right)^k \\ &+ \frac{1}{t+1} \sum_{k=1}^{n} \left(\frac{t}{t+1}\right)^k + \frac{(t+2)^2}{t^2+t} \sum_{k=1}^{n} \frac{(-1)^k}{\binom{2k}{k}} \left(\frac{t^2}{t+1}\right)^k \Bigg\}. \end{aligned} \tag{22}$$

Setting $t = -2$ in (22) and using (13), we get the following corollary:

Corollary 3.4. *Let $n \in \mathbb{N}$. Then,*

$$\sum_{k=0}^{n} \frac{2^{n+k}}{\binom{n+k}{k}} = n \sum_{k=1}^{n} \frac{2^k}{k} + \frac{2^{2n+1}}{\binom{2n}{n}} - 2^n. \tag{23}$$

Remark 3.3. *Identity (23) can be compared with the following well-known identity reported in [1]:*

$$\sum_{k=0}^{n} \frac{\binom{n+k}{k}}{2^{n+k}} = 1.$$

Setting $t = 1$ in (22), we get

Corollary 3.5. *For any $n \in \mathbb{N}$, we have*

$$\frac{1}{2^{n-1}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\binom{n+k}{k}} = n \sum_{k=1}^{n} \frac{1}{k2^k} + (3n+4) \sum_{k=1}^{n} \frac{(-1)^k}{k2^k\binom{2k}{k}} - \sum_{k=1}^{n} \frac{1}{2^k} - 9 \sum_{k=1}^{n} \frac{(-1)^k}{2^k\binom{2k}{k}}.$$

Theorem 3.4. *Let $n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then,*

$$\begin{aligned} \sum_{k=1}^{m} \frac{(-1)^k H_{k-1} t^{n+k}}{k\binom{n+k}{k}} =& (1+t)^n \Bigg\{ \frac{1}{2}\left\{H_m^2 - H_n^2 - H_m^{(2)} - H_n^{(2)}\right\} + \sum_{k=1}^{n} \frac{H_n - H_{k-1}}{k\left(1+\frac{1}{t}\right)^k} \\ &+ \frac{H_m}{\binom{m+n}{n}} \left[\mathcal{Q}_{m,n}(1+t) - \mathcal{Q}_{n,m}\left(1/(1+t)\right)\right] - \frac{1}{\binom{m+n}{n}} \left[\mathcal{G}_{n,m}\left(1/(1+t)\right) + \mathcal{G}_{m,n}(1+t)\right] \Bigg\}, \end{aligned} \tag{24}$$

where

$$\mathcal{Q}_{m,n}(x) = \sum_{k=1}^{m} \frac{(-1)^k}{k} \binom{n+m}{n+k} x^k \quad \textit{and} \quad \mathcal{G}_{m,n}(x) = \sum_{k=1}^{m} \frac{(-1)^k}{k^2} \binom{n+m}{n+k} x^k.$$

Proof. We have

$$\sum_{k=1,k\neq n}^{m+n}\frac{(-1)^k}{(n-k)^2}\binom{m+n}{k}(1+t)^k=\sum_{k=1}^{n-1}\frac{(-1)^k}{(n-k)^2}\binom{m+n}{k}(1+t)^k+\sum_{k=n+1}^{m+n}\frac{(-1)^k}{(n-k)^2}\binom{m+n}{k}(1+t)^k. \tag{25}$$

By setting $n-k=k'$ in the first sum and $k-n=k'$ in the second sum on the right-hand side of (25), and then relabeling k' as k, we obtain

$$\sum_{k=1,k\neq n}^{m+n}\frac{(-1)^k}{(n-k)^2}\binom{m+n}{k}(1+t)^k=(-1)^n(1+t)^n\left\{\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{m+n}{m+k}(1+t)^{-k}+\sum_{k=1}^{m}\frac{(-1)^k}{k^2}\binom{n+m}{n+k}(1+t)^k\right\}. \tag{26}$$

Applying (17) and (26) in (7) we get the resultant identity. □

Theorem 3.5. *Let t be any nonzero complex number and $n\in\mathbb{N}$. Then*

$$\sum_{k=1}^{n}\frac{(-1)^kH_{k-1}t^{n+k}}{k\binom{n+k}{k}}=(1+t)^n\left\{\sum_{k=1}^{n}\frac{H_n-H_{k-1}}{k}\left(\frac{t}{t+1}\right)^k+\frac{(t+2)H_n}{t}\sum_{k=1}^{n}\frac{(-1)^k}{k\binom{2k}{k}}\left(\frac{t^2}{t+1}\right)^k-\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{2k}{k}}\left(\frac{t^2}{t+1}\right)^k\right\}. \tag{27}$$

Proof. Setting $m=n$ in (24), we get

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^kH_{k-1}t^{n+k}}{k\binom{n+k}{k}}=(1+t)^n\Bigg\{&-H_n^{(2)}+\sum_{k=1}^{n}\frac{H_n-H_{k-1}}{k}\left(\frac{t}{t+1}\right)^k+\frac{H_n}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{2n}{n+k}\left\{(1+t)^k-(1+t)^{-k}\right\}\\&-\frac{1}{\binom{2n}{n}}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2n}{n+k}\left\{(1+t)^k+(1+t)^{-k}\right\}\Bigg\}.\end{aligned}$$

Applying Lemma 2.3 and Lemma 2.4 with $x=-1-t$, the proof follows. □

Setting $t=-2$ in (27), we get

Corollary 3.6. *For any $n\in\mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n}\frac{2^{n+k}H_{k-1}}{k\binom{n+k}{k}}=\sum_{k=1}^{n}\frac{(H_n-H_{k-1})2^k}{k}-\sum_{k=1}^{n}\frac{2^{2k}}{k^2\binom{2k}{k}}.$$

Setting $t=1$ in (27), we get

Corollary 3.7. *For any $n\in\mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n}\frac{(-1)^kH_{k-1}}{k\binom{n+k}{k}}=2^n\left\{\sum_{k=1}^{n}\frac{H_n-H_{k-1}}{k2^k}+3H_n\sum_{k=1}^{n}\frac{(-1)^k}{k2^k\binom{2k}{k}}-\sum_{k=1}^{n}\frac{(-1)^k}{2^kk^2\binom{2k}{k}}\right\}.$$

After dividing both sides of equation (27) by t^n, differentiating with respect to t, and then evaluating the resulting identity at $t=1$ and $t=-2$, we obtain the following two corollaries:

Corollary 3.8. *Let $n\in\mathbb{N}$. Then*

$$\sum_{k=1}^{n}\frac{(-1)^kH_{k-1}}{\binom{n+k}{k}}=2^{n-1}\left\{\sum_{k=1}^{n}\frac{(k-n)(H_n-H_{k-1})}{k2^k}-\left\{(3n+4)H_n+3\right\}\sum_{k=1}^{n}\frac{(-1)^k}{k2^k\binom{2k}{k}}+n\sum_{k=1}^{n}\frac{(-1)^k}{2^kk^2\binom{2k}{k}}+9H_n\sum_{k=1}^{n}\frac{(-1)^k}{2^k\binom{2k}{k}}\right\}.$$

Corollary 3.9. *Let $n\in\mathbb{N}$. Then*

$$\sum_{k=1}^{n}\frac{2^{n+k}H_{k-1}}{\binom{n+k}{k}}=n\sum_{k=1}^{n}\frac{(H_n-H_{k-1})2^k}{k}-n\sum_{k=1}^{n}\frac{2^{2k}}{k^2\binom{2k}{k}}+\frac{H_n2^{2n+1}}{\binom{2n}{n}}-\sum_{k=1}^{n}\frac{2^{k+1}}{k}, \tag{28}$$

where we have used (13).

Remark 3.4. *Using $H_{k-1}=H_k-\frac{1}{k}$, $\sum_{k=0}^{n}2^kH_k=2^{n+1}H_n-\sum_{k=1}^{n}\frac{2^k}{k}$ and (20) in (28), we, after a short computation, obtain*

$$\sum_{k=1}^{n}\frac{2^{n+k}H_k}{\binom{n+k}{k}}=n\sum_{k=1}^{n}\frac{(H_n-H_{k-1})2^k}{k}-n\sum_{k=1}^{n}\frac{2^{2k}}{k^2\binom{2k}{k}}-\sum_{k=1}^{n}\frac{2^k}{k}+\frac{H_n2^{2n+1}}{\binom{2n}{n}}.$$

Remark 3.5. *Theorems 3.1 and 3.4 are not new in themselves, as they can be regarded as direct consequences of Theorems 1 and 2 in [6], respectively. However, these results, together with Lemmas 2.3 and 2.4, play a crucial role in establishing Theorems 3.2 and 3.5, which constitute the most important results of the present paper. These two theorems are genuinely new, and, apart from Corollaries 3.1 and 3.3, all the other results derived from them are new as well. By specializing the parameter t to various values in Theorems 3.2, 3.3 and 3.5, we can obtain, in addition to several known identities, a number of new formulas.*

The following theorem provides an alternating analogue of Sun's identity (1), which is also a new result.

Theorem 3.6. *Let $n \in \mathbb{N}$. Then,*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2\binom{n+k}{k}} = H_n^{(2)} + \sum_{p=1}^{n-1} \frac{2^p(H_{n-p} - H_n + H_p)}{p} + 3\sum_{\ell=1}^{n} \frac{2^\ell}{\ell} \sum_{k=1}^{\ell} \frac{(-1)^k}{k2^k\binom{2k}{k}} - 2\sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell^2\binom{2\ell}{\ell}}.$$

Proof. We define $\gamma_0 = 0$ and for $n \geq 1$

$$\gamma_n := \sum_{k=1}^{n} \frac{(-1)^k}{k^2\binom{n+k}{k}}.$$

Then we have

$$\begin{aligned}\gamma_{n+1} - \gamma_n &= \sum_{k=1}^{n} \frac{(-1)^k}{k^2}\left(\frac{1}{\binom{n+k+1}{k}} - \frac{1}{\binom{n+k}{k}}\right) + \frac{(-1)^{n+1}}{(n+1)^2\binom{2n+2}{n+1}} \\ &= \sum_{k=1}^{n} \frac{(-1)^k}{k^2} \frac{\binom{n+k}{k} - \binom{n+k+1}{k}}{\binom{n+1+k}{k}\binom{n+k}{k}} + \frac{(-1)^{n+1}}{(n+1)^2\binom{2n+2}{n+1}}.\end{aligned}$$

By applying (4) and (5), we obtain after some computation that

$$\gamma_{n+1} - \gamma_n = -\sum_{k=1}^{n} \frac{(-1)^k}{k(n+k+1)\binom{n+k}{k}} + \frac{(-1)^{n+1}}{(n+1)^2\binom{2n+2}{n+1}}.$$

By partial fraction decomposition this gives

$$\gamma_{n+1} - \gamma_n = -\frac{1}{n+1}\sum_{k=1}^{n} \frac{(-1)^k}{k\binom{n+k}{k}} + \frac{1}{n+1}\sum_{k=1}^{n} \frac{(-1)^k}{(n+k+1)\binom{n+k}{k}} + \frac{(-1)^{n+1}}{(n+1)^2\binom{2n+2}{n+1}}. \tag{29}$$

Using (5) and changing the indices, we get

$$\sum_{k=1}^{n} \frac{(-1)^k}{(n+k+1)\binom{n+k}{k}} = \sum_{k=2}^{n+1} \frac{(-1)^{k-1}}{k\binom{n+k}{k}}. \tag{30}$$

Substituting the expression given in (30) into (29) we find that

$$\begin{aligned}\gamma_{n+1} - \gamma_n &= -\frac{2}{n+1}\sum_{k=1}^{n} \frac{(-1)^k}{k\binom{n+k}{k}} - \frac{(-1)^{n+1}}{(n+1)^2\binom{2n+1}{n+1}} + \frac{(-1)^{n+1}}{(n+1)^2\binom{2n+2}{n+1}} - \frac{1}{(n+1)^2} \\ &= -\frac{2}{n+1}\sum_{k=1}^{n} \frac{(-1)^k}{k\binom{n+k}{k}} + \frac{(-1)^n}{2(n+1)^2\binom{2n+1}{n}} - \frac{1}{(n+1)^2}.\end{aligned}$$

Applying Corollary 3.2 this gives

$$\gamma_{n+1} - \gamma_n = -\frac{2^{n+1}}{n+1}\left(\sum_{k=1}^{n+1} \frac{1}{k2^k} + 3\sum_{k=1}^{n+1} \frac{(-1)^k}{k2^k\binom{2k}{k}}\right) + \frac{2(-1)^{n+1}}{(n+1)^2\binom{2n+2}{n+1}}. \tag{31}$$

Replacing n in (31) by ℓ and then summing both sides of the resultant identity from $\ell = 0$ to $\ell = n-1$, we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^2\binom{n+k}{k}} = -\sum_{\ell=1}^{n} \frac{2^\ell}{\ell}\sum_{k=1}^{\ell} \frac{1}{k2^k} - 3\sum_{\ell=1}^{n} \frac{2^\ell}{\ell}\sum_{k=1}^{\ell} \frac{(-1)^k}{k2^k\binom{2k}{k}} + 2\sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell^2\binom{2\ell}{\ell}}. \tag{32}$$

By [2, Lemma 1] with $a_k = \frac{2^k}{k}$ and $b_k = \frac{1}{k2^k}$, we have

$$\sum_{\ell=1}^{n} \frac{2^\ell}{\ell}\sum_{k=1}^{\ell} \frac{1}{k2^k} = \sum_{p=0}^{n-1}\sum_{k=1}^{n-p} \frac{2^p}{k(p+k)} = H_n^{(2)} + \sum_{p=1}^{n-1} \frac{2^p}{p}\sum_{k=1}^{n-p}\left(\frac{1}{k} - \frac{1}{p+k}\right) = H_n^{(2)} + \sum_{p=1}^{n-1} \frac{2^p(H_{n-p} - H_n + H_p)}{p}. \tag{33}$$

Using (33) in (32), the proof immediately follows. □

4. Concluding Remarks

In this paper, we have developed a systematic framework for evaluating several classes of finite sums involving harmonic numbers and binomial coefficients. The main results, including Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and their corollaries, provide explicit closed-form expressions that unify and extend a number of previously known identities.

Furthermore, the finite sums studied here admit natural infinite analogues. As shown in [4], explicit evaluations can be obtained for the series

$$\sum_{k=1}^{\infty} \frac{t^k}{k^p \binom{n+k}{k}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_{k-1} t^k}{k^p \binom{n+k}{k}},$$

where $n \in \mathbb{N}$, $p = 0, 1, 2$, and $t \in \mathbb{C}$ with $|t| \leq 1$. These series may be interpreted as generating function counterparts of the finite sums considered in this work. For many other series of these types we refer the readers to [12, 15, 16, 20].

The connection between finite and infinite summation formulas suggests several promising directions for further research. For example, one may extend the present results to higher-order harmonic numbers, investigate q-analogues, or explore links with special functions such as polylogarithms. Overall, the results obtained here contribute to the growing body of literature on harmonic number identities and provide new tools for the study of finite and infinite summation problems.

Acknowledgment

The author sincerely thanks both referees for their careful reading of the manuscript and for their valuable comments and suggestions. Their constructive feedback, particularly the detailed corrections and suggestions provided by one of the referees, has helped me correct several errors and significantly improve the quality of the paper.

References

[1] H. Alzer, On a combinatorial sum, *Indag. Math.* **26** (2015) 519–525.
[2] N. Batır, Remarks on Vălean's master theorem of series, *J. Class. Anal.* **13** (2018) 79–82.
[3] N. Batır, Combinatorial identities involving harmonic numbers, *Integers* **20** (2020) #A25.
[4] N. Batır, Parametric binomial sums involving harmonic numbers, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **115** (2021) #91.
[5] N. Batır, Certain incomplete sums of reciprocals of the binomial coefficients, *Contrib. Math.* **12** (2025) 56–63.
[6] N. Batır, J. Choi, Parameterized finite binomial sums, *Mathematics* **12** (2024) #2450.
[7] N. Batır, J. Choi, Novel finite sum identities involving reciprocals of binomial coefficients, *Symmetry* **17** (2025) #1318.
[8] N. Batır, K. W. Chen, Combinatorial identities involving reciprocals of the binomial and central binomial coefficients and harmonic numbers, *J. Difference Equ. Appl.* **28** (2022) 222–243.
[9] N. Batır, A. Sofo, Finite sums involving reciprocals of the binomial and central binomial coefficients and harmonic numbers, *Symmetry* **13** (2021) #2002.
[10] N. Batır, A. Sofo, Sums involving the binomial coefficients, Bernoulli numbers of the second kind and harmonic numbers, *Notes Number Theory Discrete Math.* **29** (2023) 78–97.
[11] C. Li, W. Chu, Binomial harmonic series via parametric integration, *Integral Transforms Spec. Funct.*, In press, DOI: 10.1080/10652469.2026.2653709.
[12] T. Mansour, Combinatorial identities and inverse binomial coefficients, *Adv. Appl. Math.* **28** (2002) 196–202.
[13] J. Pla, The sum of inverses of binomial coefficients revisited, *Fibonacci Quart.* **35** (1997) 342–345.
[14] A. M. Rockett, Sums of the inverses of binomial coefficients, *Fibonacci Quart.* **19** (1981) 433–436.
[15] A. Sofo, Harmonic numbers and double binomial coefficients, *Integral Transforms Spec. Funct.* **20** (2009) 847–857.
[16] A. Sofo, New families of alternating harmonic number sums, *Tbilisi Math. J.* **8** (2015) 195–209.
[17] A. Sofo, N. Batır, On finite sums and integral representations, *Integers* **13** (2013) #A45.
[18] R. Sprugnoli, Sums of reciprocals of the central binomial coefficients, *Integers* **6** (2006) #A27.
[19] Z. W. Sun, Super congruences and Euler numbers, *Sci. China Math.* **54** (2011) 2509–2535.
[20] B. Sury, T. Wang, F.-Z. Zhao, Identities involving reciprocals of binomial coefficients, *J. Integer Seq.* **7** (2004) #04.2.8.