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Research Article

Locally Even and Locally Odd Edge 2-Colorings of Pseudographs

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Abstract

Motivated by classical results on even and odd factors, edge 2-colorings of pseudographs under local parity constraints are investigated. An edge 2-coloring is called locally even if both colors occur at every vertex and, at each vertex, at least one color appears an even number of times. A locally odd edge 2-coloring is defined analogously. It is proved that every pseudograph of minimum degree at least 3 admits a locally even edge 2-coloring. The odd analogue is also established: a pseudograph of minimum degree at least 2 admits a locally odd edge 2-coloring if and only if no component of odd order has all vertices of even degree. As an application, it is shown that every connected plane pseudograph with minimum face size at least 3 admits a weak even facial edge 2-coloring in which the boundary of every face contains both colors. This strengthens and extends the corresponding result for 2-connected simple plane graphs.

Keywords: edge coloring; factor; pseudograph.

2020 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction

All graphs considered in this paper are finite. A *pseudograph* may contain loops and parallel edges. A graph with no loops but possibly with parallel edges is called a *multigraph*, and a graph with neither loops nor parallel edges is called *simple*. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. The numbers $|V(G)|$ and $|E(G)|$ are the *order* and *size* of G. The degree of a vertex v in G, denoted by $\deg_G(v)$, is the number of edges incident with v, where a loop is counted twice. A vertex is called odd or even according to the parity of its degree. The minimum degree of G is denoted by $\delta(G)$.

Let G be a graph whose edges are colored with colors 1 and 2. For $i \in \{1, 2\}$, let $c_i(v)$ denote the number of edges of color i incident with the vertex v, where a loop of color i at v is counted twice. We say that color i occurs at v if $c_i(v) > 0$. An edge 2-coloring of G is called *locally even* if both colors occur at every vertex and, for every vertex v, at least one of $c_1(v)$ and $c_2(v)$ is even. Similarly, an edge 2-coloring is called *locally odd* if both colors occur at every vertex and, for every vertex v, at least one of $c_1(v)$ and $c_2(v)$ is odd.

A *spanning subgraph* of G is a subgraph with vertex set $V(G)$, and an *even factor* is a spanning subgraph in which every vertex has positive even degree. Thus isolated vertices are not allowed. A 2-factor is an even factor in which every vertex has degree two.

The existence of even factors has a long history. Petersen's theorem implies that every bridgeless cubic multigraph has a 2-factor. The following slightly more general form will be useful for comparison.

Theorem 1.1. *[7] Every cubic multigraph with at most two bridges has a 2-factor.*

This result is best possible with respect to the number of bridges: there are cubic graphs with three bridges and no 2-factor; one such graph is shown in Figure 1.1.

The example in Figure 1.1 has three bridges, but these bridges do not lie on a single path. Errera proved that this is the essential obstruction in the cubic setting.

Theorem 1.2. *[2] Let G be a connected cubic pseudograph such that all the bridges of G, if any, are contained in a path of G. Then G has a 2-factor.*

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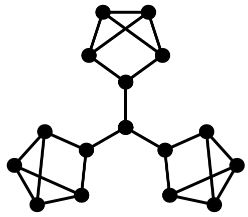

Figure 1.1: A cubic graph with no 2-factor.

It is natural to ask whether similar statements remain true beyond cubic graphs. In the bridgeless case, Fleischner gave the following positive answer.

Theorem 1.3. *[3] Every bridgeless multigraph G with $\delta(G) \geq 3$ has an even factor.*

The assumptions in Fleischner's theorem cannot be weakened in a naive way. Some graphs with bridges do have even factors, as Theorem 1.2 shows, but the bridgeless assumption cannot be omitted, as the graph in Figure 1.1 has no even factor. The condition $\delta(G) \geq 3$ is also essential: although $K_{2,2k+1}$ is bridgeless and has minimum degree 2, it has no even factor.

Fleischner's theorem may be viewed as an edge 2-coloring statement. One color class is required to induce an even factor, while the other color class is allowed to be absent at some vertices. A natural strengthening would ask whether the even factor can always be chosen properly, in the sense that the complementary color also appears at every vertex.

Problem 1.1. *Does every bridgeless multigraph G with $\delta(G) \geq 3$ admit an even factor H such that $\deg_H(v) < \deg_G(v)$ for every vertex v?*

First we show that even among bridgeless multigraphs of minimum degree at least three, an even factor may be forced to use all edges incident with some vertex.

Theorem 1.4. *Let k be a positive integer. Let G be a bipartite multigraph with partite sets V_1 and V_2 such that $|V_1| = 3k$, $|V_2| = 4k$, and*

$$\deg_G(v) = \begin{cases} 4, & \text{if } v \in V_1, \\ 3, & \text{if } v \in V_2. \end{cases}$$

Then every even factor H of G contains a vertex u such that $\deg_H(u) = \deg_G(u)$.

Proof. Let H be an even factor of G. Since H is a spanning subgraph of G and every vertex has positive even degree in H, every vertex has degree at least 2 in H. Vertices in V_2 have degree 3 in G, and hence they must have degree 2 in H.

Suppose, to the contrary, that $\deg_H(v) \neq \deg_G(v)$ for every vertex v. Then every vertex in V_1 also has degree 2 in H, because the only possible positive even degrees at such a vertex are 2 and 4. Thus $\deg_H(v) = 2$ for every vertex v of G. Since H is bipartite with partite sets V_1 and V_2, we obtain

$$\sum_{v \in V_1} \deg_H(v) = \sum_{v \in V_2} \deg_H(v).$$

Consequently, $2|V_1| = 2|V_2|$, which contradicts $|V_1| = 3k$ and $|V_2| = 4k$. □

Bridgelessness is relevant to Problem 1.1, but it is not needed in Theorem 1.4. In particular, $K_{3,4}$ is bridgeless and satisfies the assumptions of Theorem 1.4 with $k = 1$. Hence $K_{3,4}$ gives a negative answer to Problem 1.1.

The obstruction in Problem 1.1 is global: it concerns one prescribed color class that is required to be an even factor at every vertex. The local version is more flexible. At each vertex we still require both colors to occur, but the color that appears an even number of times may vary from vertex to vertex.

Our first main result shows that, under the natural minimum-degree assumption, this local flexibility removes all obstructions.

Theorem 1.5. *Every pseudograph G with $\delta(G) \geq 3$ has a locally even edge 2-coloring.*

The condition $\delta(G) \geq 3$ is best possible. A vertex of degree 1 cannot be incident with both colors. A vertex of degree 2 is also impossible: either only one color occurs at the vertex, or both colors occur exactly once, in which case neither color occurs an even number of times.

Theorem 1.5 should therefore be viewed not as a reformulation of Fleischner's theorem, but as a local analogue of it. In an even factor, one fixed color class must have positive even degree at every vertex. In a locally even coloring, both colors must be present at every vertex, but the parity requirement is local: the color satisfying the evenness condition may be different at different vertices. This shift from a global to a local parity requirement is the central idea of the paper.

We also investigate the corresponding odd version. An *odd factor* of a graph is a spanning subgraph in which every vertex has odd degree. A known characterization states that a multigraph has an odd factor if and only if every component has even order [6]. Since loops do not affect degree parity, the same characterization applies to pseudographs.

Theorem 1.6. *[6] If G is a connected pseudograph of even order, then G has an odd factor.*

A global strengthening would ask whether every connected graph G of even order has an odd factor H such that $\deg_H(v) < \deg_G(v)$ for every vertex v. Again, this is not always possible. In a cubic graph, such an odd factor would have degree 1 at every vertex, and its complement would be a 2-factor. Hence the cubic graph in Figure 1.1, which has no 2-factor, has no odd factor H satisfying $\deg_H(v) < \deg_G(v)$ for all vertices v.

This motivates the local odd analogue.

Problem 1.2. *Does every pseudograph G with $\delta(G) \geq 2$ admit a locally odd edge 2-coloring?*

Unlike the locally even case, the locally odd problem has one unavoidable obstruction. Let $\mathcal{G}$ denote the class of pseudographs that contain a component of odd order in which every vertex has even degree. Such a component cannot admit a locally odd edge 2-coloring. Indeed, if the degree of every vertex in the component is even, then requiring one color to appear an odd number of times at a vertex forces both colors to appear an odd number of times there. This would give an odd factor of a component of odd order, which is impossible.

Our second main result shows that this obvious obstruction is the only one.

Theorem 1.7. *A pseudograph G with $\delta(G) \geq 2$ has a locally odd edge 2-coloring if and only if $G \notin \mathcal{G}$.*

The lower bound $\delta(G) \geq 2$ is necessary, since a vertex of degree 1 cannot be incident with both colors. Thus Theorem 1.7 gives a complete characterization of pseudographs, under the natural minimum-degree assumption, that admit a locally odd edge 2-coloring.

Finally, we apply the locally even result to plane graphs. A *plane graph* is a planar graph with a fixed embedding. For a face f of a connected plane graph G, the *size* of f is the length of its boundary walk, and the minimum face size of G is denoted by $\delta^*(G)$.

An edge 2-coloring of a connected plane pseudograph G is called a *weak even facial edge coloring* if, for every face f, at least one color appears an even number of times on the boundary walk of f. We say that no face is monochromatic if both colors appear on the boundary walk of every face.

Theorem 1.5 implies the following facial coloring result.

Theorem 1.8. *Every connected plane pseudograph G with $\delta^*(G) \geq 3$ admits a weak even facial edge coloring with two colors in which no face is monochromatic.*

This strengthens a theorem of Czap [1], who proved that every 2-connected simple plane graph admits a weak even facial edge coloring with two colors.

2. Locally Even Edge 2-Coloring

We first deal with Eulerian pseudographs. Recall that an Euler circuit is a closed walk that traverses each edge exactly once. A connected graph is Eulerian if it admits an Euler circuit. Eulerian graphs have a simple characterization.

Theorem 2.1. *A connected pseudograph G is Eulerian if and only if every vertex of G is even.*

Lemma 2.1. *Let G be a connected Eulerian pseudograph of even size. Then G has an edge 2-coloring such that*

$$c_1(v) = c_2(v) = \frac{\deg(v)}{2} \qquad \text{for every } v \in V(G).$$

Proof. Let $C = v_0\, e_1\, v_1\, e_2 \cdots e_r\, v_r$, $v_r = v_0$, be an Euler circuit of G. Since $r = |E(G)|$ is even, we may color $e_1, e_2, \ldots, e_r$ alternately with colors 1 and 2; cyclically, the last edge and the first edge also have different colors.

Since consecutive edges of C receive different colors, whenever the circuit arrives at a vertex v along an edge of one color, it leaves v along an edge of the other color. Thus the two colors occur equally often at v. A loop causes no difficulty, since it appears once as an arriving edge and once as a departing edge at v, so it is counted twice at v. Hence $c_1(v) = c_2(v) = \frac{\deg(v)}{2}$ for every vertex v. □

We shall use Hilton's canonical edge-coloring theorem (Theorem 8 in [5]) concerning edge decompositions of graphs whose vertices all have even degree. The following formulation also appears as Theorem 1.5 of Hasanvand [4].

Theorem 2.2. *[5] Let G be a pseudograph in which every vertex has even degree, and let k be a positive integer. Then $E(G)$ can be decomposed into spanning subgraphs $G_1, \dots, G_k$ in which all vertex degrees are even and*

$$\left| \deg_{G_i}(v) - \frac{\deg_G(v)}{k} \right| < 2 \qquad \text{for every } v \in V(G) \text{ and } 1 \le i \le k.$$

Lemma 2.2 is an immediate consequence of Theorem 2.2.

Lemma 2.2. *Let G be a connected Eulerian pseudograph with $\delta(G) \ge 4$. Then G has an edge 2-coloring such that both colors occur at every vertex and each color appears an even number of times at every vertex.*

Proof. Apply Theorem 2.2 with $k = 2$, and color the edges of G_i with color i, $i \in \{1, 2\}$. For every vertex v and each i we have

$$\left| \deg_{G_i}(v) - \frac{\deg_G(v)}{2} \right| < 2.$$

Since $\deg_G(v) \ge 4$, this gives $\deg_{G_i}(v) > 0$ and $\deg_{G_i}(v) < \deg_G(v)$. Moreover, each $\deg_{G_i}(v)$ is even by Theorem 2.2. Thus both colors occur at every vertex and each appears there an even number of times. □

Lemma 2.2 has two immediate factor-theoretic consequences.

Corollary 2.1. *Every connected Eulerian pseudograph G with $\delta(G) \ge 4$ can be decomposed into two even factors.*

Proof. Apply Lemma 2.2. The two color classes are spanning subgraphs in which every vertex has positive even degree. □

Corollary 2.2. *Every connected Eulerian pseudograph G with $\delta(G) \ge 4$ admits an even factor H such that $\deg_H(v) < \deg_G(v)$ for every vertex v.*

Proof. In the decomposition given by Corollary 2.1, either color class is an even factor. Since the other color also occurs at every vertex, the chosen factor has degree strictly smaller than $\deg_G(v)$ at every vertex v. □

We now prove the locally even theorem.

Proof of Theorem 1.5. It is sufficient to color each connected component separately, so assume that G is connected.

Let X be the set of vertices of odd degree in G. By the handshaking lemma, $|X|$ is even. Pair the vertices of X arbitrarily and, for each pair, add one new edge joining its two vertices. Let G' be the resulting pseudograph.

Every vertex of G' has even degree. Moreover, $\delta(G') \ge 4$. Indeed, if $v \notin X$, then $\deg_{G'}(v) = \deg_G(v)$, which is even and at least 4; if $v \in X$, then $\deg_{G'}(v) = \deg_G(v) + 1 \ge 4$. Thus G' is a connected Eulerian pseudograph with $\delta(G') \ge 4$.

By Lemma 2.2, the graph G' has an edge 2-coloring in which both colors occur at every vertex and each color appears an even number of times at every vertex. Delete all added edges. This leaves an edge 2-coloring of the original graph G.

Let $v \in V(G)$. If $\deg_G(v)$ is even, then no added edge was incident with v; hence both colors still occur at v, and each color appears there an even number of times. If $\deg_G(v)$ is odd, then exactly one added edge was incident with v. Suppose this edge has color 1. In G', both $c_1(v)$ and $c_2(v)$ were positive and even. After deleting the added edge, $c_1(v)$ decreases by 1, so it is odd but remains positive, while $c_2(v)$ remains positive and even. Therefore both colors occur at v, and at least one color appears an even number of times.

Thus the obtained coloring is locally even. □

3. Locally Odd Edge 2-Coloring

For completeness, we first give a simple proof of Theorem 1.6. Since every connected pseudograph has a spanning tree, it suffices to show that every tree of even order has an odd factor. We proceed by induction on the order of the tree. The case $|V(T)| = 2$ is clear. Let T' be a tree of even order at least 4, and let P be a longest path in T'. Let u be an endvertex of P, and let v be its neighbor on P. If v has degree 2 in T', then $T = T' - \{u, v\}$ is a tree of even order. By induction, T has an odd factor H. Then $H \cup \{uv\}$ is an odd factor of T'. Now assume that v has degree at least 3 in T'. Since P is a longest path in T', every neighbor of v, except the next vertex of P toward the opposite end, is a leaf. Hence v is adjacent to at least two leaves. Let $w \neq u$ be a leaf adjacent to v. Then $T = T' - \{u, w\}$ is a tree of even order. By induction, T has an odd factor H. Since $v \in V(T)$, its degree in H is odd. Therefore $H \cup \{vu, vw\}$ is an odd factor of T', because the degrees of u and w become 1, the degree of v increases by 2, and the degrees of all other vertices remain unchanged. Thus T' has an odd factor, and the theorem follows.

We now turn to locally odd colorings. Let $\mathcal{G}$ be the class of pseudographs defined in the Introduction; thus $G \in \mathcal{G}$ if and only if G has a component of odd order in which every vertex has even degree.

The following lemma explains why the class $\mathcal{G}$ cannot be avoided.

Lemma 3.1. *If $G \in \mathcal{G}$, then G has no locally odd edge 2-coloring.*

Proof. Let C be a component of G of odd order in which every vertex has even degree. Suppose, to the contrary, that G has a locally odd edge 2-coloring. Every vertex of C has an even degree, hence, if one color appears an odd number of times there, then the other color also appears an odd number of times there. Since the coloring is locally odd, both colors appear an odd number of times at every vertex of C. Therefore, in the spanning subgraph of C whose edge set consists of the edges of color 1, every vertex has odd degree. This is impossible by the handshaking lemma, because C has an odd number of vertices. □

Proof of Theorem 1.7. The necessity follows from Lemma 3.1. We prove sufficiency. It is sufficient to color each connected component separately, so assume that G is connected and $G \notin \mathcal{G}$. Thus G is not an odd-order graph all of whose vertices have even degree.

First assume that every vertex of G has odd degree. Since $\delta(G) \geq 2$, in fact $\delta(G) \geq 3$. By Theorem 1.5, G has a locally even edge 2-coloring. At a vertex of odd degree, if one color appears an even number of times, then the other color appears an odd number of times. Hence the same coloring is locally odd.

Next assume that every vertex of G has even degree. Since G is connected and $G \notin \mathcal{G}$, the order of G is even. By Theorem 1.6, G has an odd factor H. For every vertex x, the degree $\deg_H(x)$ is odd, whereas $\deg_G(x)$ is even; hence $1 \leq \deg_H(x) < \deg_G(x)$. Color all edges of H with color 1 and all remaining edges with color 2. Then both colors occur at every vertex, and color 1 appears an odd number of times at every vertex. Thus the coloring is locally odd.

It remains to consider the case in which G has both even and odd vertices. We construct an auxiliary pseudograph G' as follows. For every vertex u with $\deg_G(u) \equiv 0 \pmod 4$, add one loop at u. Add a new vertex z and join z to every odd vertex of G. Since the number of odd vertices of G is even, the degree of z is even. The original vertices also have even degree in the resulting graph: vertices of degree 2 modulo 4 are unchanged, vertices of degree 0 modulo 4 receive one loop, and odd vertices receive one edge to z. If the resulting graph has odd size, add one loop at z; this preserves even degrees and makes the size even. Let G' denote the final pseudograph.

Because G is connected and has at least one odd vertex in the present case, the graph G' is connected. Every vertex of G' has even degree, and $|E(G')|$ is even. Therefore G' is an Eulerian pseudograph of even size. By Lemma 2.1, it has an edge 2-coloring such that $c_1(x) = c_2(x) = \frac{\deg_{G'}(x)}{2}$ for every $x \in V(G')$.

Delete the vertex z, and delete all loops that were added at vertices of G. The remaining coloring is an edge 2-coloring of the original graph G.

Let $x \in V(G)$. If $\deg_G(x) = 4k + 2$ for some $k \geq 0$, then no added edge or loop was incident with x. Hence each color appears $(4k + 2)/2 = 2k + 1$ times at x, so both colors appear an odd number of times.

If $\deg_G(x) = 4k$ for some $k \geq 1$, then one loop was added at x. In G', each color appears $2k + 1$ times at x. After deleting this loop, one color appears $2k - 1$ times at x, while the other appears $2k + 1$ times. Thus both colors occur at x, and both appear an odd number of times.

Finally, suppose that $\deg_G(x) = 2k+1$ for some $k \geq 1$. Then exactly one added edge xz was incident with x. In G', each color appears $k+1 \geq 2$ times at x. After deleting xz, one color decreases by 1, so it appears k times at x, while the other color still appears $k+1$ times. Since both colors appeared at least twice in G', both colors still occur at x in G. Moreover, one of k and $k+1$ is odd, so at least one color appears an odd number of times at x.

Therefore the resulting coloring is locally odd in all cases. □

4. An Application to Plane Graphs

Let G be a connected plane pseudograph. The *plane dual* G^* of G is defined as follows. Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*. If the edge e of G is incident with faces f and g in G, then the endvertices of the dual edge $e^* \in E(G^*)$ are the vertices f^*, g^* of G^* that correspond to the faces f, g in G. Observe that if e is a bridge in G, then $f = g$, so e^* is a loop in G^*. We place each vertex f^* in the corresponding face f of G, and then draw each edge e^* in such a way that it crosses the corresponding edge e of G exactly once and crosses no other edge of G. This drawing of G^* is the plane dual of G.

Proof of Theorem 1.8. Let G^* be the plane dual of G. By the definition of G^*, we have $\delta(G^*) = \delta^*(G)$. Since $\delta^*(G) \geq 3$, we have $\delta(G^*) \geq 3$. By Theorem 1.5, the pseudograph G^* has a locally even edge 2-coloring.

Transfer this coloring to the edges of G via the natural one-to-one correspondence between $E(G)$ and $E(G^*)$. Let f be a face of G, and let f^* be the corresponding vertex of G^*. Since both colors appear at f^*, both colors occur on the boundary walk of f. Moreover, if color i appears an even number of times at f^*, then color i appears an even number of times on the boundary walk of f. Thus the induced coloring of G is a weak even facial edge coloring with two colors in which no face is monochromatic. □

For simple plane graphs we obtain the following immediate corollary.

Corollary 4.1. *Every connected simple plane graph with at least two edges admits a weak even facial edge coloring with two colors in which no face is monochromatic.*

Proof. A connected simple plane graph with at least two edges has no face of size 1 or 2. Hence every face has size at least 3, so $\delta^*(G) \geq 3$, and the result follows from Theorem 1.8. □

The assumption $\delta^*(G) \geq 3$ in Theorem 1.8 is necessary. Indeed, let G consist of two vertices joined by several parallel edges. Then every face of G has size 2. In any edge coloring of G satisfying the weak even condition, the two edges on the boundary of each face must receive the same color; otherwise each color would appear exactly once on that face. Consecutive faces then force all edges to receive the same color, so every weak even facial edge coloring of G uses only one color.

5. Open Problems

We conclude the paper by proposing two open problems.

Problem 5.1. *Characterize all pseudographs G that have an even factor H such that $\deg_H(v) < \deg_G(v)$ for every vertex v.*

Problem 5.2. *Characterize all pseudographs G that have an odd factor H such that $\deg_H(v) < \deg_G(v)$ for every vertex v.*

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