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Research Article

Graphs with Small Triameters

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Abstract

The triameter of a graph is the maximum sum of pairwise distances over all triples of vertices. Structural properties of graphs with small triameters are investigated. It is shown that all admissible pairs of diameter and triameter can occur, and that the triameter cannot be bounded below in terms of the maximum and minimum vertex degrees. The paper then focuses on the case where the triameter equals six, obtaining structural characterizations for several natural graph classes, including block graphs, graphs with a unique cut vertex, and bipartite graphs, together with further corollaries for modular and median graphs.

Keywords: triameter; graph diameter; block graphs; biconnected graphs; bipartite graphs; modular graphs; median graphs.

2020 Mathematics Subject Classification: 05C12, 05C75.

1. Introduction

The vertex set of every connected simple undirected graph G can be endowed with a discrete metric structure via the natural graph distance d_G. Namely, for two vertices $u, v \in V(G)$, let $d_G(u,v)$ be the length of a shortest $u-v$ path in G. Despite the classical characterization of graph metrics due to Kay and Chartrand [6], metric graph theory remains an active area of research, with close connections to other branches of mathematics such as discrete geometry, convexity spaces, simplicial and cubical complexes, median algebras, lopsided sets, and others (see the survey paper [1] by Bandelt and Chepoi).

Apart from qualitative properties, for finite graphs it is natural to consider quantitative parameters related to the graph metric. One of the simplest such parameters is the diameter $\mathrm{diam}(G)$ of a connected graph G, defined as the maximum distance between pairs of vertices in G. A closely related parameter, called the triameter, was introduced by Das [3]. It is the number $\mathrm{tr}(G)$ defined as the maximum of the sums $d_G(u,v)+d_G(u,w)+d_G(v,w)$ taken over all triples $u,v,w \in V(G)$.

In [3], various lower and upper bounds on $\mathrm{tr}(G)$ were established, together with characterizations of graphs attaining extremal values of the triameter. Nordhaus–Gaddum-type bounds relating $\mathrm{tr}(G)$ and $\mathrm{tr}(\overline{G})$ were also obtained. In particular, the work [3] provided a lower bound on the triameter of trees in terms of the number of vertices and leaves. This bound was later refined to an optimal one by Hak, Kozerenko, and Oliynyk in [5]. Furthermore, in [5], two other open questions from [3] were answered: it was shown that in block graphs, any diametral pair of vertices can be extended to a triametral triple, and conversely, any triametral triple contains a diametral pair. Several related questions were also posed for the classes of median graphs and distance-hereditary graphs. Finally, in [4], among other results, Das and Abdioğlu characterized graphs with triameter 4 and 5, and posed the problem of characterizing graphs G with $\mathrm{tr}(G)=\mathrm{tr}(\overline{G}) \in \{6,7,8\}$.

The aim of this work is to further investigate the structure of graphs with small triameters. In particular, we focus on graphs with triameter 6. We provide characterizations of such graphs within the following graph classes: block graphs, non-biconnected graphs, bipartite graphs, modular graphs, and median graphs.

This paper is organized as follows. In Section 2, we provide all necessary definitions and preliminary results used throughout the paper. The main results are presented in Section 3. First, we prove a realization theorem for the diameter and triameter (see Proposition 3.1), showing that there are no nontrivial inequalities between these two parameters for general graphs. In Proposition 3.2, we give a negative answer to the final problem from [3] concerning the existence of a lower bound on the triameter in terms of the maximum and minimum degrees. Theorem 3.1 provides a characterization of block graphs with triameter at most 8. A criterion for graphs with a unique cut vertex and triameter 6 is established in Theorem 3.2. Finally, bipartite graphs with triameter 6 are characterized in Theorem 3.3, together with corresponding corollaries for the related classes of modular and median graphs.

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2. Main Definitions

General Notions

All graphs in this work are assumed to be finite, simple, and undirected. For a vertex $u \in V(G)$ in a graph G, we denote by $N_G(u) = \{x \in V(G) : \{u,x\} \in E(G)\}$ and $N_G[u] = N_G(u) \cup \{u\}$ the *neighborhood* and *closed neighborhood* of u, respectively. The *degree* of a vertex u is the number $\deg_G(u) = |N_G(u)|$. By $\Delta(G)$ and $\delta(G)$ we denote the largest and the smallest vertex degrees in G, respectively. A vertex u is called *universal* if $\deg_G(u) = |V(G)| - 1$, *isolated* if $\deg_G(u) = 0$, and a *leaf* if $\deg_G(u) = 1$. The unique neighbor of a leaf vertex is called its *support*, and the corresponding edge is called a *leaf edge*.

For a subset of vertices $A \subseteq V(G)$, we denote by $G[A]$ the corresponding induced subgraph. For a pair of disjoint subsets $A, B \subseteq V(G)$, we denote by $G[A,B]$ the corresponding *induced bipartite subgraph* with vertex set $A \cup B$ and edge set $\{\{a,b\} \in E(G) : a \in A,\ b \in B\}$.

A graph is *connected* if every pair of its vertices is joined by a path. A *connected component* of a graph is a maximal connected subgraph.

By K_n we denote the complete graph on n vertices. A graph is a *cluster graph* if each of its connected components is a complete graph. By $\overline{G}$ we denote the complement of a graph G. The complement $\overline{K}_n$ of a complete graph is called an *empty graph*. By P_n and C_n we denote the *path graph* and the *cycle graph* on n vertices, respectively.

A set of vertices $A \subseteq V(G)$ is called *independent* if no two vertices in A are adjacent (equivalently, $G[A]$ is empty). A set of edges $E' \subseteq E(G)$ is called a *matching* if no two distinct edges in E' share a vertex.

Let G and H be two graphs. Their *join* $G + H$ is the graph with vertex set $V(G) \sqcup V(H)$ (where $\sqcup$ denotes disjoint union) and edge set $E(G) \cup E(H) \cup \{\{u,v\} : u \in V(G), v \in V(H)\}$. A graph is *bipartite* if its vertex set can be partitioned into two independent sets. By $K_{m,n} = \overline{K}_m + \overline{K}_n$ we denote the *complete bipartite graph* with parts of cardinalities $m, n \in \mathbb{N}$.

The *Cartesian product* of two graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and edge set $\{\{(u,v),(x,y)\} : u = x,\ \{v,y\} \in E(H)$ or $\{u,x\} \in E(G),\ v = y\}$. The *grid graph* is the Cartesian product of two paths $P_n \square P_m$. The *hypercube* Q_n is the Cartesian product of n copies of K_2. For example, $Q_2 = C_4$, and Q_3 is the *cube*.

A *tree* is a connected graph without cycles. Path graphs P_n and *stars* $K_{1,n-1}$ are prominent examples of trees. A *spider* is a tree which contains at most one vertex with degree at least 3. For example, path graphs and stars are examples of spiders. If a spider T contains a vertex u of degree at least 3, then u is called its *center*, and all paths from u to the leaf vertices of T are called the *legs* of T.

A vertex in a connected graph is called a *cut vertex* if its deletion results in a disconnected graph. The *block-degree* $\mathrm{bdeg}_G(u)$ of a vertex $u \in V(G)$ is the number of connected components in $G - u$. A graph is *biconnected* if it does not have cut vertices. A *block* in a graph is a maximal biconnected subgraph. A graph is a *block graph* if all of its blocks are complete.

The *metric interval* between a pair of vertices $u, v \in V(G)$ in a graph G is the set $[u,v]_G = \{x \in V(G) : d_G(u,x) + d_G(x,v) = d_G(u,v)\}$. For a triple of vertices $x, y, z \in V(G)$, by $M_G(x,y,z) = [x,y]_G \cap [x,z]_G \cap [y,z]_G$ we denote their *median set*. A graph G is called *modular* if $M_G(x,y,z) \neq \emptyset$ for all $x, y, z \in V(G)$. For example, complete bipartite graphs $K_{m,n}$ are modular. Also, modular graphs are necessarily bipartite. A graph G is called *median* provided $|M_G(x,y,z)| = 1$ for all $x, y, z \in V(G)$. Of course, every median graph is modular. Trees and hypercubes are prominent examples of median graphs. Finally, note that the Cartesian product of two median graphs is also median.

Radius, Diameter and Triameter of Graphs

In what follows, all graphs will be assumed to be *connected* unless stated otherwise.

The *radius* of a graph G is the number $\mathrm{rad}(G) = \min\{\max\{d_G(u,v) : v \in V(G)\} : u \in V(G)\}$. The *diameter* of G is the number $\mathrm{diam}(G) = \max\{d_G(u,v) : u, v \in V(G)\}$. One can observe that $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$. Apart from this inequality, there are no further restrictions relating these two parameters.

Proposition 2.1 (folklore)**.** *For all numbers $r, d \in \mathbb{N}$ with $r \leq d \leq 2r$, there exists a graph G having $\mathrm{rad}(G) = r$ and $\mathrm{diam}(G) = d$. Moreover, one can choose G to be a path graph or a unicyclic graph.*

For a triple of vertices $u, v, w \in V(G)$ define $d_G(u,v,w) = d_G(u,v) + d_G(u,w) + d_G(v,w)$. The *triameter* [3] of a graph G is defined as $\mathrm{tr}(G) = \max\{d_G(u,v,w) : u, v, w \in V(G)\}$.

It is not hard to show that $2\,\mathrm{diam}(G) \leq \mathrm{tr}(G) \leq 3\,\mathrm{diam}(G)$ for any graph G. Further, it is clear that $\mathrm{tr}(G) = 2$ if and only if $G = K_2$; and $\mathrm{tr}(G) = 3$ if and only if G is a complete graph with $n \geq 3$ vertices. In [4], Das and Abdioğlu completely characterized graphs with triameters 4 and 5.

Theorem 2.1. [4, Theorems 4.1, 4.2] *Let G be a graph with $n \geq 3$ vertices. Then*

1. $\mathrm{tr}(G) = 4$ *if and only if* $G = K_n - M$*, where* $M \subseteq E(K_n)$ *is a non-empty matching.*
2. $\mathrm{tr}(G) = 5$ *if and only if* $\mathrm{diam}(G) = 2$ *and* $G = K_n - T$*, where* $T \subseteq E(K_n)$ *is a set of edges containing at least two adjacent edges but no triangle.*

A set of vertices $A \subseteq V(G)$ is called *total dominating* provided for any $x \in V(G)$ there is $a \in A$ with $\{a, x\} \in E(G)$. The *total domination number* $\gamma_t(G)$ of G is defined as the cardinality of a smallest total dominating set in G.

Proposition 2.2. [3, Corollary 3.12] *For any graph G, we have* $\mathrm{tr}(G) \leq 4\gamma_t(G)$.

It is well known that almost every graph G is connected and satisfies $\mathrm{diam}(G) = 2$ (see [7]). In particular, this implies that almost every graph G has $\mathrm{tr}(G) \leq 6$. Combining this observation with Theorem 2.1, we are naturally led to the following problem.

Main Problem. Characterize graphs G with $\mathrm{tr}(G) = 6$.

Observe that any graph G with $\mathrm{tr}(G) = 6$ necessarily satisfies $2 \leq \mathrm{diam}(G) \leq 3$. Moreover, in view of Theorem 2.1, the case $\mathrm{diam}(G) = 2$ is essentially trivial. Indeed, graphs G with $\mathrm{diam}(G) = 2$ and $\mathrm{tr}(G) = 6$ are precisely those graphs of diameter 2 that do not satisfy any of the conditions in Theorem 2.1 (here we use the fact that every graph of diameter 2 has triameter at most 6). Therefore, the main problem reduces to the characterization of graphs G with $\mathrm{tr}(G) = 6$ and $\mathrm{diam}(G) = 3$. In the next section, we obtain criteria for such graphs within several graph classes.

3. Main Results

We start by proving a realization theorem for the diameter and triameter, analogous to Proposition 2.1 for the radius and diameter.

Proposition 3.1. *For all numbers $d, t \in \mathbb{N}$ with $2d \leq t \leq 3d$, there exists a block graph G with* $\mathrm{diam}(G) = d$ *and* $\mathrm{tr}(G) = t$.

Proof. We consider two cases depending on the parity of t.

Case 1: t is even. If d is also even, consider a spider G with three legs of lengths $\frac{d}{2}$, $\frac{d}{2}$, and $\frac{t}{2} - d \geq 0$. Then $\mathrm{tr}(G) = 2(\frac{d}{2} + \frac{d}{2} + \frac{t}{2} - d) = t$ and $\mathrm{diam}(G) = \max\{\frac{d}{2} + \frac{d}{2}, \frac{d}{2} + \frac{t}{2} - d\} = \max\{d, \frac{t-d}{2}\} = d$ as $t \leq 3d$.

If d is odd, then again, let G be a spider with three legs of lengths $\frac{d+1}{2}$, $\frac{d-1}{2}$, and $\frac{t}{2} - d$. We have $\mathrm{tr}(G) = 2(\frac{d+1}{2} + \frac{d-1}{2} + \frac{t}{2} - d) = t$ and $\mathrm{diam}(G) = \max\{\frac{d+1}{2} + \frac{d-1}{2}, \frac{d+1}{2} + \frac{t}{2} - d, \frac{d-1}{2} + \frac{t}{2} - d\} = \max\{d, \frac{t+1-d}{2}, \frac{t-1-d}{2}\} = \max\{d, \frac{t+1-d}{2}\} = d$ as $t \leq 3d - 1$ (recall that here t is even and d is odd).

Case 2: t is odd. If d is also odd, consider a unicyclic graph G obtained from K_3 by attaching three paths to the three vertices of K_3 with the lengths of these paths $\frac{d-1}{2}$, $\frac{d-1}{2}$, and $\frac{t-1}{2} - d \geq 0$ (note that here $t \geq 2d + 1$). Then $\mathrm{tr}(G) = 2(\frac{d-1}{2} + \frac{d-1}{2} + \frac{t-1}{2} - d) + 3 = 2d - 2 + t - 1 - 2d + 3 = t$. And $\mathrm{diam}(G) = \max\{\frac{d-1}{2} + \frac{d-1}{2} + 1, \frac{d-1}{2} + \frac{t-1}{2} - d + 1\} = \max\{d, \frac{t-d}{2}\} = d$ as $t \leq 3d$.

If d is even, then construct the same unicyclic graph G but with lengths of the corresponding paths $\frac{d}{2}$, $\frac{d}{2} - 1$, $\frac{t-1}{2} - d$. Here $\mathrm{tr}(G) = 2(\frac{d}{2} + \frac{d}{2} - 1 + \frac{t-1}{2} - d) + 3 = 2d - 2 + t - 1 - 2d + 3 = t$. Finally, $\mathrm{diam}(G) = \max\{\frac{d}{2} + \frac{d}{2} - 1 + 1, \frac{d}{2} + \frac{t-1}{2} - d + 1, \frac{d}{2} - 1 + \frac{t-1}{2} - d + 1\} = \max\{d, \frac{t-d+1}{2}\} = d$ as $t + 1 \leq 3d$ (recall that here t is odd and d is even).

Since in both cases we constructed block graphs G, the claim is proved. □

In [3], the following simple lower bound for the triameter was obtained: $\mathrm{tr}(G) \geq g(G)$, where $g(G)$ is the length of the smallest cycle in G. After that, Das [3] posed the problem of establishing lower bounds for the triameter of general graphs in terms of other parameters, in particular, $\Delta(G)$ and $\delta(G)$. The next proposition shows that no such lower bound exists in terms of these parameters.

Proposition 3.2. *For all numbers $\Delta \geq \delta \geq 2$, there exists a graph G with $3 \leq \mathrm{tr}(G) \leq 5$ having $\Delta(G) = \Delta$ and $\delta(G) = \delta$.*

Proof. Take a complete graph K_Δ and fix an arbitrary subset $A \subseteq V(K_\Delta)$ with $|A| = \delta$ (we can do this since $\Delta \geq \delta$). Now add to K_Δ a new vertex u_0 with edges $\{u_0, x\}$ for $x \in A$. Denote the resulting graph as G. It is clear that $\Delta(G) = \Delta$ and $\delta(G) = \delta$. Also, by construction, $3 \leq \mathrm{tr}(G) \leq 5$:

- if $\Delta = \delta \geq 2$, then $G = K_{\Delta+1}$, implying $\mathrm{tr}(G) = 3$;
- if $\Delta - \delta = 1$, then $G = K_{\Delta+1} - e$, implying $\mathrm{tr}(G) = 4$;
- if $\Delta - \delta \geq 2$, then $\mathrm{tr}(G) = 5$ (a triametral triple is given by u_0 together with an arbitrary pair of vertices from $V(K_\Delta) \setminus A$).

□

Our first theorem presents a characterization of block graphs (which form an important class of graphs as suggested by Proposition 3.1) with triameters up to 8. To prove this, we first need a preliminary result concerning the number of cut vertices in graphs of a given triameter.

Lemma 3.1. *If a graph G has at least three cut vertices, then $G = P_5$ or $\mathrm{tr}(G) \geq 9$.*

Proof. Assume that a graph G has at least three distinct cut vertices. First, suppose G contains three cut vertices $x, y, z \in V(G)$ that lie in a common block B. Then fix three neighbors $x' \in N_G(x) \setminus V(B)$, $y' \in N_G(y) \setminus V(B)$, and $z' \in N_G(z) \setminus V(B)$. We have $d_G(x', y', z') \geq 9$.

Now assume that every block in G has at most two cut vertices. Then there exist two adjacent blocks B_1 and B_2, each having two cut vertices. Let x, y be the cut vertices of B_1 and y, z be the cut vertices of B_2. Fix two neighbors $x' \in N_G(x) \setminus V(B_1)$ and $z' \in N_G(z) \setminus V(B_2)$. It is clear that $d_G(x', z') \geq 4$. If $d_G(x', z') \geq 5$, then $\mathrm{tr}(G) \geq 10$. Otherwise, assume $d_G(x', z') = 4$. Then $\{x, y\}, \{y, z\} \in E(G)$. If $[x', z']_G = V(G)$, then $G = P_5$. If $[x', z']_G \neq V(G)$, then fix a vertex $w \in V(G) \setminus [x', z']_G$. Since $w \notin [x', z']_G$, we have $d_G(x', z', w) \geq 2d_G(x', z') + 1 \geq 9$. □

Theorem 3.1. *Let G be a block graph. Then*

1. $\mathrm{tr}(G) = 4$ *if and only if* $G = P_3$.
2. $\mathrm{tr}(G) = 5$ *if and only if* $G = K_1 + (K_a \cup K_b)$ *for* $\max\{a, b\} \geq 2$.
3. $\mathrm{tr}(G) = 6$ *if and only if either* $G = P_4$*, or G has a unique cut vertex of block-degree at least* 3.
4. $\mathrm{tr}(G) = 7$ *if and only if* $G \neq P_4$ *and G has exactly two cut vertices, each of block-degree* 2.
5. $\mathrm{tr}(G) = 8$ *if and only if either* $G = P_5$*, or G has exactly two cut vertices and at least one of them has block-degree at least* 3.

Proof. 1. It follows from Theorem 2.1 that P_3 is the only block graph having triameter 4.

2. **Necessity.** If $\mathrm{tr}(G) = 5$, then $\mathrm{diam}(G) = 2$. For block graphs G, this means that G has a unique cut vertex u_0. If $\mathrm{bdeg}_G(u_0) \geq 3$, then G contains an independent set of three vertices, implying that $\mathrm{tr}(G) = 6$. Thus, $\mathrm{bdeg}_G(u_0) = 2$, and $G = K_1 + (K_a \cup K_b)$ for some $a, b \in \mathbb{N}$. Finally, if $a = b = 1$, then $G = P_3$, which contradicts the condition $\mathrm{tr}(G) = 5$. Therefore, $\max\{a, b\} \geq 2$.

Sufficiency. Let $G = K_1 + (K_a \cup K_b)$ with $\max\{a, b\} \geq 2$. We can assume that $a \geq 2$. Clearly, $\mathrm{tr}(G) \leq 3\,\mathrm{diam}(G) = 6$. Since G has diameter 2 and contains no independent set of size 3, we have $\mathrm{tr}(G) \leq 5$. Finally, taking two non-cut vertices x, y from the complete subgraph K_a together with a vertex z from K_b produces a triple with $d_G(x, y, z) = 5$.

3. **Necessity.** If $\mathrm{tr}(G) = 6$, then $\mathrm{diam}(G) \in \{2, 3\}$. First, let $\mathrm{diam}(G) = 3$. Fix a diametral pair $u, v \in V(G)$ and a shortest path $u - x - y - v$. Since $\mathrm{tr}(G) = 6$, we have $t \in [u, v]_G$ for all $t \in V(G)$. Thus, G contains only these 4 vertices, so $G = P_4$. Further, assume $\mathrm{diam}(G) = 2$. Then G has a unique cut vertex u_0. If $\mathrm{bdeg}_G(u_0) = 2$, then by the previous case, we would have $\mathrm{tr}(G) \leq 5$. Hence, $\mathrm{bdeg}_G(u_0) \geq 3$.

Sufficiency. It is clear that $\mathrm{tr}(P_4) = 6$. Now assume that G has a unique cut vertex u_0 with $\mathrm{bdeg}_G(u_0) \geq 3$. Then $\mathrm{diam}(G) \leq 2$, implying $\mathrm{tr}(G) \leq 6$. But taking three non-cut vertices from three different blocks of G yields $\mathrm{tr}(G) = 6$.

4. **Necessity.** Let G be a block graph with $\mathrm{tr}(G) = 7$. Clearly, $G \neq P_4$. Moreover, $\mathrm{diam}(G) = 3$. Fix a diametral pair $u, v \in V(G)$ and a shortest path $u - x - y - v$. Then x, y are both cut vertices. Using Lemma 3.1, we conclude that x, y are the only cut vertices in G. Moreover, if $\mathrm{bdeg}_G(x) \geq 3$, then taking u, v and a vertex t from a block containing x distinct from the two blocks containing the edges $\{u, x\}$ and $\{x, y\}$, we obtain $d_G(u, v, t) \geq 8$ (see Figure 3.1). This contradiction implies that $\mathrm{bdeg}_G(x) = 2$. Similarly, $\mathrm{bdeg}_G(y) = 2$.

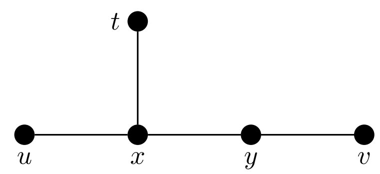

Figure 3.1: Illustration for the proof of part 4 of Theorem 3.1.

Sufficiency. Suppose that $G \neq P_4$ is a block graph with exactly two cut vertices x, y with $\mathrm{bdeg}_G(x) = \mathrm{bdeg}_G(y) = 2$. Then x and y lie in a common block B_0 (implying that $\{x, y\} \in E(G)$), and there exist two blocks B_x and B_y, distinct from B_0, such that $x \in V(B_x)$ and $y \in V(B_y)$. Fix two non-cut vertices $u \in V(B_x)$ and $v \in V(B_y)$. Since $G \neq P_4$, it follows that B_0 has a non-cut vertex t, or one of B_x, B_y has at least two non-cut vertices. In the first case, $d_G(u, v, t) = 7$. In the second case, assume without loss of generality that there is another non-cut vertex $t' \in V(B_x) \setminus \{u\}$. Then $d_G(u, v, t') = 7$. Thus, in both cases, $\mathrm{tr}(G) \geq 7$.

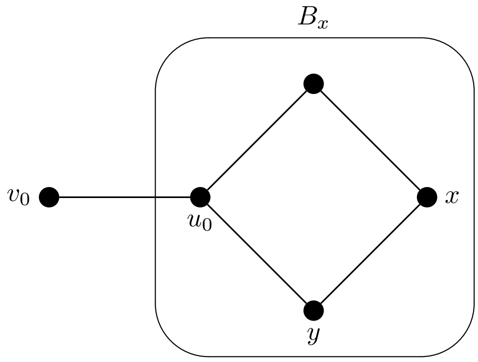

Figure 3.2: The vertices u_0 and x are adjacent to all other vertices of the block B_x.

Now Proposition 2.2 implies that $\mathrm{tr}(G) \leq 8$, since $\gamma_t(G) = 2$ (with $\{x, y\}$ being a total dominating set in G). However, the equality $\mathrm{tr}(G) = 8$ would imply that there are three vertices $p, q, r \in V(G)$ with the three corresponding distances equal to $3, 3, 2$. If $d_G(p, q) = d_G(p, r) = 3$, then both q, r lie in a common block, implying that $d_G(q, r) = 1$. This contradiction shows that $\mathrm{tr}(G) = 7$.

5. **Necessity.** Let G be a block graph with $\mathrm{tr}(G) = 8$. If G has at least three cut vertices, then Lemma 3.1 implies that $G = P_5$. Thus, assume that G has at most two cut vertices. It is clear that G is neither biconnected nor has a unique cut vertex. Thus, let x, y be the two cut vertices of G. If $\mathrm{bdeg}_G(x) = \mathrm{bdeg}_G(y) = 2$, then by part 4 we have $\mathrm{tr}(G) \leq 7$. Therefore, at least one of x, y has block-degree at least 3.

Sufficiency. Clearly, $\mathrm{tr}(P_5) = 8$. Now assume that G has exactly two cut vertices x, y and $\mathrm{bdeg}_G(x) \geq 3$. It is clear that x and y lie in a common block B_0 (in particular, $\{x, y\} \in E(G)$). Also, fix two other blocks $B_1, B_2 \neq B_0$ containing the vertex x, and a block $B_y \neq B_0$ that contains y. Taking three non-cut vertices $u \in V(B_1)$, $v \in V(B_2)$, $w \in V(B_y)$, we obtain $d_G(u, v, w) = 8$. This means that $\mathrm{tr}(G) \geq 8$. Finally, Proposition 2.2 again implies that $\mathrm{tr}(G) \leq 8$ as $\gamma_t(G) = 2$ (with $\{x, y\}$ being a total dominating set in G). Therefore, $\mathrm{tr}(G) = 8$. □

We now return to the main problem of characterizing graphs with triameter 6. Apart from block graphs, we do this for several other natural graph classes. First, observe that Lemma 3.1 implies that a graph with triameter 6 can have at most two cut vertices. Moreover, P_4 is the only graph with triameter 6 that has exactly two cut vertices. Indeed, for any such graph G, we can take its *block closure* G^β, which is obtained from G by making all blocks in G complete (see [2]). It is clear that $\mathrm{tr}(G^\beta) \leq \mathrm{tr}(G)$ and G^β is a block graph with two cut vertices. Using the first three statements from Theorem 3.1, we conclude that $G^\beta = P_4$. However, this immediately implies that $G = P_4$. The case of a unique cut vertex is addressed in the theorem below.

Theorem 3.2. *A graph G has a unique cut vertex and $\mathrm{tr}(G) = 6$ if and only if G is of one of the following two types:*

1. *$G = K_1 + H$, where H is a disconnected graph that is not a disjoint union of two complete graphs;*
2. *G is obtained from a graph of the form $\overline{K}_2 + H$, with $|V(H)| \geq 2$, by adding a leaf vertex adjacent to one of the two vertices of the $\overline{K}_2$ subgraph.*

Proof. Necessity. Let $u_0 \in V(G)$ be the unique cut vertex of G. First, assume that $\mathrm{diam}(G) = 2$. Then it is easy to see that u_0 must be a universal vertex in G (otherwise, we would have $\mathrm{diam}(G) \geq 3$). Hence, $G = K_1 + H$ for some disconnected graph H. If $H = K_a \cup K_b$ for some $a, b \in \mathbb{N}$, then $\mathrm{tr}(G) = 5$ by part 2 of Theorem 3.1. Thus, in this case, $H \neq K_a \cup K_b$ and G is of the first type.

Now assume that $\mathrm{diam}(G) = 3$. In this case, u_0 is not a universal vertex in G. Fix a vertex $x \in V(G)$ with $d_G(u_0, x) = 2$. Denote by B_x the block of G containing the edge incident to x on a shortest $u_0 - x$ path; in particular, $u_0 \in V(B_x)$. If $|V(G) \setminus V(B_x)| \geq 2$, then take two distinct vertices $y, z \in V(G) \setminus V(B_x)$ and consider the triple x, y, z. It is easy to see that $d_G(x, y, z) \geq 7$. Thus, $|V(G) \setminus V(B_x)| \leq 1$. Since the case $G = B_x$ is impossible, as G is not biconnected, we have $|V(G) \setminus V(B_x)| = 1$. Therefore, $\mathrm{bdeg}_G(u_0) = 2$, and the block different from B_x is a leaf edge, say $\{u_0, v_0\}$. Note that x, v_0 is a diametral pair in G.

Let us further examine the structure of the block B_x. We claim that $N_{B_x}(u_0) = N_{B_x}(x) = V(B_x) \setminus \{u_0, x\}$. Indeed, fix a vertex $y \in V(B_x) \setminus \{u_0, x\}$ and consider the triple x, y, v_0. We have $d_G(x, y, v_0) \leq 6$, $d_G(x, v_0) = 3$, and $d_G(y, v_0) \geq 2$. This means that y is adjacent to x and $d_G(y, v_0) = 2$. But the latter equality asserts that y is adjacent to u_0 (see Figure 3.2).

Combining the above, we can conclude that $B_x = \overline{K}_2 + H$, where $V(\overline{K}_2) = \{x, u_0\}$ and $H = G[V(B_x) \setminus \{u_0, x\}]$ (note that $|V(H)| \geq 2$ as B_x is biconnected). This means that G is of the second type.

Sufficiency. It is clear that both types produce graphs G with a unique cut vertex. We need to prove that $\operatorname{tr}(G) = 6$ in both cases.

Let $G = K_1 + H$ for some disconnected graph $H \neq K_a \cup K_b$. It is clear that $\operatorname{diam}(G) = 2$. Hence, $\operatorname{tr}(G) \leq 6$. If H has at least three connected components, then it is clear that $\operatorname{tr}(G) = 6$. Otherwise, assume that H has exactly two connected components H_1, H_2. By our condition, at least one of them is non-complete. Without loss of generality, assume that H_1 is not complete. Fix two non-adjacent vertices $u, v \in V(H_1)$ and a vertex $w \in V(H_2)$. Then $d_G(u, v, w) = 6$, implying $\operatorname{tr}(G) = 6$.

Now consider the second type. Let $V(\overline{K}_2) = \{u, v\}$, and suppose that the vertex u is adjacent to a leaf vertex $w \in V(G)$. It is clear that $d_G(v, w) = \operatorname{diam}(G) = 3$ and v, w is the unique diametral pair in G. This implies that $\operatorname{tr}(G) \geq 6$. However, each vertex $y \in V(H) \cup \{u\}$ lies in the interval $[v, w]_G$. Therefore, $\operatorname{tr}(G) = 6$. $\square$

Remark 3.1. *Theorem 3.2 and the preceding discussion show that the main problem further reduces to finding a criterion for biconnected graphs G with $\operatorname{tr}(G) = 6$ and $\operatorname{diam}(G) = 3$.*

Remark 3.2. *We make a useful general observation about the structure of graphs G with $\operatorname{tr}(G) = 6$ and $\operatorname{diam}(G) = 3$. For such a graph G, fix an arbitrary diametral pair $u, v \in V(G)$. It is clear that $[u, v]_G = V(G)$, i.e., every vertex of G lies between u and v. Conversely, if G is a graph with $\operatorname{diam}(G) = 3$, then this condition (the equality $[u, v]_G = V(G)$ for any diametral pair $u, v \in V(G)$) implies that $\operatorname{tr}(G) = 6$.*

We now aim to characterize bipartite graphs with triameter 6. Note that complete bipartite graphs having at least three vertices in one part are the only bipartite graphs with diameter 2 and triameter 6. Indeed, for a bipartite graph G, the condition $\operatorname{diam}(G) = 2$ already implies that G is complete bipartite; moreover, one needs at least three vertices in the same part to ensure that $\operatorname{tr}(G) = 6$. Thus, as before, we restrict ourselves to bipartite graphs G with $\operatorname{tr}(G) = 6$ and $\operatorname{diam}(G) = 3$.

Theorem 3.3. *A graph G is bipartite with $\operatorname{tr}(G) = 6$ and $\operatorname{diam}(G) = 3$ if and only if there exist two vertices $u_0, v_0 \in V(G)$ such that the following conditions hold:*

1. *$V(G) = N_G[u_0] \sqcup N_G[v_0]$;*
2. *induced subgraphs $G[N_G(u_0)]$ and $G[N_G(v_0)]$ are empty;*
3. *induced bipartite subgraph $G[N_G(u_0), N_G(v_0)]$ is complete bipartite minus a matching and has no isolated vertices.*

Proof. Necessity. Fix a diametral pair $u_0, v_0 \in V(G)$. Then the second condition follows from the bipartiteness of G. Also, the equality $\operatorname{diam}(G) = 3$ implies that $N_G[u_0] \cap N_G[v_0] = \emptyset$. If there exists a vertex $w \in V(G) \setminus (N_G[u_0] \cup N_G[v_0])$, then $d_G(u_0, v_0, w) \geq 8$, which is a contradiction. Therefore, the first condition also holds.

To show that $G[N_G(u_0), N_G(v_0)]$ indeed has the desired structure, we need to work closely with the condition $\operatorname{tr}(G) = 6$. Namely, fix a vertex $x \in N_G(u_0)$. Since $x \in [u_0, v_0]_G$ (see Remark 3.2), it follows that x is adjacent to at least one vertex in $N_G(v_0)$. We aim to show that x is adjacent to all vertices in $N_G(v_0)$, with at most one exception. Suppose, for a contradiction, that x is not adjacent to two vertices $y, z \in N_G(v_0)$. Then $d_G(x, y) = d_G(x, z) = 3$, so x, y and x, z are diametral pairs in G. This contradicts the fact that $\operatorname{tr}(G) = 6$ (see Remark 3.2, or observe that $d_G(x, y, z) \geq 7$). Therefore, the induced bipartite subgraph $G[N_G(u_0), N_G(v_0)]$ is a complete bipartite graph minus a matching.

Moreover, if there exists an isolated vertex w in $G[N_G(u_0), N_G(v_0)]$, then one of u_0 or v_0 is a support of the leaf vertex w in G, which implies that $d_G(w, v_0) \geq 4$ or $d_G(w, u_0) \geq 4$. Hence, the third condition holds as well.

Sufficiency. It is clear that the first two conditions ensure that G is bipartite (with $V(G) = (N_G(u_0) \cup \{v_0\}) \sqcup (N_G(v_0) \cup \{u_0\})$ being its bipartition). Now we show that the first part of the third condition implies $\operatorname{diam}(G) = 3$. Indeed, it is easy to see that $d_G(u_0, v_0) = 3$. Furthermore, we have $d_G(u_0, y) = 2$ for all $y \in N_G(v_0)$: since y is non-isolated in $G[N_G(u_0), N_G(v_0)]$, fix a neighbor $x \in N_G(u_0)$ of y and obtain a path $y - x - u_0$ in G. Similarly, $d_G(v_0, x) = 2$ for all $x \in N_G(u_0)$. Further, $d_G(x, x') = 2$ for all $x, x' \in N_G(u_0)$, and $d_G(y, y') = 2$ for all $y, y' \in N_G(v_0)$. Finally, fix $x \in N_G(u_0)$ and $y \in N_G(v_0)$. Since there exists a neighbor $x' \in N_G(u_0)$ of y, we have a path $x - u_0 - x' - y$. Thus, $d_G(x, y) \leq 3$. The above cases cover all pairs of vertices in G, which shows that $\operatorname{diam}(G) = 3$.

Finally, we show that $\operatorname{tr}(G) = 6$. Since we have already established that $\operatorname{diam}(G) = 3$, we use Remark 3.2 to prove that $[x, y]_G = V(G)$ for any diametral pair $x, y \in V(G)$. Indeed, this clearly holds for the diametral pair u_0, v_0. Other possible diametral pairs are of the form x, y, where $x \in N_G(u_0)$, $y \in N_G(v_0)$, and $\{x, y\} \notin E(G)$. The third condition implies that x is adjacent to all vertices in $N_G(v_0) \setminus \{y\}$ and y is adjacent to all vertices in $N_G(u_0) \setminus \{x\}$. Thus, for every $y' \in N_G(v_0) \setminus \{y\}$, we have a path $x - y' - v_0 - y$, which shows that $y' \in [x, y]_G$. Similarly, $N_G(u_0) \setminus \{x\} \subseteq [x, y]_G$. Moreover, $u_0, v_0 \in [x, y]_G$. Hence, $[x, y]_G = V(G)$, and therefore $\operatorname{tr}(G) = 6$. $\square$

Remark 3.3. *Theorem 3.3 suggests that each bipartite graph with triameter* 6 *and diameter* 3 *can be encoded by two simple parameters: the multiset of degrees* $\{\deg_G(u_0), \deg_G(v_0)\} = \{d_1, d_2\}$ *and the size* m *of the matching from the third condition. Thus, we obtain a pair of parameters* $(\{d_1, d_2\}, m)$*. Note that* $d_1, d_2 \in \mathbb{N}$, $0 \leq m \leq \min\{d_1, d_2\}$*; and if* $\min\{d_1, d_2\} = 1$*, then* $m = 0$*. We call such parameters* admissible.

Remark 3.4. *For a bipartite graph* G *with* $\mathrm{tr}(G) = 6$ *and* $\mathrm{diam}(G) = 3$*, the admissible parameters* $(\{d_1, d_2\}, m)$ *are independent of the choice of a diametral pair* u_0, v_0*. Indeed, as established in the proof of Theorem 3.3, any other diametral pair in* G *must be of the form* x, y*, where* $x \in N_G(u_0)$*,* $y \in N_G(v_0)$*, and* $\{x, y\} \notin E(G)$*. One can see that* $\deg_G(x) = \deg_G(v_0)$*,* $\deg_G(y) = \deg_G(u_0)$*, and the size* m *of the corresponding matching remains the same.*

Remark 3.5. *Given a pair of admissible parameters* $(\{d_1, d_2\}, m)$ *(satisfying the conditions from Remark 3.3), it is clear that we can always construct the corresponding bipartite graph by a straightforward procedure. For example, consider the admissible parameters* $(\{3, 4\}, 3)$*. The corresponding bipartite graph* G *is depicted in Figure 3.3.*

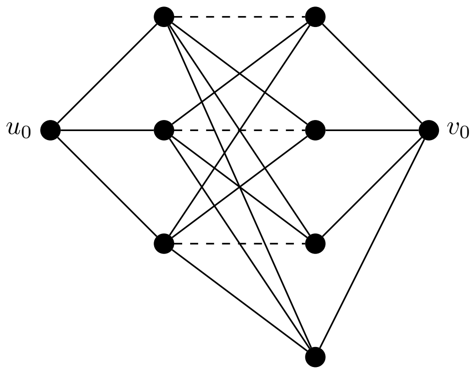

Figure 3.3: The bipartite graph with admissible parameters $(\{3, 4\}, 3)$.

Using Theorem 3.3, we can characterize graphs from the main problem in the related classes of modular and median graphs. Again, note that complete bipartite graphs having at least three vertices in one part are the only modular graphs with diameter 2 and triameter 6. The case of diameter 3 is addressed in the next result.

Corollary 3.1. *Let* G *be a bipartite graph with* $\mathrm{tr}(G) = 6$ *and* $\mathrm{diam}(G) = 3$*. Then* G *is not modular if and only if its admissible parameters are of the form* $(\{2, d_2\}, 2)$*.*

Proof. First, assume that G is a bipartite graph with admissible parameters $(\{2, d_2\}, 2)$ (in particular, $d_2 \geq 2$). Then for any fixed diametral pair $u_0, v_0 \in V(G)$, without loss of generality, we have $N_G(u_0) = \{x_1, x_2\}$ and $|N_G(v_0)| \geq 2$ (see Remark 3.4). Moreover, since the size of the corresponding matching is $m = 2$, there exist $y_1, y_2 \in N_G(v_0)$ with $\{x_1, y_1\}, \{x_2, y_2\} \notin E(G)$.

We claim that u_0, y_1, y_2 form a 'bad triple', i.e. $M_G(u_0, y_1, y_2) = \emptyset$. Indeed, $[u_0, y_1]_G \cap [u_0, y_2]_G = \{u_0, x_2, y_1\} \cap \{u_0, x_1, y_2\} = \{u_0\}$, but $[y_1, y_2]_G = \{y_1, y_2, v_0\}$. Thus, G is non-modular.

Conversely, suppose that G is a bipartite graph with $\mathrm{tr}(G) = 6$, $\mathrm{diam}(G) = 3$, and admissible parameters $(\{d_1, d_2\}, m) \neq (\{2, d_2\}, 2)$. We aim to show that G is modular. To this end, we examine all possible triples of vertices in G.

First, it is clear that if a triple of vertices contains the pair u_0, v_0, then its median set is non-empty. Also, any triple of vertices from $N_G(u_0)$ (or $N_G(v_0)$) has u_0 (or v_0) in its median set. Similarly, $u_0 \in M_G(u_0, x_1, x_2)$ for all $x_1, x_2 \in N_G(u_0)$, and analogously for triples of the form v_0, y_1, y_2 with $y_1, y_2 \in N_G(v_0)$.

Now let $x_1, x_2 \in N_G(u_0)$ and $y \in N_G(v_0)$. If $\{x_1, y\}, \{x_2, y\} \in E(G)$, then $y \in M_G(x_1, x_2, y)$. If one of x_1, x_2 is not adjacent to y, say x_1, then $d_G(x_1, y) = 3$, and hence $x_2 \in M_G(x_1, x_2, y)$.

Next, consider triples of the form u_0, x, y, where $\{u_0, x\}, \{v_0, y\} \in E(G)$. If $\{x, y\} \in E(G)$, then $x \in M_G(u_0, x, y)$. If $\{x, y\} \notin E(G)$, then $d_G(x, y) = 3$, implying that $u_0 \in M_G(u_0, x, y)$. In each case, the corresponding triple has a non-empty median set. The same argument applies to triples of the form v_0, x, y.

Finally, we consider triples of the form u_0, y_1, y_2 with $y_1, y_2 \in N_G(v_0)$ (the case of triples v_0, x_1, x_2 for $x_1, x_2 \in N_G(u_0)$ is analogous). If $\deg_G(u_0) = 1$, then $M_G(u_0, y_1, y_2)$ contains the unique neighbor of u_0. If $\deg_G(u_0) \geq 3$, then, recalling the structure of the subgraph $G[N_G(u_0), N_G(v_0)]$ from Theorem 3.3, we conclude that there exists a vertex $x' \in N_G(u_0)$ with $\{x', y_1\}, \{x', y_2\} \in E(G)$. Hence, $x' \in M_G(u_0, y_1, y_2)$.

The final case is $\deg_G(u_0) = 2$; in this situation, $m \neq 2$. Clearly, we have $0 \leq m \leq 1$ (see Remark 3.3). In either case, there exists a vertex $x' \in N_G(u_0)$ with $\{x', y_1\}, \{x', y_2\} \in E(G)$, and hence $x' \in M_G(u_0, y_1, y_2)$.

Therefore, every triple of vertices has a non-empty median set. $\square$

For median graphs, the situation is simpler. Note that stars and the cycle C_4 are the only median graphs with diameter 2. Hence, stars with at least four vertices are exactly the median graphs with diameter 2 and triameter 6. The case of diameter 3 is addressed in the next result.

Corollary 3.2. *The only median graphs with triameter* 6 *and diameter* 3 *are the following: the path* P_4*, the cycle* C_4 *with the attached leaf vertex, the grid graph* $P_2 \square P_3$*, and the cube* Q_3*.*

Proof. One can easily observe that the listed graphs are indeed median and have triameter 6 and diameter 3. Thus, we need to prove the converse. Suppose that G is such a graph. Let $(\{d_1, d_2\}, m)$ be its admissible parameters. Without loss of generality, assume $d_1 \leq d_2$. Fix a diametral pair $u_0, v_0 \in V(G)$ with $\deg_G(u_0) = d_1$ and $\deg_G(v_0) = d_2$. In what follows, we consider several cases.

Case 1: $d_1 \geq 3$ and $d_2 \geq 4$. Here, for any triple of vertices $x_1, x_2, x_3 \in N_G(u_0)$, the structure of the subgraph $G[N_G(u_0), N_G(v_0)]$ from Theorem 3.3 implies the existence of $y' \in N_G(v_0)$ adjacent to every x_i, $1 \leq i \leq 3$. This implies $u_0, y' \in M_G(x_1, x_2, x_3)$, which is a contradiction.

Case 2: $d_1 = d_2 = 3$. In order for the argument in the previous case to fail, one needs to have $m = 3$. Then G has parameters $(\{3, 3\}, 3)$, which implies that $G = Q_3$.

Case 3: $d_1 = 2$. Let $N_G(u_0) = \{x_1, x_2\}$. Here we consider several subcases.

Subcase 3.1: $d_2 \geq 4$. Then the triple v_0, x_1, x_2 has at least two medians (see Figure 3.4), since the vertices x_1 and x_2 have at least two common neighbors in $N_G(v_0)$.

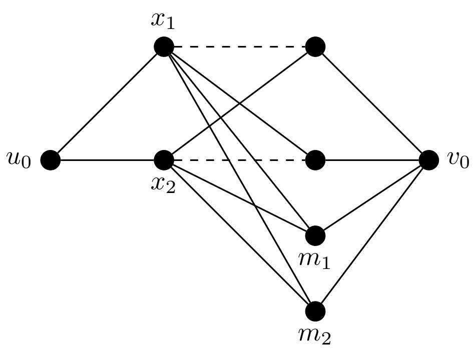

Figure 3.4: The triple v_0, x_1, x_2 has two medians, being m_1 and m_2.

Subcase 3.2: $d_2 = 3$. In this subcase, the triple of neighbors of v_0 has v_0 as its unique median if and only if $m = 2$. Then G has parameters $(\{2, 3\}, 2)$, implying that it is non-modular (and hence non-median) by Corollary 3.1.

Subcase 3.3: $d_2 = 2$. Here G has parameters $(\{2, 2\}, m)$ with $0 \leq m \leq 2$. If $m = 2$, then G is non-modular by Corollary 3.1 (in fact, $G = C_6$). If $m = 1$, then $G = P_2 \square P_3$ is a grid graph. If $m = 0$, then the triple v_0, x_1, x_2 has two medians, namely the two neighbors of v_0.

Case 4: $d_1 = 1$. If $d_2 \geq 3$, then any triple of neighbors of v_0 has two medians, namely v_0 and the unique neighbor of u_0. If $d_2 = 2$, then G is the graph obtained from C_4 by attaching a leaf vertex. If $d_2 = 1$, then $G = P_4$. $\square$

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