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Research Article

Generalizations of a Sum Rule for Derangements

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Abstract

In this paper, some generalizations of a recent summation formula involving derangement numbers are proven. One of the generalizations is in terms of the q-derangement polynomials of Garsia and Remmel [*European J. Combin.* **1** (1980) 47–59], and another involves what are termed the r-derangement numbers. A related summation identity is generalized and provided a combinatorial proof. Finally, analogous results involving the joint distribution for the number of fixed points and cycles on the set of all permutations of a given length are established.

Keywords: q-derangement polynomial; r-derangement number; combinatorial statistic; combinatorial proof.

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1. Introduction

Let $\mathcal{S}_n$ denote the set of all permutations of $[n] = \{1, \ldots, n\}$. A member of $\mathcal{S}_n$ that contains no fixed points (i.e., cycles of length one) is called a *derangement*. Let $\mathcal{D}_n$ denote the subset of $\mathcal{S}_n$ consisting of the derangements and $d(n) = |\mathcal{D}_n|$ for $n \geq 0$. Then $d(n)$ is given explicitly by

$$d(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}, \qquad n \geq 0, \tag{1}$$

(see, e.g., [12, p. 67]) and also by

$$d(n) = \|n!/e\| = \lfloor (n!+1)/e \rfloor, \qquad n \geq 1, \tag{2}$$

where $\|x\|$ and $\lfloor x \rfloor$ represent the nearest integer and integer part of a real number x, respectively. See entry A000166 in [11] for further properties of the sequence $d(n)$.

In [8], it was shown

$$\sum_{i=1}^{n} i d(i) = \lfloor (n+1)!/e \rfloor, \qquad n \geq 0. \tag{3}$$

Three proofs of (3) are provided in [8]. The first two proofs are inductive and depend upon respectively certain integral representations and the Hermite identity, while the third one is direct and makes use of the recurrence relation $d(n) = nd(n-1) + (-1)^n$. Further, in the third proof, the derangement number summation formula

$$d(n+1) = \frac{1-(-1)^n}{2} + \sum_{i=1}^{n} i d(i), \qquad n \geq 0, \tag{4}$$

plays a key role.

In this paper, we establish a generalization of (3) in terms of a q-analogue for $d(n)$ and another involving the r-derangement numbers. Our proofs for these results in the cases when $q = 1$ or $r = 0$, respectively, will be seen to differ from those presented in [8] for (3). Indeed, we make use of elementary properties of alternating series in both proofs, which in the cases $q = 1$ or $r = 0$ will be seen to provide a simplification of the third proof of (3) in [8]. We also supply combinatorial proofs of extensions of (4) involving the aforementioned generalizations of $d(n)$. For combinatorial proofs of other related derangement formulas, see, e.g., [1, 4, 10, 15].

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The organization of this paper is as follows. In the next section, we prove generalizations of (3) and (4) in terms of the q-derangement polynomials and r-derangement numbers, providing combinatorial proofs for the extensions of (4) as well. In the third section, we consider comparable results for the sequence $n!$. In particular, we establish analogues of (4) involving the joint distribution for the statistics on $\mathcal{S}_n$ which track the number of cycles and fixed points. By setting the parameter marking the number of fixed points equal to zero within this distribution, one obtains another generalization of (4) in terms of the number of cycles distribution on $\mathcal{D}_n$.

2. Generalizations of a Derangement Sum Formula

Before proving our first result, let us recall some notation and a q-generalization of $d(n)$. Given an indeterminate q and a positive integer k, define k_q by $k_q = 1 + q + \cdots + q^{k-1}$, with $0_q = 0$. Let $k_q^! = \prod_{i=1}^k i_q$ for $k \geq 1$, with $0_q^! = 1$. Garsia and Remmel [5] defined a q-analogue of $d(n)$ as follows. First, express each $\pi \in \mathcal{D}_n$ by $\pi = (\alpha_1)(\alpha_2)\cdots$, where the (α_i) are the (disjoint) cycles of π ordered by increasing second smallest elements, with the second smallest element last in each cycle. This is termed the *ordered cycle factorization* (OCF, for short) in [5]. Define the statistic inv_o on $\mathcal{D}_n$ by writing out the cycles of π, expressed in OCF form, removing each pair of enclosing parentheses and counting inversions in the resulting word. For example, if $\pi \in \mathcal{D}_7$ is given in the one-line notation by $\pi = 4321756$, then the OCF of π is $(23)(14)(576)$ and $\text{inv}_o(\pi) = 3$, the number of inversions in the word 2314576.

Let

$$d_q(n) = \sum_{\pi \in \mathcal{D}_n} q^{\text{inv}_o(\pi)}, \qquad n \geq 1,$$

represent the inv_o distribution polynomial on $\mathcal{D}_n$, with $d_q(0) = 1$. Garsia and Remmel [5] showed

$$d_q(n) = n_q^! \sum_{k=0}^{n} \frac{(-1)^k}{k_q^!}, \qquad n \geq 0, \tag{5}$$

which reduces to (1) when $q = 1$.

Let $e_q(x) = \sum_{k\geq 0} \frac{x^k}{k_q^!}$ denote the q-exponential function. Our first result extends (3) in terms of the polynomials $d_q(n)$.

Theorem 2.1. *If $n \geq 0$ and q is a positive integer, then*

$$q \sum_{i=1}^{n} i_q d_q(i) = \lfloor (n+1)_q^! e_q(-1) \rfloor. \tag{6}$$

Proof. We first show, for any q,

$$d_q(n) = \frac{1 + (-1)^n}{2} + q \sum_{i=1}^{n-1} i_q d_q(i), \qquad n \geq 1. \tag{7}$$

We proceed inductively and may assume $n \geq 2$, since the $n = 1$ case is clear as $d_q(1) = 0$. Upon considering separately the even and odd cases for n, we have that the n-case of (7) follows from the $(n-1)$-case if and only if

$$d_q(n) - d_q(n-1) = q(n-1)_q d_q(n-1) + (-1)^n, \quad \text{i.e.,} \quad d_q(n) = n_q d_q(n-1) + (-1)^n,$$

where we have made use of the fact $1 + q(n-1)_q = n_q$. The latter equality is seen to follow directly from (5), which completes the induction.

Note if $q > 0$ is any real number, then the sequence $\left(\frac{1}{k_q^!}\right)_{k\geq 0}$ is decreasing with limit zero, whence $e_q(-1)$ is a convergent alternating series. Thus, we have

$$d_q(n) - 1 < d_q(n) - \frac{1}{(n+1)_q} < n_q^! e_q(-1) < d_q(n), \tag{8}$$

if n is even, and

$$d_q(n) < n_q^! e_q(-1) < d_q(n) + \frac{1}{(n+1)_q} < d_q(n) + 1, \tag{9}$$

if n is odd, by properties of alternating series, upon noting

$$n_q^! \sum_{k=0}^{n+1} \frac{(-1)^k}{k_q^!} = n_q^! \sum_{k=0}^{n} \frac{(-1)^k}{k_q^!} + \frac{(-1)^{n+1} n_q^!}{(n+1)_q^!} = d_q(n) + \frac{(-1)^{n+1}}{(n+1)_q}.$$

If q is a positive integer, then so is $d_q(n)$, which implies for such q,

$$d_q(n) = \frac{1+(-1)^n}{2} + \lfloor n_q^! e_q(-1) \rfloor, \qquad n \geq 0, \tag{10}$$

by (8) and (9). Formula (6) now follows from combining (7) and (10). □

One can provide a combinatorial explanation of the identity (7) used above in the proof of (6).

Combinatorial proof of (7)*:*

We first establish the odd case of (7), and let $n = 2m+1$. The $m = 0$ case is trivial, so assume $m \geq 1$. We enumerate $\pi \in \mathcal{D}_{2m+1}$ where i is the largest $j \in [m]$ for which the 2-cycle $(2j, 2j+1)$ does not occur. Note that such an i must exist, for otherwise, π would contain the 1-cycle (1), contradicting that it is a derangement. Thus, we have

$$\pi = \pi'(2i+2, 2i+3)\cdots(2m, 2m+1),$$

when expressed in the OCF form, where π' is a member of $\mathcal{D}_{2i+1}$ for which $(2i, 2i+1)$ fails to occur. Note that the $m-i$ terminal 2-cycles of π are not responsible for any inversions and hence do not contribute to $\text{inv}_o(\pi)$. We consider cases on the position of $2i+1$ within π'. If the cycle $(a, 2i+1)$ occurs in π' for some $a \in [2i-1]$, then a is responsible for $2i-a$ inversions since it lies to the right of the elements of $[a+1, 2i] = \{a+1, a+2, \ldots, 2i\}$ in the OCF, as $(a, 2i+1)$ would be the last cycle of π'. Thus, the choice of a is accounted for by $q + q^2 + \cdots + q^{2i-1} = q(2i-1)_q$, with the remaining cycles of π' being accounted for by $d_q(2i-1)$, yielding a weight of $q(2i-1)_q d_q(2i-1)$ in this case.

Now suppose $2i+1$ occurs in a cycle of π' with at least two other elements. Then π' can be obtained in this case by inserting $2i+1$ into one of the cycles within a precursor $\rho \in \mathcal{D}_{2i}$ such that $2i+1$ directly precedes an element of $[2i]$. Note $2i+1$ cannot be inserted at the end of a cycle of ρ since it would not be the second smallest element in its cycle. Thus, the placement of $2i+1$ introduces anywhere from 1 to $2i$ inversions when inserted into ρ. As there are $d_q(2i)$ possibilities for ρ, we get a contribution towards the weight of $q(2i)_q d_q(2i)$ for π where $2i+1$ occurs in a cycle of length at least three. Combining with the prior case, we obtain a weight of $q(2i-1)_q d_q(2i-1) + q(2i)_q d_q(2i)$ for each i. Summing this over all $1 \leq i \leq m$ implies that the inv_o-weight of the members of $\mathcal{D}_{2m+1}$ is given by $q\sum_{i=1}^{2m} i_q d_q(i) = q\sum_{i=1}^{n-1} i_q d_q(i)$, which completes the proof of (7) in the odd case.

A similar argument applies to the even case of (7), upon letting i denote the largest $j \in [m]$ such that $(2j-1, 2j)$ does not occur within a member of $\mathcal{D}_{2m}$. Then one obtains the weight of all members of $\mathcal{D}_{2m} - \{(1,2)\cdots(2m-1,2m)\}$, and hence we must add 1 to this in order to account for the weight of the excluded derangement. □

Note that (6) reduces to (3) when $q = 1$. We proceed to show a second generalization of (3) as follows. Given $n \geq r \geq 0$, let $D_r(n)$ denote the number of derangements of $[r+n]$ such that the elements $1, 2, \ldots, r$ belong to distinct cycles. The $D_r(n)$ are referred to as r-derangement numbers and have been studied, for example, in [6, 14]. The $D_r(n)$ reduce to d_n when $r = 0$, with $D_1(n) = d_{n+1}$. A summation formula involving a convolution of binomial coefficients with r-derangement numbers was found in [14, Theorem 9].

Formula (4) above may be generalized as follows to $D_r(n)$.

Lemma 2.1. *If* $r, m \geq 0$*, then*

$$D_r(r+2m+1) = \sum_{i=0}^{m}(2r+2i)D_r(r+2i) + \sum_{i=1}^{m}(r+2i-1)D_r(r+2i-1) + \sum_{i=0}^{m} rD_{r-1}(r+2i), \tag{11}$$

$$D_r(r+2m) - r! = \sum_{i=1}^{m}(2r+2i-1)D_r(r+2i-1) + \sum_{i=0}^{m-1}(r+2i)D_r(r+2i) + \sum_{i=1}^{m} rD_{r-1}(r+2i-1). \tag{12}$$

Proof. We provide a combinatorial explanation by extending the $q = 1$ case of the argument given in the preceding proof of (7). We start with (11) and let i denote the largest j where $0 \leq j \leq m$ such that the 2-cycle $(2r+2j, 2r+2j+1)$ does not occur within an r-derangement of $[2r+2m+1]$. Note that such an i must exist, for otherwise the elements of $[2r-1]$ would form a derangement wherein $1, \ldots, r$ belong to distinct cycles, which is impossible. Then $2r+2i+1$ occurs in either (i) a cycle with two or more other elements of $[2r+2i]$, (ii) a 2-cycle with some member of $[r+1, 2r+2i-1]$, or (iii) a 2-cycle with a member of $[r]$. It is seen that there are $(2r+2i)D_r(r+2i)$, $(r+2i-1)D_r(r+2i-1)$ and $rD_{r-1}(r+2i)$ possibilities for (i), (ii) and (iii), respectively. Considering all possible values of i then yields (11). A comparable proof applies to (12) upon considering the largest $j \in [m]$ such that $(2r+2j-1, 2r+2j)$ does not occur, provided such a j exists. The subtracted $r!$ term then accounts for the r-derangements of $[2r+2m]$ in which the 2-cycles $(2r+2j-1, 2r+2j)$ for $1 \leq j \leq m$ all occur (leaving a matching between the members of $[r]$ and $[r+1, 2r]$). □

Note that (11) and (12) reduce when $r = 0$ to the even and odd n cases of (4), respectively. Given a positive integer m, we indicate the rising and falling factorials $x(x+1)\cdots(x+m-1)$ and $x(x-1)\cdots(x-m+1)$ by $x^{\overline{m}}$ and $x^{\underline{m}}$, respectively, with $x^{\overline{0}} = x^{\underline{0}} = 1$. Recall the r-Stirling numbers of the first kind (see, e.g., [3, 7]) given recursively by

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r + (r+n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r, \qquad n,k \geq 1,$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_r = r^{\overline{n}}$ and $\begin{bmatrix} 0 \\ k \end{bmatrix}_r = \delta_{k,0}$ for $n,k \geq 0$, and the complementary Bell numbers (see [9]), which will be denoted here by B_n^*, defined as the coefficients in the expansion

$$\exp(1-e^x) = \sum_{n\geq 0} B_n^* \frac{x^n}{n!}.$$

We have the following formula for $D_r(n)$.

Theorem 2.2. *Let $n \geq r \geq 0$ be fixed. Then we have*

$$D_r(n) = \lfloor \ell_{n,r} \rfloor + \frac{1+(-1)^{r+n}}{2}, \tag{13}$$

where

$$\ell_{n,r} = \frac{n!}{r!}\sum_{k\geq 0}(k-n)^{\overline{r}}\frac{(-1)^{k+r}}{k!}.$$

Moreover, $\ell_{n,r}$ is given explicitly by

$$\ell_{n,r} = \frac{n!}{r!e}\sum_{j=0}^{r}(-1)^j \begin{bmatrix} r \\ j \end{bmatrix}_{n-r+1} B_j^*. \tag{14}$$

Proof. First recall that $D_r(n)$ has the explicit formula

$$D_r(n) = n^{\underline{r}} \sum_{k=0}^{n-r}(-1)^k \binom{n-r}{k}(r+1)^{\overline{n-r-k}},$$

see [6, Theorem 1], which may be rewritten as

$$D_r(n) = \binom{n}{r}(n-r)!\sum_{k=0}^{n-r}(-1)^k\frac{(n-k)!}{k!(n-r-k)!} = \frac{n!}{r!}\sum_{k=0}^{n-r}(k-n)^{\overline{r}}\frac{(-1)^{k+r}}{k!}, \qquad n \geq r \geq 0. \tag{15}$$

Note that the sequence $(k-n)^{\overline{r}}/k!$ for $k \geq n+1$, where $n \geq r$ are fixed, is decreasing with limit zero. Define $S_m = S_m^{(n,r)}$ by

$$S_m = \frac{n!}{r!}\sum_{k=0}^{m}(k-n)^{\overline{r}}\frac{(-1)^{k+r}}{k!}, \qquad m \geq 0, \tag{16}$$

and hence $\lim_{m\to\infty}(S_m)$ exists, being a convergent alternating series. We denote this limit by $\ell = \ell_{n,r}$. Note $S_n = S_{n-r} = D_r(n)$, by (15), since the terms for $k = n-r+1,\ldots,n$ appearing in the summation on the right-hand side of (16) are all zero.

We consider cases based on the parity of n and r. If n and r are of different parity, then we have

$$D_r(n) = S_{n-r} = S_n < \ell < S_{n+1} = S_n + \frac{1}{n+1} < S_n + 1,$$

and hence $D_r(n) = \lfloor \ell \rfloor$, as S_m for $m \geq n$ is a convergent alternating series. On the other hand, if n and r have the same parity, then

$$S_n - 1 \leq S_{n+1} = S_n - \frac{1}{n+1} < \ell < S_n,$$

which implies $D_r(n) = \lfloor \ell \rfloor + 1$ and completes the proof of (13).

For (14), first recall the fact

$$\sum_{j=0}^{m}(-1)^{m-j}\begin{bmatrix} m \\ j \end{bmatrix}_r x^j = (x-r)^{\underline{m}}, \qquad m \geq 0,$$

see, e.g., [3]. Thus, we may write

$$\frac{r!}{n!}\ell_{n,r} = \sum_{k\geq 0}(k-n+r-1)^{\underline{r}}\frac{(-1)^{k+r}}{k!} = \sum_{k\geq 0}\frac{(-1)^{k+r}}{k!}\sum_{j=0}^{r}(-1)^{r-j}\begin{bmatrix} r \\ j \end{bmatrix}_{n-r+1} k^j = \sum_{j=0}^{r}(-1)^j\begin{bmatrix} r \\ j \end{bmatrix}_{n-r+1}\sum_{k\geq 0}\frac{(-1)^k k^j}{k!}. \tag{17}$$

By [9, Proposition 1], we have $B_m^* = e\sum_{k\geq 0}\frac{(-1)^k k^m}{k!}$ for each $m \geq 0$. Using this fact in (17) yields (14) and completes the proof. $\square$

Combining Lemma 2.1 and Theorem 2.2 implies the following summation formula for $D_r(n)$, which reduces to (3) when $r=0$.

Corollary 2.1. *If $n \geq r \geq 0$, then*

$$\sum_{i=r}^{n} iD_r(i) + r\sum_{i=0}^{\lfloor (n-r)/2\rfloor}(D_r(n-2i)+D_{r-1}(n-2i)) = \lfloor \ell_{n+1,r}\rfloor + \frac{(-1)^{r+n}-1}{2}(r!-1), \tag{18}$$

where $\ell_{n,r}$ is given by (14).

Remark: Further examination of the inequalities used in the proof of (13) shows the following formula that reduces to (2) when $r=0$: $D_r(n) = \|\ell_{n,r}\| = \lfloor \ell_{n,r}+e^{-1}\rfloor$ for $n \geq r \geq 0$, where it is assumed $n \geq 1$ if $r=0$. See [14, Theorem 7] for a related asymptotic formula for $D_r(n)$ with error bound.

3. Further Related Results

An analogous formula to (4) for permutations is given by

$$\sum_{i=1}^{n} i\cdot i! = (n+1)!-1, \qquad n \geq 1. \tag{19}$$

See, for instance, [2, Identity 181]. We prove here a polynomial extension of (19) as follows. Assume members of $\mathcal{S}_n$ are expressed in disjoint cycle form. Define $a_n(p,q)$ for $n \geq 1$ by

$$a_n(p,q) = \sum_{\pi\in\mathcal{S}_n} p^{\mu(\pi)}q^{\mathrm{fix}(\pi)}, \qquad n \geq 1, \tag{20}$$

with $a_0(p,q)=1$, where $\mu(\pi)$ and fix(π) denote the number of cycles of length at least two and number of fixed points, respectively, in the permutation π. Let $d_n(p)=\sum_{\pi\in\mathcal{D}_n}p^{\mathrm{cyc}(\pi)}$ for $n \geq 1$, with $d_0(p)=1$, where cyc(π) represents the number of cycles of π. Note that $d_n(p)=a_n(p,0)$, by the definitions.

There is the following explicit formula for $a_n(p,q)$.

Theorem 3.1. *We have*

$$a_{n+1}(p,q)-q^{n+1} = \sum_{i=1}^{n}\sum_{j=0}^{i}\binom{i}{j}d_j(p)\left(jq^{n-j}+(i-j)pq^{n-j-1}\right), \qquad n \geq 1. \tag{21}$$

Proof. We first provide an algebraic proof using generating functions; see below for a combinatorial argument. Let

$$r_n = \sum_{i=1}^{n} q^{n-i}\sum_{j=0}^{i}\binom{i}{j}d_j(p)\left(jq^{i-j}+(i-j)pq^{i-j-1}\right),$$

and note

$$r_n - qr_{n-1} = \sum_{j=0}^{n}\binom{n}{j}d_j(p)\left(jq^{n-j}+(n-j)pq^{n-j-1}\right), \qquad n \geq 1,$$

with $r_0=0$. Computing the exponential generating function of r_n-qr_{n-1}, we obtain

$$\begin{aligned}
\sum_{n\geq 1}(r_n-qr_{n-1})\frac{x^n}{n!} &= \sum_{n\geq 0}\frac{x^n}{n!}\sum_{j=0}^{n}\binom{n}{j}d_j(p)\left(jq^{n-j}+(n-j)pq^{n-j-1}\right)\\
&= \left(1-\frac{p}{q}\right)\sum_{n\geq 0}\frac{x^n}{n!}\sum_{j=0}^{n}\binom{n}{j}jq^{n-j}d_j(p) + \sum_{n\geq 0}\frac{x^n}{n!}\sum_{j=0}^{n}\binom{n}{j}npq^{n-j-1}d_j(p)\\
&= \left(1-\frac{p}{q}\right)\sum_{j\geq 1}\frac{d_j(p)q^{-j}}{(j-1)!}\sum_{n\geq j}\frac{(qx)^n}{(n-j)!} + p\sum_{j\geq 0}\frac{d_j(p)q^{-j-1}}{j!}\sum_{n\geq j}\left(\frac{n-j}{(n-j)!}+\frac{j}{(n-j)!}\right)(qx)^n\\
&= \left(1-\frac{p}{q}\right)\sum_{j\geq 1}\frac{d_j(p)x^j}{(j-1)!}\cdot e^{qx} + p\sum_{j\geq 0}\frac{d_j(p)x^{j+1}}{j!}\cdot e^{qx} + \frac{p}{q}\sum_{j\geq 1}\frac{d_j(p)x^j}{(j-1)!}\cdot e^{qx}\\
&= xe^{qx}\sum_{j\geq 1}\frac{d_j(p)x^{j-1}}{(j-1)!} + pxe^{qx}\sum_{j\geq 0}\frac{d_j(p)x^j}{j!}.
\end{aligned}$$

By the exponential formula (see, e.g., [13, Section 7.1]), we have

$$\sum_{j\geq 0} \frac{d_j(p)x^j}{j!} = \exp\left(-px + p\sum_{i\geq 1}\frac{x^i}{i}\right) = e^{-px-p\ln(1-x)} = \frac{e^{-px}}{(1-x)^p}, \tag{22}$$

and thus

$$\sum_{n\geq 1}(r_n - qr_{n-1})\frac{x^n}{n!} = xe^{qx}\left(\frac{pe^{-px}}{(1-x)^p} + \frac{d}{dx}\left(\frac{e^{-px}}{(1-x)^p}\right)\right) = \frac{pxe^{(q-p)x}}{(1-x)^{p+1}}. \tag{23}$$

Let $g(x) = \sum_{n\geq 0} r_n \frac{x^{n+1}}{(n+1)!}$. By (23), we have

$$g'(x) - qg(x) = \frac{pxe^{(q-p)x}}{(1-x)^{p+1}}. \tag{24}$$

Solving the first-order linear differential equation (24) with initial condition $g(0) = 0$ yields

$$g(x) = \frac{e^{(q-p)x}}{(1-x)^p} - e^{qx}.$$

Note

$$\sum_{n\geq 0} a_n(p,q)\frac{x^n}{n!} = \exp\left((q-p)x + p\sum_{i\geq 1}\frac{x^i}{i}\right) = \frac{e^{(q-p)x}}{(1-x)^p},$$

by the exponential formula (or by the fact $a_n(p,q) = \sum_{k=0}^{n}\binom{n}{k}q^{n-k}d_k(p)$, together with (22)). Extracting the coefficient of $\frac{x^{n+1}}{(n+1)!}$ in $g(x)$ then gives

$$r_n = a_{n+1}(p,q) - q^{n+1}, \qquad n \geq 0,$$

which completes the proof. □

Taking $p = q = 1$, one has that (21) reduces to (19), by the fact

$$i! = \sum_{j=0}^{i}\binom{i}{j}d_j.$$

Formula (21) reduces when $p = 1$, $q = 0$ to the well-known derangement number recurrence $d_{n+1} = n(d_n + d_{n-1})$, since $a_n(1,0) = d_n(1) = d_n$, as the only non-zero terms within the summation on the right-hand side of (21) in this case are those for which $i = n$ and $j = n,\ n-1$. One may provide a combinatorial proof of (21) as follows by extending the argument given for (19) in [2].

Combinatorial proof of (21)*:*

First note that the left side of (21) gives the sum of the weights of all the members of $\mathcal{S}_{n+1}$ in the sense of (20) except for the identity permutation $12\cdots(n+1)$, which accounts for the subtracted term q^{n+1}. Let $\mathcal{A}_i$ for $1 \leq i \leq n$ denote the subset of $\mathcal{S}_{n+1} - \{12\cdots(n+1)\}$ wherein $i+1$ is the largest member of $[2, n+1]$ that is not a fixed point. It suffices to show

$$\sum_{\pi\in\mathcal{A}_i} p^{\mu(\pi)}q^{\text{fix}(\pi)} = q^{n-i}\sum_{j=0}^{i}\binom{i}{j}d_j(p)\left(jq^{i-j} + (i-j)pq^{i-j-1}\right), \qquad 1 \leq i \leq n, \tag{25}$$

from which (21) will follow upon considering all possible i.

Let $\pi \in \mathcal{A}_i$. Then the elements in $[i+2, n+1]$ each occur as 1-cycles and hence account for the initial factor of q^{n-i} in (25). First suppose $i+1$ occurs in a cycle of size at least three. Let $i-j$ denote the number of fixed points amongst the elements of $[i]$ within π, where $2 \leq j \leq i$. Then there are $\binom{i}{j}q^{i-j}$ possibilities for these elements, with the remaining j members of $[i]$ comprising a derangement of size j, and hence are accounted for by $d_j(p)$. We insert $i+1$ into one of the cycles of this derangement. Considering all possible j then implies that the weight of such π is given by $q^{n-i}\sum_{j=2}^{i}\binom{i}{j}d_j(p)jq^{i-j}$.

Now assume $i+1$ occurs within a 2-cycle of π. Suppose exactly j elements amongst those in $[i]$ occur in cycles of length at least two, excluding the cycle containing $i+1$, where $0 \leq j \leq i-1$. Then these elements contribute weight $\binom{i}{j}d_j(p)$, as they comprise a member of $\mathcal{D}_j$. From the other $i-j$ members of $[i]$, we select one to go into a cycle with $i+1$, which is accounted for by $(i-j)p$. Then the remaining $i-j-1$ members of $[i]$ occur as 1-cycles, which contributes an additional factor of q^{i-j-1}. Considering all possible j then implies that the weight of all $\pi \in \mathcal{A}_i$ where $i+1$ occurs in a 2-cycle is given by $q^{n-i}\sum_{j=0}^{i-1}\binom{i}{j}d_j(p)(i-j)pq^{i-j-1}$. Combining this case with the prior one, and noting that both sums may be extended to $0 \leq j \leq i$, gives the weight of all the members of $\mathcal{A}_i$ and implies (25), as desired. □

The following formulas for $a_n(p,q)$ will be seen to simultaneously generalize formulas (4) and (19).

Theorem 3.2. *We have*

$$a_{2m+1}(p,q)=\sum_{i=1}^{m}(2i-1)p^{m-i+1}a_{2i-1}(p,q)+\sum_{i=0}^{m}\sum_{j=0}^{2i}\binom{2i}{j}(j+q)p^{m-i}q^{2i-j}d_j(p),\quad m\geq 0, \tag{26}$$

and

$$a_{2m}(p,q)-p^m=\sum_{i=1}^{m-1}2ip^{m-i}a_{2i}(p,q)+\sum_{i=1}^{m}\sum_{j=0}^{2i-1}\binom{2i-1}{j}(j+q)p^{m-i}q^{2i-j-1}d_j(p),\quad m\geq 1. \tag{27}$$

Proof. Consider those $\pi\in\mathcal{S}_{2m+1}$ for which i is the largest $j\in[m]$ such that the 2-cycle $(2j,2j+1)$ does not occur. Then it must be the case that $2i+1$ either (i) belongs to the cycle $(a,2i+1)$ for some $a\in[2i-1]$, (ii) lies in a cycle containing at least two elements of $[2i]$, or (iii) occurs as a 1-cycle. Note that there are $(2i-1)p^{m-i+1}a_{2i-1}(p,q)$ possibilities for π satisfying (i), with the factor p^{m-i+1} accounting for the $m-i+1$ cycles $(a,2i+1),(2i+2,2i+3),\ldots,(2m,2m+1)$ and $a_{2i-1}(p,q)$ giving the contribution from the elements of $[2i]-\{a\}$ for each $a\in[2i-1]$. If (ii) or (iii) occurs, let $2i-j$ denote the number of 1-cycles occurring amongst the members of $[2i]$, where $0\leq j\leq 2i$. Then the remaining j elements of $[2i]$ contribute $d_j(p)$, as they comprise a derangement ρ of size j (marked by the number of cycles), with the factor of p^{m-i} attributable to the cycles $(2i+2,2i+3),\ldots,(2m,2m+1)$. The element $2i+1$ may then directly follow a letter within a cycle of ρ (assuming cycles are arranged in standard form) or occur as a 1-cycle, which gives $j+q$ possibilities regarding the placement of $2i+1$. Thus, for each $0\leq j\leq 2i$, we get a contribution towards the weight of $\binom{2i}{j}(j+q)p^{m-i}q^{2i-j}d_j(p)$. Considering all possible j yields $\sum_{j=0}^{2i}\binom{2i}{j}(j+q)p^{m-i}q^{2i-j}d_j(p)$ for those π wherein $2i+1$ satisfies (ii) or (iii). Summing this contribution and the prior one over $1\leq i\leq m$ gives the weight of all members of $\mathcal{S}_{2m+1}-\{(1)(2,3)\cdots(2m,2m+1)\}$ and accounts for the two summations on the right side of (26). The permutation $(1)(2,3)\cdots(2m,2m+1)$ has weight p^mq corresponding to the $i=0$ term in the second summation, which completes the proof of (26). A similar proof applies to (27), upon considering the largest $j\in[m]$ for which the cycle $(2j-1,2j)$ fails to occur within a member of $\mathcal{S}_{2m}-\{(1,2)\cdots(2m-1,2m)\}$. □

Letting $q=0$ in (26) and (27), and noting $a_n(p,0)=d_n(p)$, yields

$$d_{2m+1}(p)=\sum_{i=1}^{m}(2i-1)p^{m-i+1}d_{2i-1}(p)+\sum_{i=1}^{m}2ip^{m-i}d_{2i}(p),\qquad m\geq 0,$$

$$d_{2m}(p)-p^m=\sum_{i=1}^{m-1}2ip^{m-i}d_{2i}(p)+\sum_{i=1}^{m}(2i-1)p^{m-i}d_{2i-1}(p),\qquad m\geq 1.$$

Letting $p=q$, one obtains comparable formulas for the cycles statistic distribution on $\mathcal{S}_n$. Note that (26) and (27) reduce to (4) when $p=1$, $q=0$. When $p=q=1$, on the other hand, note that the inner sum $\sum_{j=0}^{k}\binom{k}{j}(j+q)q^{k-j}d_j(p)$ within the second summation reduces to $\sum_{j=0}^{k}\binom{k}{j}jd(j)+k!$. The latter sum is seen to enumerate all of the letters occurring in cycles of length greater than one within the members of $\mathcal{S}_k$, with $k!$ enumerating the fixed points in $\mathcal{S}_k$, and hence their total is given by $k\cdot k!$ representing the number of letters altogether in $\mathcal{S}_k$. Therefore, the right side of (26) reduces to

$$\sum_{i=1}^{m}(2i-1)(2i-1)!+1+\sum_{i=1}^{m}(2i)(2i)!=1+\sum_{i=1}^{2m}i\cdot i!,$$

and similarly the right side of (27) reduces to $\sum_{i=1}^{2m-1}i\cdot i!$. Thus, when $p=q=1$, we have that (26) and (27) correspond to (19) in the n even and n odd cases, respectively.

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